Random Networks

Complex Networks
CSYS/MATH 303, Spring, 2011

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Center for Complex Systems
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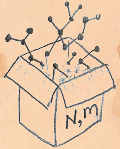
University of Vermont

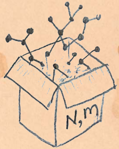


Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Random friends are strange
- Largest component
- Simple, physically-motivated analysis

References



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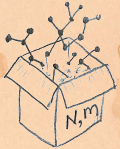
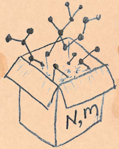


Random networks

Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = one randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.



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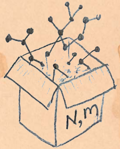

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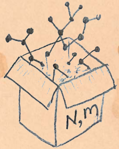

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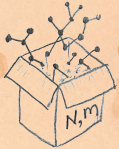

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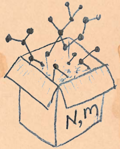

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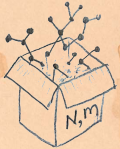

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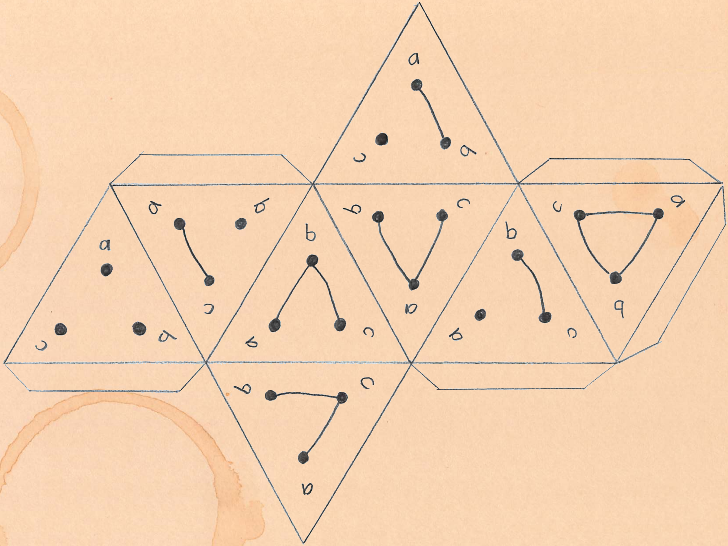
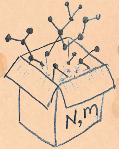
Random network generator for $N = 3$:



- Get your own exciting generator here (⊞).
- As $N \nearrow$, our polyhedral die rapidly becomes a ball...



Random networks—basic features:

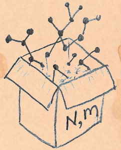
- Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- Limit of $m = 0$: empty graph.
- Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- Number of possible networks with N labelled nodes:

$$2^{\binom{N}{2}} \sim e^{\frac{\ln 2}{2} N^2}.$$

- Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Real world: links are usually costly so real networks are almost always sparse.



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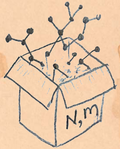

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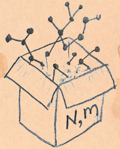

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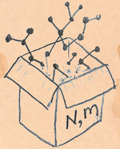

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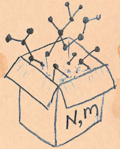

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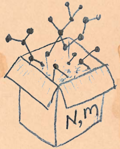

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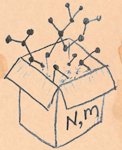

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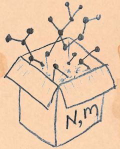

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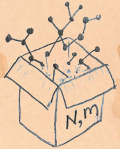
How to build standard random networks:

- Given N and m.
- Two probablistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.
   - Useful for theoretical work.
2. Take N nodes and add exactly m links by selecting edges without replacement.
   - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
   - Best for adding relatively small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.



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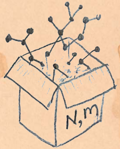

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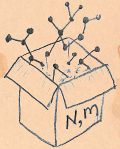

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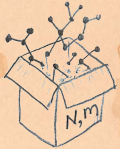

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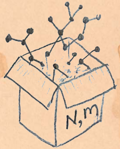

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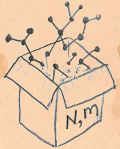

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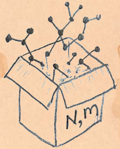

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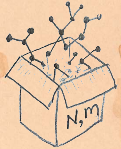

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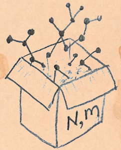

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A few more things:

- For method 1, # links is probablistic:

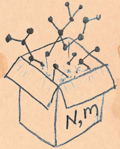
$$\langle m\rangle = p\binom{N}{2} = p\frac{1}{2}N(N-1)$$

- So the expected or average degree is

$$\langle k\rangle = \frac{2\langle m\rangle}{N}$$

$$= \frac{2}{N}p\frac{1}{2}N(N-1) = \frac{\not{2}}{\not{N}}p\frac{1}{\not{2}}\not{N}(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k\rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.



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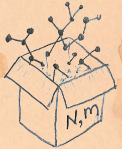

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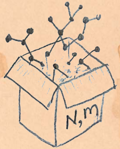

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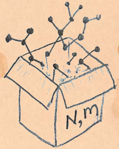

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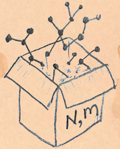

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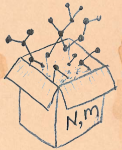

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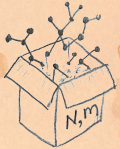
$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

- So the expected or average degree is

$$\langle k \rangle = \frac{2 \langle m \rangle}{N}$$

$$= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{\not{2}}{\not{N}} p \frac{1}{\not{2}} \not{N}(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.



