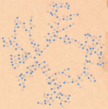
Generating Functions for Random Networks

Complex Networks
CSYS/MATH 303, Spring, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont



Outline

Generating Functions
- Definitions
- Basic Properties
- Giant Component Condition
- Component sizes
- Useful results
- Size of the Giant Component
- Average Component Size

References

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Generating functions

- Idea: Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^{∞} into a function defined on R^1.
- Related to Fourier, Laplace, Mellin, …

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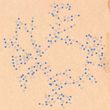

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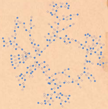

Simple example

Rolling dice:

- $p_k^{(\Box)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6.$
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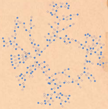
$$F^{(\Box)}(x) = \sum_{k=1}^{6} p_k x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- We'll come back to this simple example as we derive various delicious properties of generating functions.



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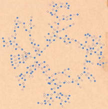

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Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where $c = 1 - e^{-\lambda}$.

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

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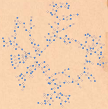

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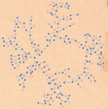

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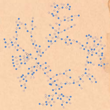

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Properties of generating functions

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1}$$

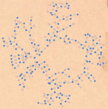
$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \Big|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda}) e^{-\lambda}}{(1 - x e^{-\lambda})^2}.$$

- So:

$$\langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$



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Properties of generating functions

Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

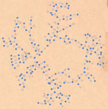
$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$$

- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \Big|_{x=0}$$



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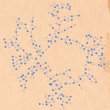

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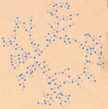

Outline

Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's reexpress our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.

- Condition in terms of g.f. is:

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- Now find how F_R is related to F_P...

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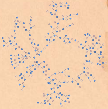

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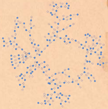

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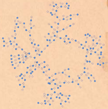

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Edge-degree distribution

- We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k+1$ and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{\mathrm{d}}{\mathrm{d}x} x^j$$

$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0\right) = \frac{1}{\langle k \rangle} F_P'(x).$$

Finally, since $\langle k \rangle = F_P'(1)$,

$$\boxed{F_R(x) = \frac{F_P'(x)}{F_P'(1)}}$$

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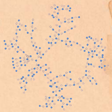

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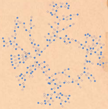

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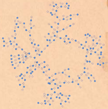

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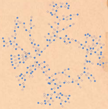

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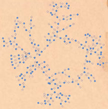

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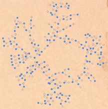

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Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
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Local-global connection:

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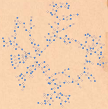

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- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
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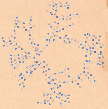

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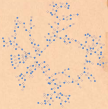

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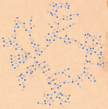

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- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V.
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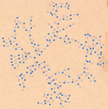

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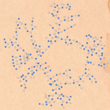

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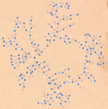

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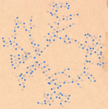

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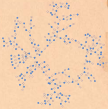

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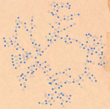

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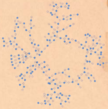

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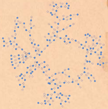

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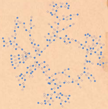

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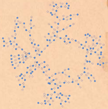

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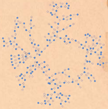

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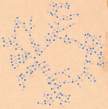

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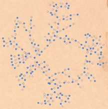

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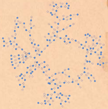

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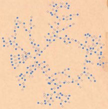

Connecting generating functions

- Goal: figure out forms of the component generating functions, F_π and F_ρ.
- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c}\text{sum of sizes of subcomponents}\\ \text{at end of } k \text{ random links} = n-1\end{array}\right)$$

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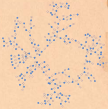

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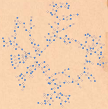

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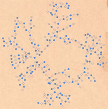

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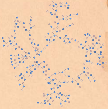

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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
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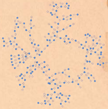

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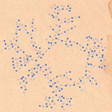

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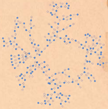

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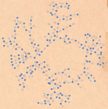

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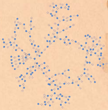

- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F_P'(x)/F_P'(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_ρ and F_R.
- The first equation then immediately gives us F_π in terms of F_ρ and F_R.

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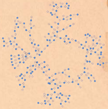

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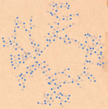

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Component sizes

- Remembering vaguely what we are doing:
  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.
- Set $x = 1$ in our two equations:

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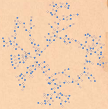

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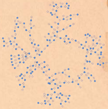

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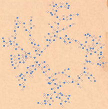

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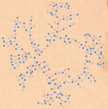

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- $$\therefore F_\pi(x) = x e^{-\langle k \rangle (1 - F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

Or: $\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$

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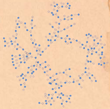

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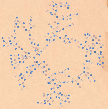

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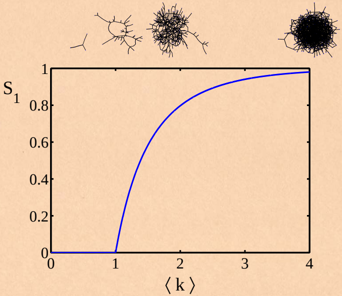

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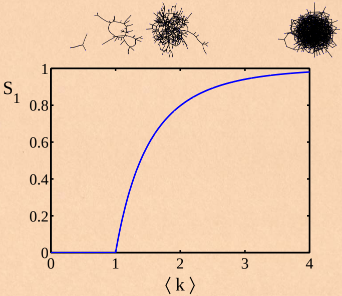

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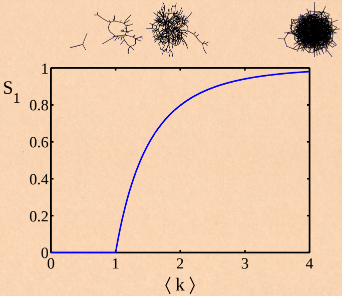

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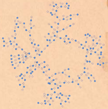

Average component size

- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$...
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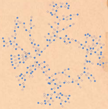

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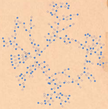

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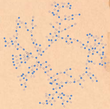

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- Try to avoid finding $F_{\pi}(x)$...
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$$F'_{\rho}(x) = F_R\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_R\left(F_{\rho}(x)\right)$$

- Now set $x = 1$ in both equations.
- We solve the second equation for $F'_{\rho}(1)$ (we must already have $F_{\rho}(1)$).
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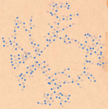

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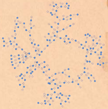

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Example: Standard random graphs.

- Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
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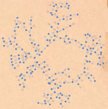

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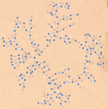

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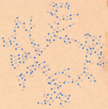

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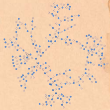

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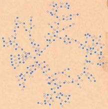

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- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

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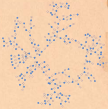

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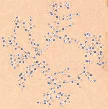

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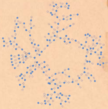

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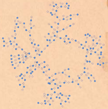

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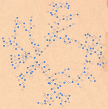

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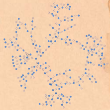

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- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}$.
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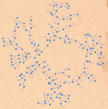

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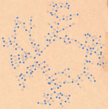

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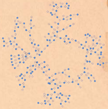

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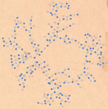

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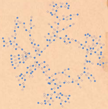

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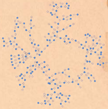

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