

# Overview of Complex Networks

Complex Networks  
CSYS/MATH 303, Spring, 2011

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University of Vermont

- Class admin
- Basic definitions
- Popularity
- Examples of Complex Networks
- Properties of Complex Networks
- Modelling Complex Networks
- Nutshell
- References



# Outline

Class admin

Basic definitions

Popularity

Examples of Complex Networks

Properties of Complex Networks

Modelling Complex Networks

Nutshell

References

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

- ▶ Office hours:
  - ▶ 1:00 pm to 3:00 pm, Wednesday;  
Farrell Hall, second floor, Trinity Campus.
  - ▶ Appointments by email (peter.dodds@uvm.edu).
- ▶ Course outline
- ▶ Projects
- ▶ Assignments (about 8)
- ▶ Assignment 1 appears today and involves:
  - ▶ dolphins
  - ▶ a Karate club
  - ▶ political blogs
  - ▶ a worm's brain
  - ▶ the Internet
  - ▶ jazz musicians



# Exciting details regarding these slides:

- ▶ Three versions (all in pdf):
  1. Presentation,
  2. Flat Presentation,
  3. Handout (2x2).
- ▶ Presentation versions are [navigable](#) and hyperlinks are [clickable](#).
- ▶ Web links look [like this](#) (☐).
- ▶ References in slides link to full citation at end. [2]
- ▶ Citations contain links to papers in pdf (if available).
- ▶ Brought to you by a troubling concoction of  $\text{\LaTeX}$ , Beamer, and perl.

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Bonus materials:

## Textbooks:

- ▶ Mark Newman (Physics, Michigan)  
“Networks: An Introduction” (田)
- ▶ David Easley and Jon Kleinberg (Economics and Computer Science, Cornell)  
“Networks, Crowds, and Markets: Reasoning About a Highly Connected World” (田)

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



## Review articles:

- ▶ S. Boccaletti et al.

“Complex networks: structure and dynamics” [5]

Times cited: 1,028 (as of June 7, 2010)

- ▶ M. Newman

“The structure and function of complex networks” [16]

Times cited: 2,559 (as of June 7, 2010)

- ▶ R. Albert and A.-L. Barabási

“Statistical mechanics of complex networks” [1]

Times cited: 3,995 (as of June 7, 2010)

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Basic definitions:

Complex: (Latin = with + fold/weave (com + plex))

## Adjective

- ▶ Made up of multiple parts; intricate or detailed.
- ▶ Not simple or straightforward.

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Basic definitions:

## Complex System—Some ingredients:

- ▶ Distributed system of many interrelated parts
- ▶ No centralized control
- ▶ Nonlinear relationships
- ▶ Existence of feedback loops
- ▶ Complex systems are open (out of equilibrium)
- ▶ Presence of Memory
- ▶ Modular (nested)/multiscale structure
- ▶ Opaque boundaries
- ▶ Emergence—‘More is Different’<sup>[2]</sup>
- ▶ Many phenomena can be complex: social, technical, informational, geophysical, meteorological, fluidic, ...

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

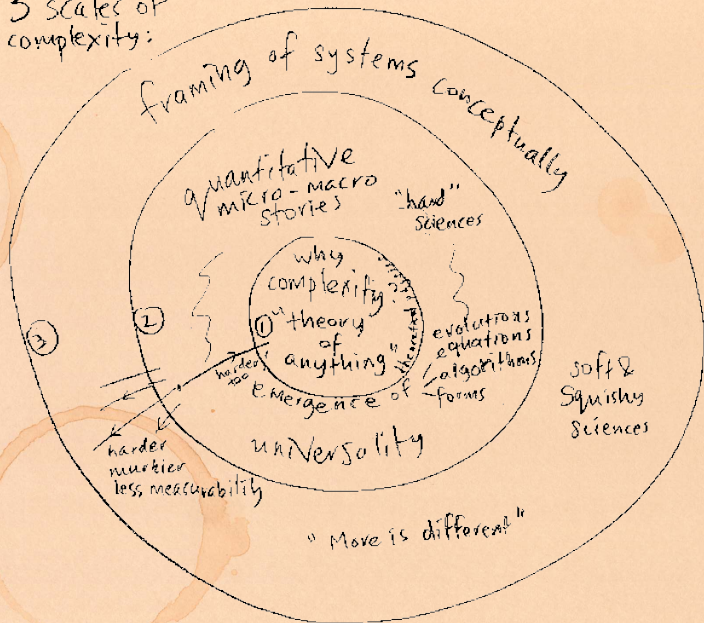
Nutshell

References





3 scales of complexity:



Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

# net•work |'net,wərk|

noun

- 1 an arrangement of intersecting horizontal and vertical lines.
  - a complex system of roads, railroads, or other transportation routes : *a network of railroads.*
- 2 a group or system of interconnected people or things : *a trade network.*
  - a group of people who exchange information, contacts, and experience for professional or social purposes : *a support network.*
  - a group of broadcasting stations that connect for the simultaneous broadcast of a program : *the introduction of a second TV network* | [as adj. ] *network television.*
  - a number of interconnected computers, machines, or operations : *specialized computers that manage multiple outside connections to a network* | *a local cellular phone network.*
  - a system of connected electrical conductors.

verb [ trans. ]

connect as or operate with a network : *the stock exchanges have proven to be resourceful in networking these deals.*

- link (machines, esp. computers) to operate interactively : [as adj. ] ( **networked** ) *networked workstations.*
- [ intrans. ] [often as n. ] ( **networking** ) interact with other people to exchange information and develop contacts, esp. to further one's career : *the skills of networking, bargaining, and negotiation.*

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Thesaurus deliciousness:

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

## network

noun

- 1** *a network of arteries* WEB, lattice, net, matrix, mesh, crisscross, grid, reticulum, reticulation; Anatomy plexus.
- 2** *a network of lanes* MAZE, labyrinth, warren, tangle.
- 3** *a network of friends* SYSTEM, complex, nexus, web, webwork.



# Ancestry:

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

From Keith Briggs's excellent etymological  
investigation: (田)

- ▶ Opus reticulatum:
- ▶ A Latin origin?



[<http://serialconsign.com/2007/11/we-put-net-network>]



# Ancestry:

First known use: Geneva Bible, 1560

‘And thou shalt make unto it a grate like networke of brass (Exodus xxvii 4).’

From the OED via Briggs:

- ▶ 1658—: reticulate structures in animals
- ▶ 1839—: rivers and canals
- ▶ 1869—: railways
- ▶ 1883—: distribution network of electrical cables
- ▶ 1914—: wireless broadcasting networks

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



## Net and Work are venerable old words:

- ▶ **'Net'** first used to mean spider web (King Ælfréd, 888).
- ▶ **'Work'** appears to have long meant purposeful action.

Class admin

Basic definitions

Popularity

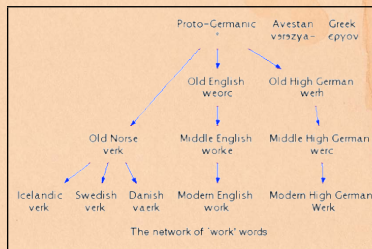
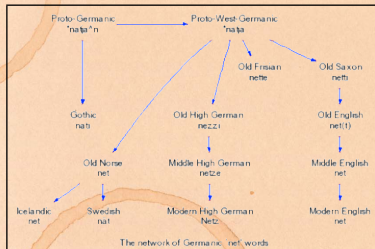
Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



- ▶ **'Network'** = something built based on the idea of natural, flexible lattice or web.
- ▶ c.f., ironwork, stonework, fretwork.

# Key Observation:

- ▶ Many **complex systems** can be viewed as **complex networks** of physical or abstract interactions.
- ▶ Opens door to mathematical and numerical analysis.
- ▶ Dominant approach of last decade of a **theoretical-physics/stat-mechish** flavor.
- ▶ Mindboggling amount of work published on complex networks since 1998...
- ▶ ... largely due to your typical theoretical physicist:



- ▶ *Piranha physicus*
- ▶ Hunt in packs.
- ▶ Feast on new and interesting ideas (see chaos, cellular automata, ...)