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Random networks

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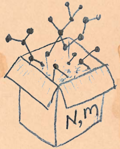
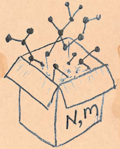


Outline

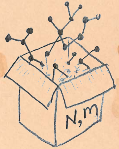


Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.



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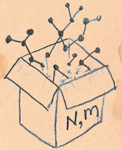

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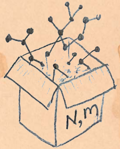

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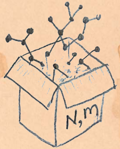

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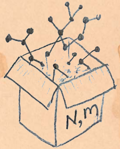
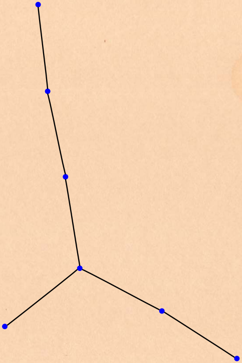


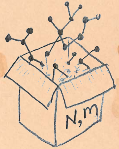
Random networks: examples

entire network:

largest component:

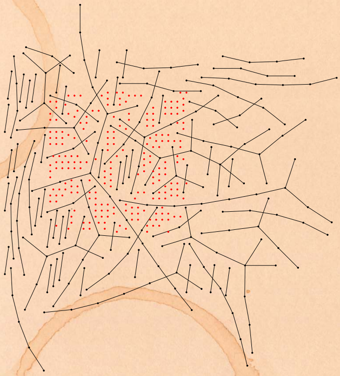


$N = 500$, number of edges $m = 100$
average degree $\langle k \rangle = 0.4$

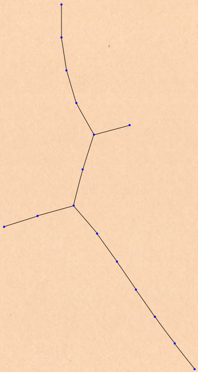


Random networks: examples

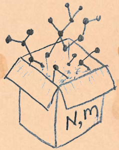
entire network:



largest component:



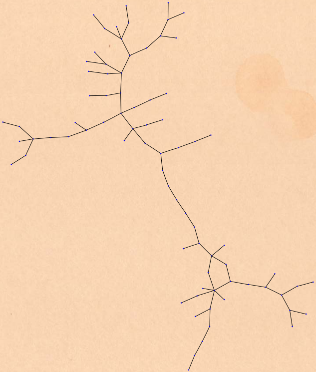
$N = 500$, number of edges $m = 200$
average degree $\langle k \rangle = 0.8$



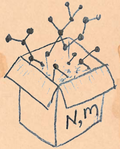
Random networks: examples

entire network:

largest component:



$N = 500$, number of edges $m = 230$
average degree $\langle k \rangle = 0.92$



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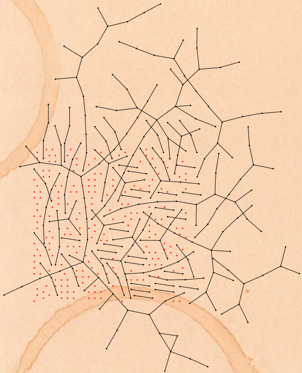
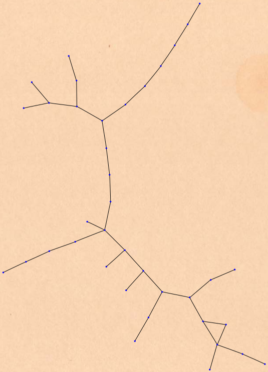
Random networks: examples

entire network:

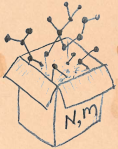


largest component:



$N = 500$, number of edges $m = 240$

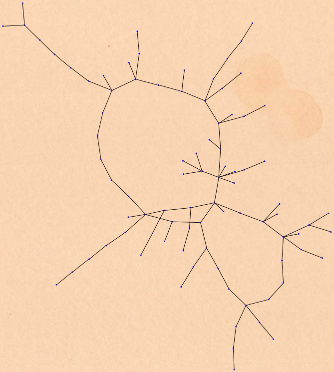
average degree $\langle k \rangle = 0.96$



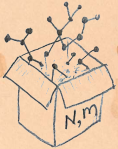
Random networks: examples

entire network:

largest component:

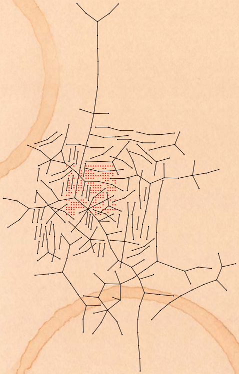


$N = 500$, number of edges $m = 250$
average degree $\langle k \rangle = 1$

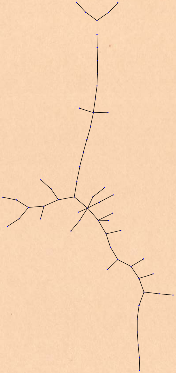


Random networks: examples

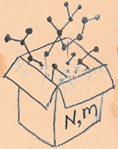
entire network:



largest component:

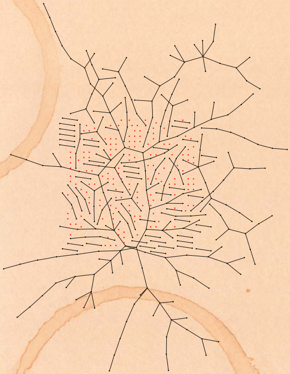


$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$

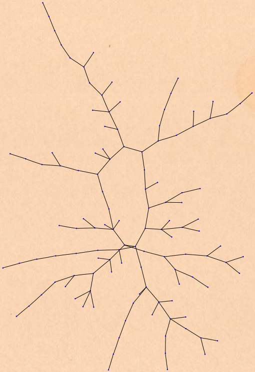


Random networks: examples

entire network:

