Chapter 3/4: Lecture 15

Matrixology (Linear Algebra)—Lecture 14/25
MATH 124, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
  1. Big Picture of $\mathbb{A}\vec{x} = \vec{b}$
     Must be able to draw the big picture!
  2. Projections and the normal equation
- As always, want ‘doing’ and ‘understanding’ abilities.

Stuff to know/understand

Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.
- Concept of orthogonal complement.
- Various techniques for finding bases and orthogonal complements.

Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $\mathbb{A}$.
- Symmetry of $\mathbb{A}$ and $\mathbb{A}^{\mathrm{T}}$.

- Column space $C(\mathbb{A}) \subset R^m$.
- Left Nullspace $N(\mathbb{A}^{\mathrm{T}}) \subset R^m$.
- dim $C(\mathbb{A})$ + dim $N(\mathbb{A}^{\mathrm{T}})$ = $r + (m - r) = m$
- Orthogonality: $C(\mathbb{A}) \bigotimes N(\mathbb{A}^{\mathrm{T}}) = R^m$

- Row space $C(\mathbb{A}^{\mathrm{T}}) \subset R^n$.
- (Right) Nullspace $N(\mathbb{A}) \subset R^n$.
- dim $C(\mathbb{A}^{\mathrm{T}})$ + dim $N(\mathbb{A})$ = $r + (n - r) = n$
- Orthogonality: $C(\mathbb{A}^{\mathrm{T}}) \bigotimes N(\mathbb{A}) = R^n$

Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $\mathbb{A}$ to $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{A}^{\mathrm{T}}$ to $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$.
- Understand crucial nature of $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$.
- Identify pivot columns and free columns.
- Rank r of $\mathbb{A}$ = # pivot columns.
- Know that relationship between $\mathbb{R}_{\mathbb{A}}$’s columns hold for $\mathbb{A}$’s columns.

Stuff to know/understand:

Bases for column space—three ways:

1. Find when $\mathbb{A}\vec{x} = \vec{b}$ has a solution:
   - Reduce $[\mathbb{A} \,|\, \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$'s elements for a solution to $\mathbb{A}\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(\mathbb{A})$.
2. Use $\mathbb{R}_{\mathbb{A}}$:
   - Find pivot columns in $\mathbb{R}_{\mathbb{A}}$—same columns in $\mathbb{A}$ form a basis for $C(\mathbb{A})$.
   - Warning: $\mathbb{R}_{\mathbb{A}}$'s columns do not give a basis for $C(\mathbb{A})$
3. Use $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$:
   - Best and easiest way: basis for column space = non-zero rows in $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$, the reduced form of $\mathbb{A}^{\mathrm{T}}$.

Basis for row space:

- Take non-zero rows in $\mathbb{R}_{\mathbb{A}}$ (easy!).
- Matches way 3 for column space.

Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b} = \vec{p} + \vec{e}$
- $\vec{p}$ = that part of $\vec{b}$ that lies in the line:

$$\vec{p} = \frac{\vec{a}^{\mathrm{T}}\vec{b}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{a} \left(= \frac{\vec{a}\vec{a}^{\mathrm{T}}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{b} \right)$$

- $\vec{e}$ = that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
- Understand construction and use of subspace projection operator $\mathbb{P}$:

$$\mathbb{P} = \mathbb{A}(\mathbb{A}^{\mathrm{T}}\mathbb{A})^{-1}\mathbb{A}^{\mathrm{T}},$$

  where $\mathbb{A}$'s columns form a subspace basis.

Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $\mathbb{A}\vec{x} = \vec{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- # free variables = n - # pivot variables = $n - r$ = dim $N(\mathbb{A})$.
- Similarly find basis for $N(\mathbb{A}^{\mathrm{T}})$ by solving $\mathbb{A}^{\mathrm{T}}\vec{y} = \vec{0}$.
- dim $N(\mathbb{A}^{\mathrm{T}})$ = $m - r$.
- Key: Find bases for both nullspaces directly from $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$.

