Singular Value Decomposition

Matrixology (Linear Algebra)—Lecture 25/25
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Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Fundamental Theorem of Linear Algebra

- Applies to any $m \times n$ matrix A.
- Symmetry of A and A^{T}.

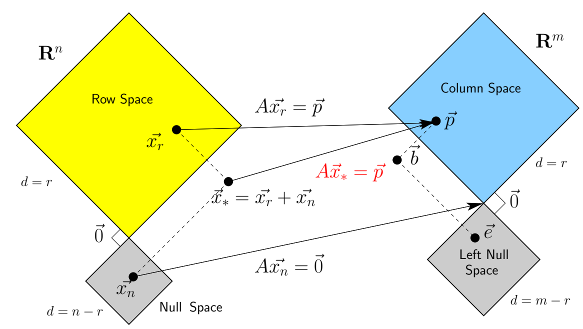
Where $\vec{x}$ lives:

- Row space $C(A^{\mathrm{T}}) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- $\dim C(A^{\mathrm{T}}) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^{\mathrm{T}}) \bigotimes N(A) = R^n$

Where $\vec{b}$ lives:

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^{\mathrm{T}}) \subset R^m$.
- $\dim C(A) + \dim N(A^{\mathrm{T}}) = r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^{\mathrm{T}}) = R^m$

Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:



Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a ‘best’ orthonormal basis
- The $\hat{v}_i$ span R^n
- We find the $\hat{v}_i$ as eigenvectors of $A^{\mathrm{T}}A$.
- The $\hat{u}_i$ span R^m
- We find the $\hat{u}_i$ as eigenvectors of AA^{T}.

Happy bases

- $\{\hat{v}_1, \ldots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space

Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

- $$\boxed{A\hat{v}_i = \sigma_i \hat{u}_i} \text{ for } i = 1, \ldots, r.$$
  and
  $$\boxed{A\hat{v}_i = \hat{0}} \text{ for } i = r + 1, \ldots, n.$$
- Matrix version:
  $$\boxed{A = U\Sigma V^{\mathrm{T}}}$$
- A sends each $\hat{v}_i \in C(A^{\mathrm{T}})$ to its partner $\hat{u}_i \in C(A)$ with a positive stretch/shrink factor $\sigma_i > 0$.
- A is diagonal with respect to these bases.
- When viewed in the right way, every A is a diagonal matrix Σ.

The complete big picture:

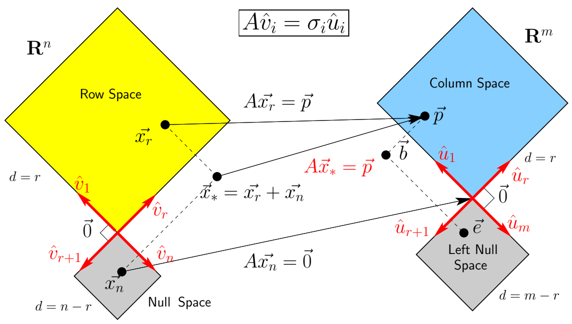


Image approximation (80x60)

Idea: use SVD to approximate images

- Interpret elements of matrix A as color values of an image.
- Truncate series SVD representation of A:

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

- Use fact that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$.
- Rank $r = \min(m, n)$.
- Rank $r = \#$ of pixels on shortest side (usually).
- For color: approximate 3 matrices (RGB).