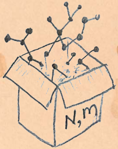


largest component:



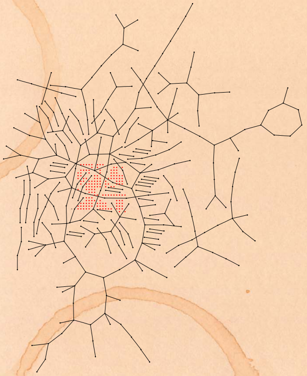
$N = 500$, number of edges $m = 280$

average degree $\langle k \rangle = 1.12$

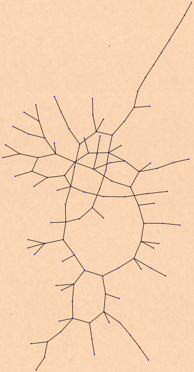


Random networks: examples

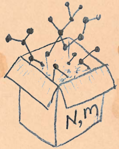
entire network:



largest component:



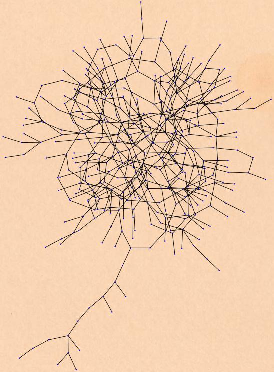
$N = 500$, number of edges $m = 300$
average degree $\langle k \rangle = 1.2$



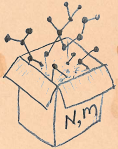
Random networks: examples

entire network:

largest component:



$N = 500$, number of edges $m = 500$
average degree $\langle k \rangle = 2$



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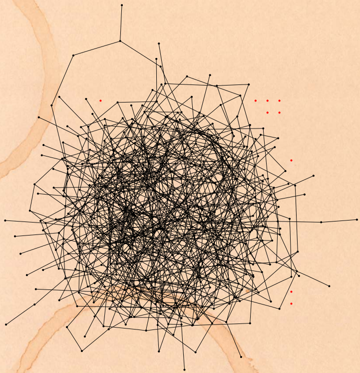
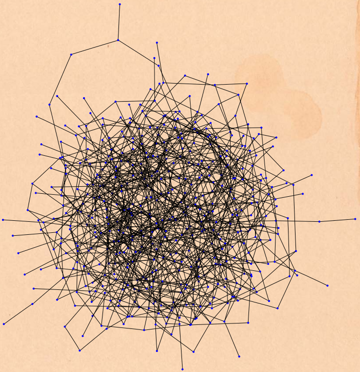
Random networks: examples

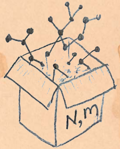
entire network:



largest component:



$N = 500$, number of edges $m = 1000$
average degree $\langle k \rangle = 4$



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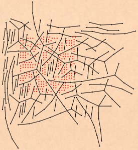
Random networks: examples for N=500

$m = 100$
$\langle k \rangle = 0.4$



$m = 200$
$\langle k \rangle = 0.8$

$m = 230$
$\langle k \rangle = 0.92$

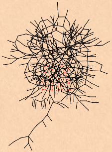
$m = 240$
$\langle k \rangle = 0.96$

$m = 250$
$\langle k \rangle = 1$

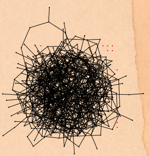
$m = 260$
$\langle k \rangle = 1.04$

$m = 280$
$\langle k \rangle = 1.12$

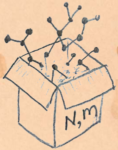
$m = 300$
$\langle k \rangle = 1.2$



$m = 500$
$\langle k \rangle = 2$

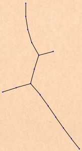


$m = 1000$
$\langle k \rangle = 4$



Random networks: largest components

$m = 100$
$\langle k \rangle = 0.4$



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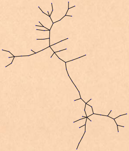

$m = 230$
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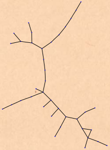
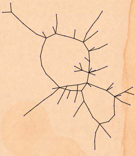
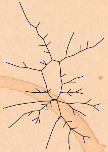


$m = 240$
$\langle k \rangle = 0.96$

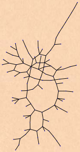


$m = 250$
$\langle k \rangle = 1$

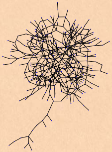
$m = 260$
$\langle k \rangle = 1.04$



$m = 280$
$\langle k \rangle = 1.12$



$m = 300$
$\langle k \rangle = 1.2$



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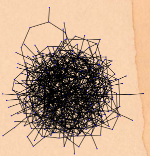

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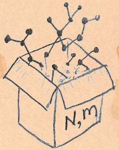
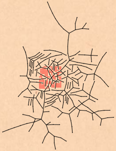


Random networks: examples for N=500

m = 250
$\langle k \rangle$ = 1

m = 250
$\langle k \rangle$ = 1

m = 250
$\langle k \rangle$ = 1



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$\langle k \rangle$ = 1

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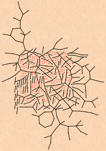
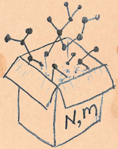


m = 250
$\langle k \rangle$ = 1

m = 250
$\langle k \rangle$ = 1

m = 250
$\langle k \rangle$ = 1



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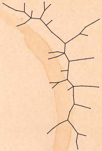
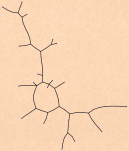
Random networks: largest components



