Lognormals and friends

Principles of Complex Systems
CSYS/MATH 300, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics | Center for Complex Systems | Vermont Advanced Computing Center | University of Vermont

Outline

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Lognormals
Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

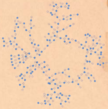
$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞)

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} \mathrm{d}x$$

   CCDF = stretched exponential (⊞).

3. Gamma distributions (⊞), and more.



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lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(- \frac{(\ln x - \mu)^2}{2\sigma^2} \right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

- All moments of lognormals are finite.

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Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi\sigma}}\mathrm{d}y \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

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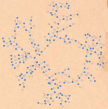

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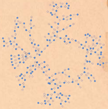

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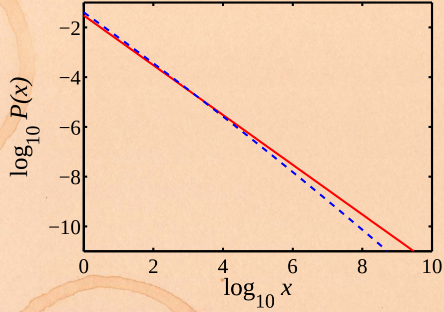
Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

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What's happening:

- $$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$
- $$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$
- $$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$
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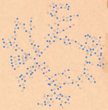

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- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- This happens when (roughly)
- $$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$
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Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
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- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
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- One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

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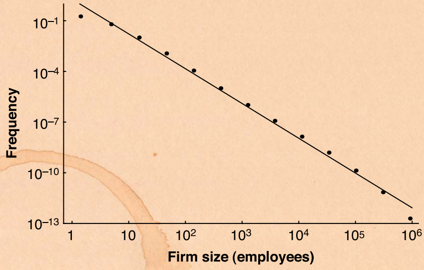

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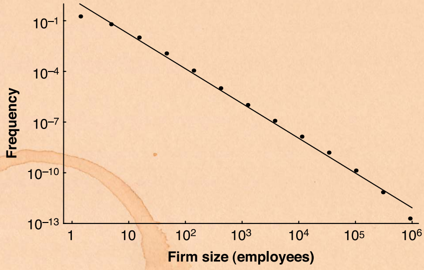

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An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 1$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c\langle x_i \rangle)$$

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- where γ is implicitly given by
$$N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{(c/N)^{\gamma-1}-1}{(c/N)^{\gamma-1}-(c/N)}\right]$$
N = total number of firms.
- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$
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Outline

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

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- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
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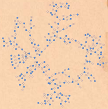

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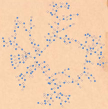

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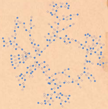

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