Stuff to know/understand

Normal equation for $\mathbb{A}\vec{x} = \vec{b}$:

- If $\vec{b} \notin C(\mathbb{A})$, project $\vec{b}$ onto $C(\mathbb{A})$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(\mathbb{A})$ so $\exists\, \vec{x}_*$ such that $\mathbb{A}\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $\mathbb{A}^{\mathrm{T}}\vec{e} = \mathbb{A}^{\mathrm{T}}(\vec{b} - \vec{p}) = \vec{0}$.
- Rearrange:

$$\mathbb{A}^{\mathrm{T}}\vec{p} = \mathbb{A}^{\mathrm{T}}\vec{b}$$

  .
- Since $\mathbb{A}\vec{x}_* = \vec{p}$, we end up with

$$\mathbb{A}^{\mathrm{T}}\mathbb{A}\vec{x}_* = \mathbb{A}^{\mathrm{T}}\vec{b}.$$

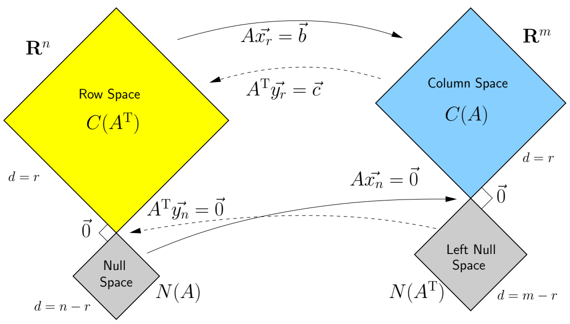
- This is linear algebra's normal equation; $\vec{x}_*$ is our best solution to $\mathbb{A}\vec{x} = \vec{b}$.

Stuff to know/understand:

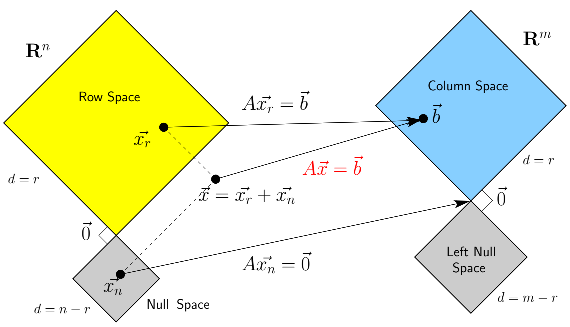
Number of solutions to $\mathbb{A}\vec{x} = \vec{b}$:

1. If $\vec{b} \notin C(\mathbb{A})$, there are no solutions.
2. If $\vec{b} \in C(\mathbb{A})$ there is either one unique solution or infinitely many solutions.
   - Number of solutions now depends entirely on $N(\mathbb{A})$.
   - If dim $N(\mathbb{A}) = n - r > 0$, then there are infinitely many solutions.
   - If dim $N(\mathbb{A}) = n - r = 0$, then there is one solution.

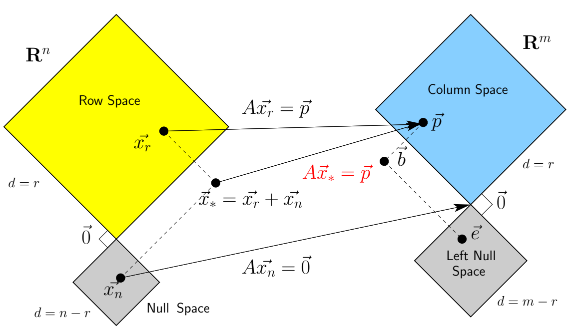
The symmetry of $\mathbb{A}\vec{x} = \vec{b}$ and $\mathbb{A}^{\mathrm{T}}\vec{y} = \vec{c}$:



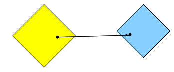
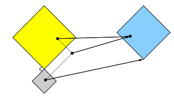
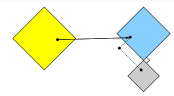
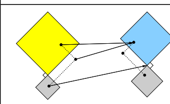
How $\mathbb{A}\vec{x} = \vec{b}$ works:



Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:



The fourfold ways of $\mathbb{A}\vec{x} = \vec{b}$:

case	example R	big picture	# solutions
$m = r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		1 always
$m = r$, $n > r$	$\begin{bmatrix} 1 & 0 & ☕_1 \\ 0 & 1 & ☕_2 \end{bmatrix}$		∞ always
$m > r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$		0 or 1
$m > r$, $n > r$	$\begin{bmatrix} 1 & 0 & 🚲_1 \\ 0 & 1 & 🚲_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		0 or ∞