$m = 250$
$\langle k \rangle = 1$



$m = 250$
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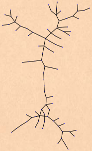

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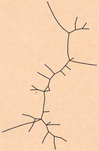
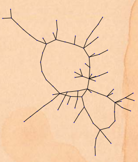


$m = 250$
$\langle k \rangle = 1$



$m = 250$
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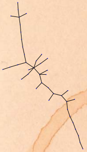
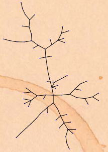
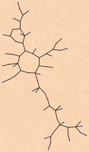


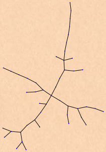
$m = 250$
$\langle k \rangle = 1$



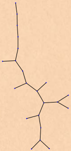
$m = 250$
$\langle k \rangle = 1$



$m = 250$
$\langle k \rangle = 1$



$m = 250$
$\langle k \rangle = 1$



$m = 250$
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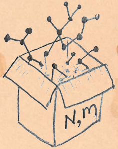

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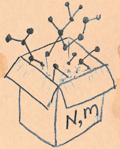
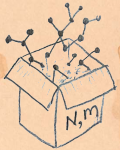


Clustering in random networks:

- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient:[1]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$



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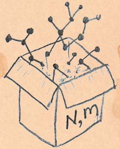

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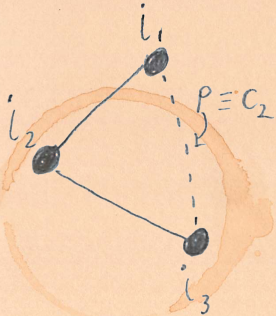
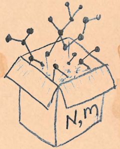


- Recall: C_2 = probability that two friends of a node are also friends.
- Or: C_2 = probability that a triple is part of a triangle.
- For standard random networks, we have simply that

$$C_2 = p.$$



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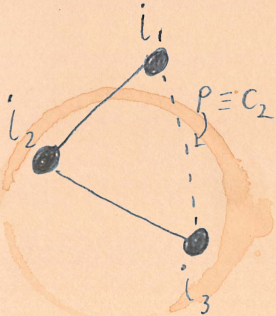

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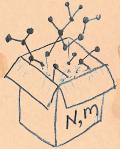

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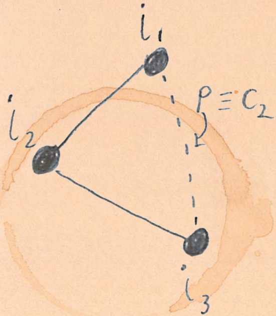

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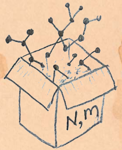


Other ways to compute clustering:

- Expected number of triples in entire network:

$$\frac{1}{2}N(N-1)(N-2)p^2$$

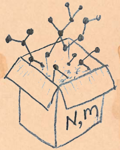
(Double counting dealt with by $\frac{1}{2}$.)

- Expected number of triangles in entire network:

$$\frac{1}{6}N(N-1)(N-2)p^3$$

(Over-counting dealt with by $\frac{1}{6}$.)

- $$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}} = \frac{3 \times \frac{1}{6}N(N-1)(N-2)p^3}{\frac{1}{2}N(N-1)(N-2)p^2} = p.$$



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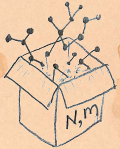

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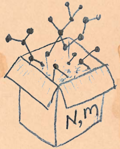

Other ways to compute clustering:

- **Or:** take any three nodes, call them a, b, and c.
- Triple a-b-c centered at b occurs with probability $p^2 \times (1-p) + p^2 \times p = p^2$.
- Triangle occurs with probability p^3.
- Therefore,

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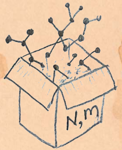

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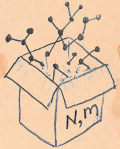

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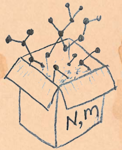

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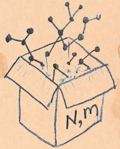
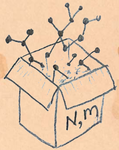


Clustering in random networks:

- So for large random networks ($N \to \infty$), clustering drops to zero.
- Key structural feature of random networks is that they locally look like pure branching networks
- No small loops.



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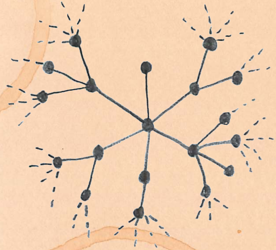

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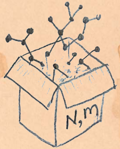

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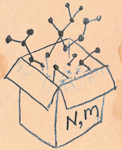

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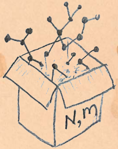

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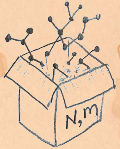
Random networks

Degree distribution:

- Recall P_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$



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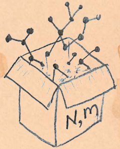

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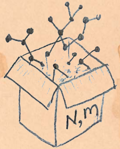

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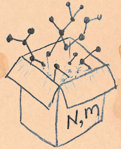

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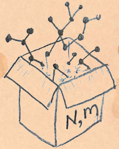

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- Our degree distribution: $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \to \infty$?
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- If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \to \infty$.
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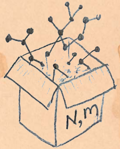

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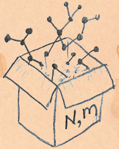

