Solving Linear Equations

Matrixology (Linear Algebra)—Lecture 2/25
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Outline

Solving $A\,\vec{x} = \vec{b}$

Solving $A\vec{x} = \vec{b}$:

- We (people + computers) solve systems of linear equations by a systematic method of Elimination followed by Back substitution
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

$$\begin{aligned} -x_1 &+ 3x_2 &= 1 \\ 2x_1 &+ x_2 &= 5 \end{aligned} \tag{1}$$

$\rightsquigarrow$ chalkage

Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an 'upper triangular form'

   e.g.

$$\begin{array}{rcrcr} & & x_2 & = & 3 \\ 2x_1 & - & x_2 & = & -1 \end{array} \rightarrow \begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ & & x_2 & = & 3 \end{array}$$

Gaussian elimination:

Solve:

$$2x_1 - 3x_2 = 3$$
$$4x_1 - 5x_2 + x_3 = 7$$
$$2x_1 - x_2 - 3x_3 = 5$$

Gaussian elimination:

Summary:

Using row operations, we turned this problem:

$$A\vec{x} = \vec{b}: \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

into this problem:

$$U\vec{x} = \vec{d}: \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

and the latter is easy to solve using back substitution.

Defn:

The entries along U's main diagonal are the pivots of A. (The pivots are hidden—elimination finds them.)

Defn:

A matrix with only zeros below the main diagonal is called upper triangular. A matrix with only zeros above the main diagonal is called lower triangular. We travel from A to U and the latter is always upper triangular.

Defn:

Singular means a system has no unique solution.

- It may have no solutions or infinitely many solutions.
- Singular = archaic way of saying 'messed up.'

Truth:

If at least one pivot is zero, the matrix will be singular. (but the reverse is not necessarily true).

Gaussian elimination:

The one true method:

- We simplify A using elimination in the same way every time.
- Eliminate entries one column at a time, moving left to right, and down each column.

$$\begin{array}{ccccccccc}
X & + & X & + & X & + & X & = & X \\
1\downarrow & + & X & + & X & + & X & = & X \\
2\downarrow & + & 4\downarrow & + & X & + & X & = & X \\
3\nearrow & + & 5\rightarrow & + & 6 & + & X & = & X
\end{array}$$

Gaussian elimination:

- To eliminate entry in row i of jth column, subtract a multiple ℓ_{ij} of the jth row from i.
- For example:

$$\begin{array}{ccccccccc}
2x_1 & + & 3x_2 & + & -2x_3 & + & x_4 & = & 1 \\
x_1 & - & 7x_2 & + & 3x_3 & + & x_4 & = & 1 \\
-x_1 & - & 3x_2 & - & x_3 & + & 5x_4 & = & -2 \\
2x_1 & + & x_2 & - & 2x_3 & + & 2x_4 & = & 0
\end{array}$$

$\ell_{21} = 1/2$, $\ell_{31} = -1/2$, $\ell_{41} = ?$.

- Note: we cannot find ℓ_{32} etc., until we are finished with row 1. Pivots are hidden!
- Note: the denominator of each ℓ_{ij} multiplier is the pivot in the jth column.