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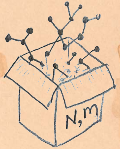

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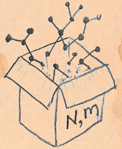

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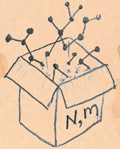

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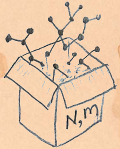

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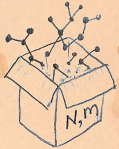

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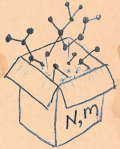

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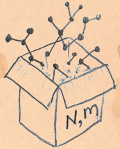

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- Substitute $p = \frac{\langle k \rangle}{N-1}$ into $P(k; p, N)$ and hold k fixed:

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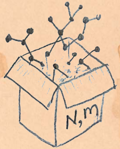

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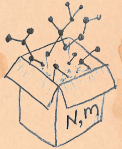

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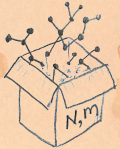

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- We are now here:

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k}$$

- Now use the excellent result:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

(Use l'Hôpital's rule to prove.)

- Identifying $n = N - 1$ and $x = -\langle k \rangle$:

$$P(k; \langle k \rangle) \simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

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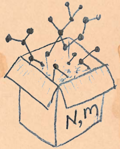

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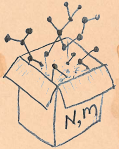

The University of Vermont

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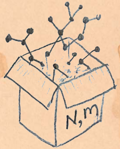

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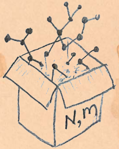
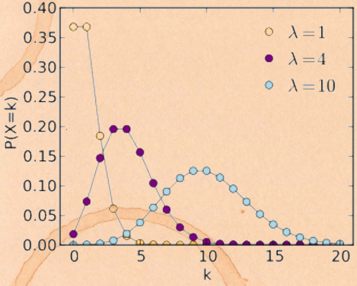
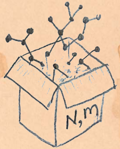


Poisson basics:

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$$



- $\lambda > 0$
- $k = 0, 1, 2, 3, \ldots$
- Classic use: probability that an event occurs k times in a given time period, given an average rate of occurrence.
- e.g.: phone calls/minute, horse-kick deaths.
- ‘Law of small numbers’



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- Normalization: we must have

$$\sum_{k=0}^{\infty} P(k;\langle k \rangle) = 1$$

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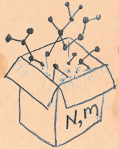

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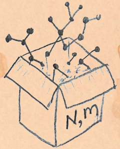

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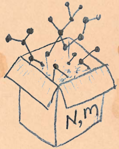

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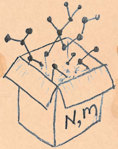

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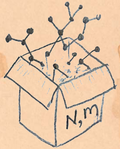

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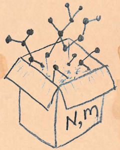

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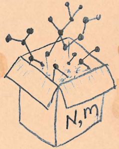

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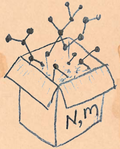

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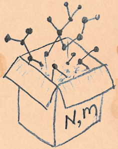

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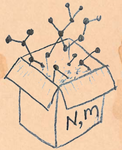

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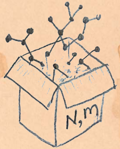

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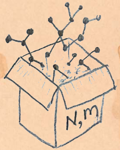

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Poisson basics:

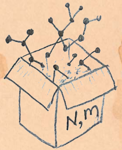
- The variance of degree distributions for random networks turns out to be very important.
- Use calculation similar to one for finding $\langle k \rangle$ to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

- So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.
- Note: This is a special property of Poisson distribution and can trip us up...



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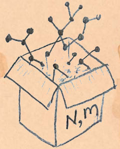

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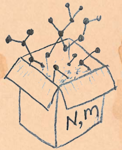

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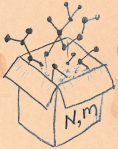
- So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.
- Note: This is a special property of Poisson distribution and can trip us up...



Poisson basics:

- The variance of degree distributions for random networks turns out to be very important.
- Use calculation similar to one for finding $\langle k \rangle$ to find the second moment:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

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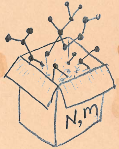

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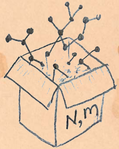

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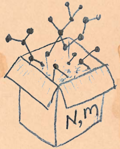

Outline

Basics
- Definitions
- How to build
- Some visual examples

Structure
- Clustering
- Degree distributions
- Configuration model
- Random friends are strange
- Largest component
- Simple, physically-motivated analysis

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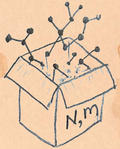

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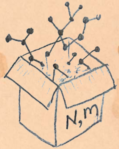
General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k.
- Also known as the configuration model.[1]
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- But we'll be more interested in
  1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
  2. Examining mechanisms that lead to networks with certain degree distributions.



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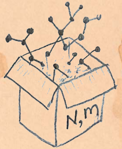

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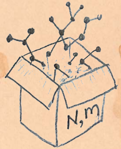

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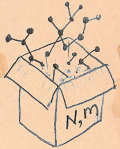

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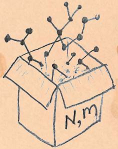

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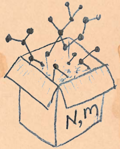

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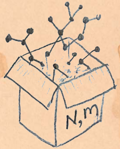

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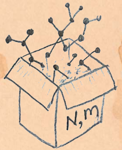
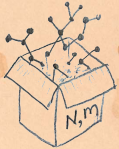


Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent γ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.



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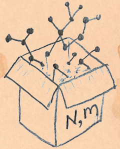

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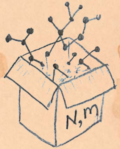

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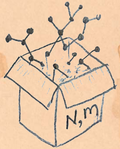

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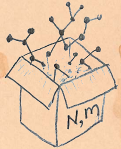

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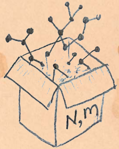

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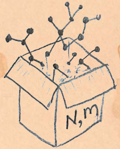
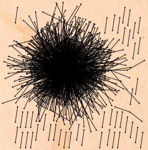
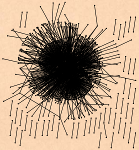


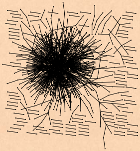
Random networks: examples for N=1000



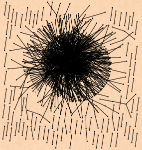
γ = 2.1
$\langle k \rangle$ = 3.448



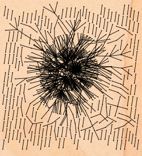
γ = 2.19
$\langle k \rangle$ = 2.986



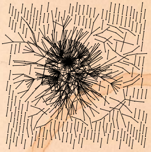
γ = 2.28
$\langle k \rangle$ = 2.306



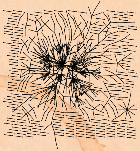
γ = 2.37
$\langle k \rangle$ = 2.504



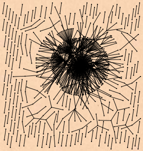
γ = 2.46
$\langle k \rangle$ = 1.856



γ = 2.55
$\langle k \rangle$ = 1.712

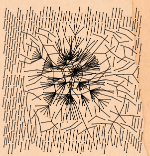


γ = 2.64
$\langle k \rangle$ = 1.6

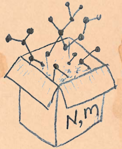


γ = 2.73
$\langle k \rangle$ = 1.862

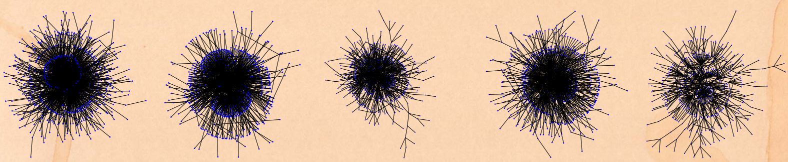
γ = 2.82
$\langle k \rangle$ = 1.386



γ = 2.91
$\langle k \rangle$ = 1.49



Random networks: largest components



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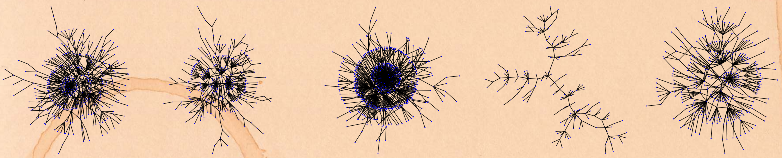

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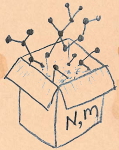

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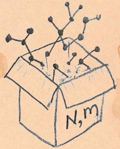


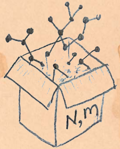
The edge-degree distribution:

- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
- Now choosing nodes based on their degree (i.e., size):

$$\boxed{Q_k \propto k P_k}$$

- Normalized form:

$$Q_k = \frac{k P_k}{\sum_{k'=0}^{\infty} k' P_{k'}} = \frac{k P_k}{\langle k \rangle}.$$



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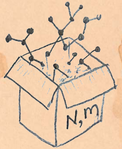

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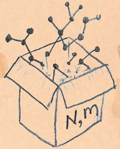

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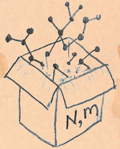

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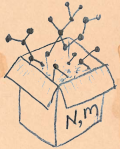

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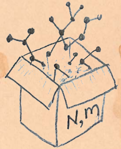

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The edge-degree distribution:

- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
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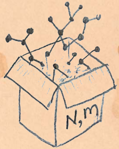

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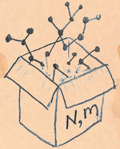

The edge-degree distribution:

- For random networks, Q_k is also the probability that a friend (neighbor) of a random node has *k* friends.
- Useful variant on Q_k:

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- Equivalent to friend having degree $k+1$.
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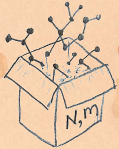

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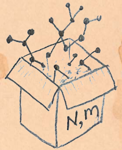

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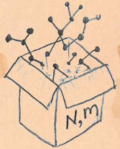

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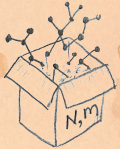

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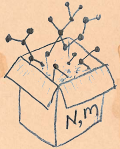

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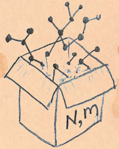

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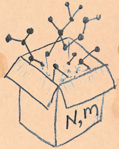

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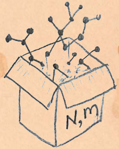

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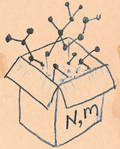

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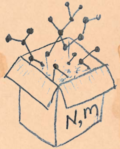

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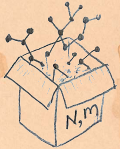

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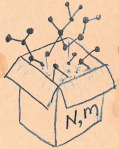

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- Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle}\left(\langle k^2 \rangle - \langle k \rangle\right)$, is true for all random networks, independent of degree distribution.
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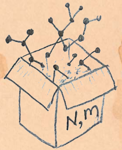
$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle}\left(\langle k \rangle^2 + \langle k \rangle - \langle k \rangle\right) = \langle k \rangle$$

- Again, neatness of results is a special property of the Poisson distribution.
- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...



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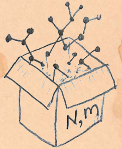

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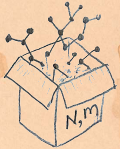

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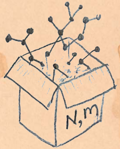

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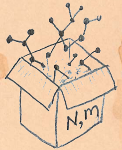

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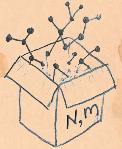

Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle}\left(\langle k^2 \rangle - \langle k \rangle\right) = \langle k^2 \rangle - \langle k \rangle.$$

- Key: Average depends on the 1st and 2nd moments of P_k and not just the 1st moment.
- Three peculiarities:
  1. We might guess $\langle k_2 \rangle = \langle k \rangle(\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
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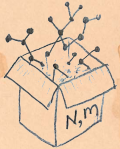

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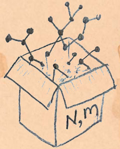

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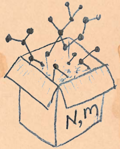

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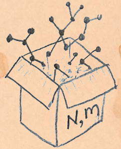

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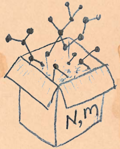

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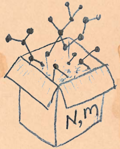

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- Average # friends of friends per node is

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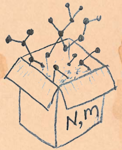

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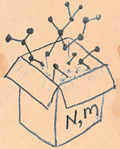

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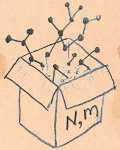

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More on peculiarity #3:

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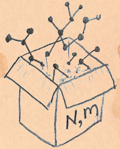

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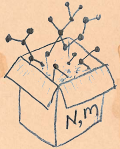

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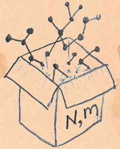

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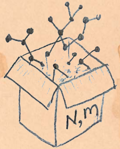

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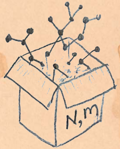

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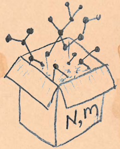

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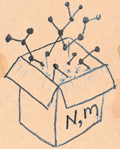

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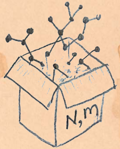

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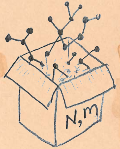
(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- Defn: Giant component = component that comprises a non-zero fraction of a network as $N \to \infty$.
- Note: Component = Cluster



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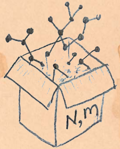

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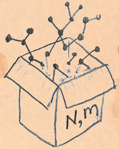

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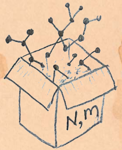

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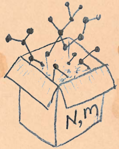

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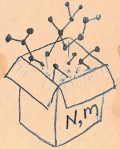
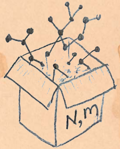


Outline



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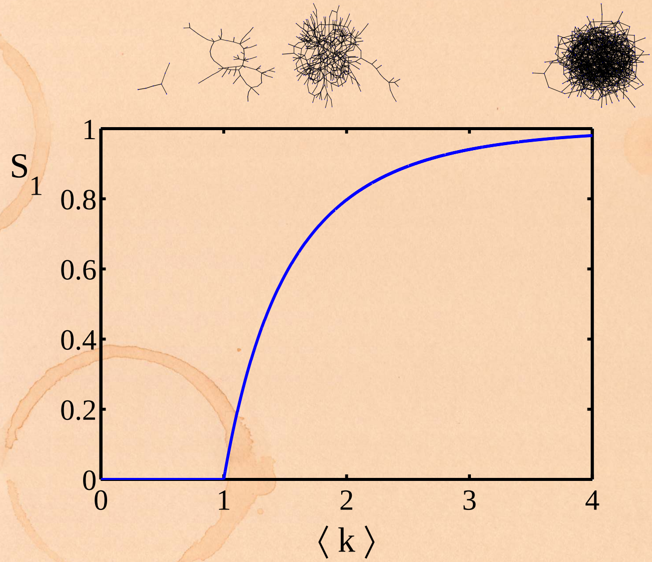
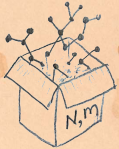




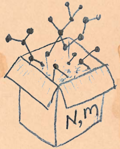
Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.
- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$



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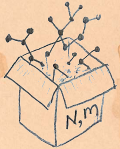

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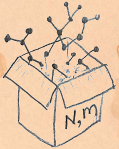

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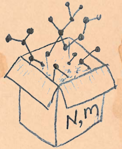

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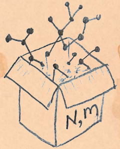

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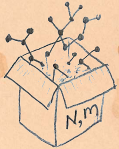

Giant component

Standard random networks:

- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Condition for giant component:

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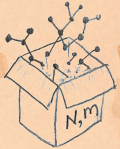
- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
- When $\langle k \rangle < 1$, all components are finite.
- Fine example of a continuous phase transition (⊞).
- We say $\langle k \rangle = 1$ marks the critical point of the system.



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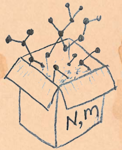

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- Fine example of a continuous phase transition (⊞).
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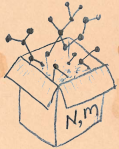

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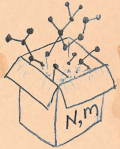

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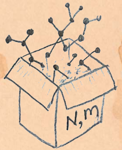

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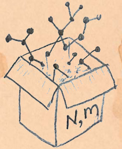

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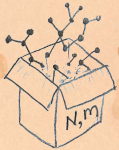

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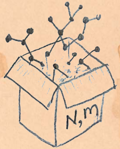

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- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$, $k \geq 1$, then

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- So giant component always exists for these kinds of networks.
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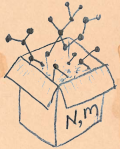

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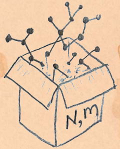

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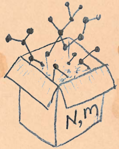

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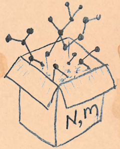

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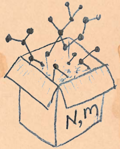

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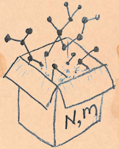

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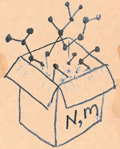

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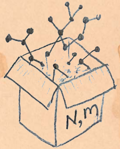

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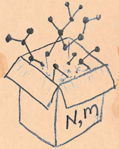
Giant component

And how big is the largest component?

- Define S_1 as the size of the largest component.
- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let's find S_1 with a back-of-the-envelope argument.
- Define δ as the probability that a randomly chosen node does not belong to the largest component.
- Simple connection: $\delta = 1 - S_1$.
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.
- So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

- Substitute in Poisson distribution...



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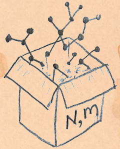

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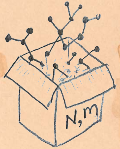

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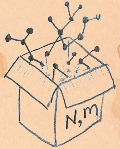

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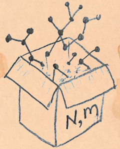

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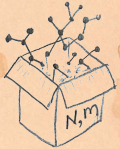

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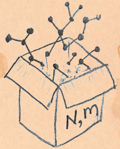

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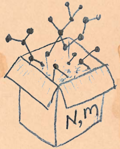

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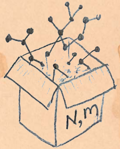

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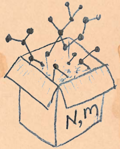

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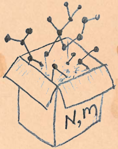

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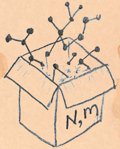

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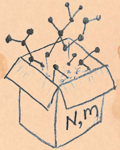

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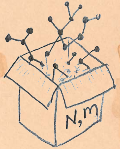

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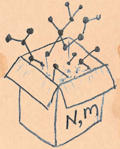
- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1:

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
- Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- Only solvable for $S_1 > 0$ when $\langle k \rangle > 1$.
- Really a transcritical bifurcation. [2]



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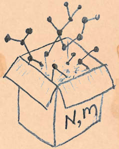

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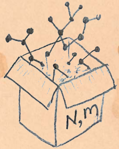

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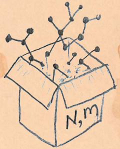

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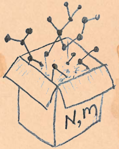

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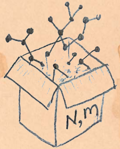

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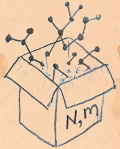

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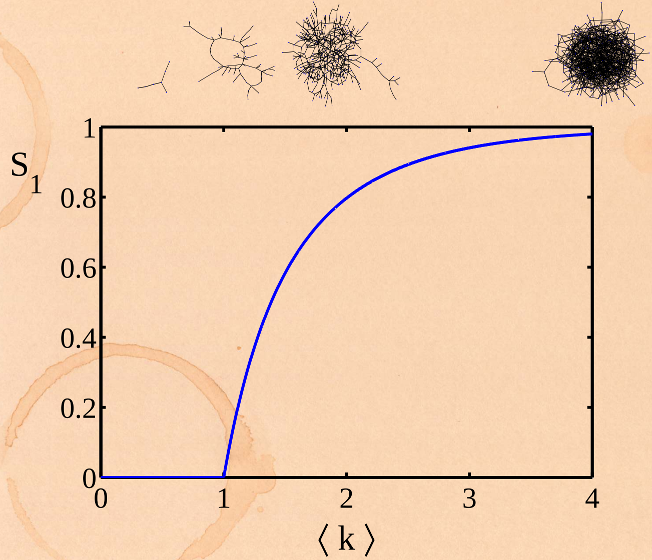

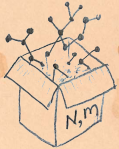

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- Our dirty trick only works for ER random networks.
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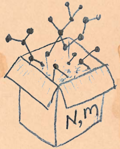

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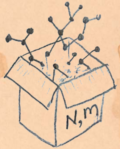

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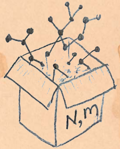

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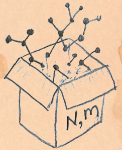

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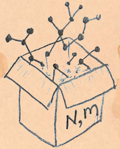

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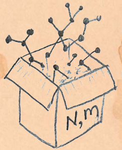

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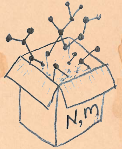
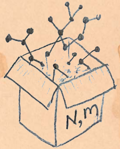
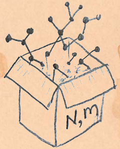


Outline

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[3] H. S. Wilf.
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