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Popularity (according to ISI)

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

## “Collective dynamics of ‘small-world’ networks” [23]

- ▶ Watts and Strogatz  
Nature, 1998
- ▶  $\approx 4677$  citations (as of January 18, 2011)
- ▶ Over 1100 citations in 2008 alone.

## “Emergence of scaling in random networks” [3]

- ▶ Barabási and Albert  
Science, 1999
- ▶  $\approx 5270$  citations (as of January 18, 2011)
- ▶ Over 1100 citations in 2008 alone.





# Popularity according to books:

Overview

Class admin

Basic definitions

Popularity

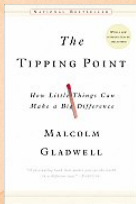
Examples of  
Complex Networks

Properties of  
Complex Networks

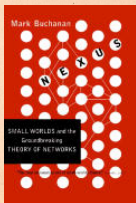
Modelling Complex  
Networks

Nutshell

References



The Tipping Point: How Little Things can  
make a Big Difference—Malcolm Gladwell<sup>[11]</sup>



Nexus: Small Worlds and the Groundbreaking  
Science of Networks—Mark Buchanan



# Popularity according to books:

Overview

Class admin

Basic definitions

Popularity

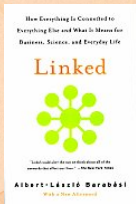
Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



Linked: How Everything Is Connected to  
Everything Else and What It  
Means—Albert-Laszlo Barabási



Six Degrees: The Science of a Connected  
Age—Duncan Watts<sup>[21]</sup>



# Numerous others:

- ▶ [Complex Social Networks](#)—F. Vega-Redondo <sup>[20]</sup>
- ▶ [Fractal River Basins: Chance and Self-Organization](#)—I. Rodríguez-Iturbe and A. Rinaldo <sup>[17]</sup>
- ▶ [Random Graph Dynamics](#)—R. Durrett
- ▶ [Scale-Free Networks](#)—Guido Caldarelli
- ▶ [Evolution and Structure of the Internet: A Statistical Physics Approach](#)—Romu Pastor-Satorras and Alessandro Vespignani
- ▶ [Complex Graphs and Networks](#)—Fan Chung
- ▶ [Social Network Analysis](#)—Stanley Wasserman and Kathleen Faust
- ▶ [Handbook of Graphs and Networks](#)—Eds: Stefan Bornholdt and H. G. Schuster <sup>[7]</sup>
- ▶ [Evolution of Networks](#)—S. N. Dorogovtsev and J. F. F. Mendes <sup>[10]</sup>

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# More observations

- ▶ But surely **networks aren't new**...
- ▶ Graph theory is well established...
- ▶ Study of social networks started in the 1930's...
- ▶ So why all this 'new' research on networks?
- ▶ **Answer:** **Oodles of Easily Accessible Data.**
- ▶ We can now inform (alas) our theories with a much more measurable reality.\*
- ▶ Real networks occupy a tiny, low entropy part of all network space and require specific attention.
- ▶ A worthy goal: establish **mechanistic explanations.**
- ▶ What kinds of dynamics lead to these real networks?  
*\* If this is upsetting, maybe string theory is for you...*

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# More observations

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

- ▶ Web-scale data sets can be overly **exciting**.

## Witness:

- ▶ The End of Theory: The Data Deluge Makes the Scientific Theory Obsolete (Anderson, Wired) (田)
- ▶ “The Unreasonable Effectiveness of Data,” Halevy et al. [12]
- ▶ c.f. Wigner’s “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [24]

## But:

- ▶ For scientists, description is only part of the battle.
- ▶ We still need to **understand**.



# Super basic definitions

**Nodes** = A collection of entities which have properties that are somehow related to each other

- ▶ e.g., people, forks in rivers, proteins, webpages, organisms,...

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Basic definitions:

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

## Links = Connections between nodes

- ▶ **links**
  - ▶ may be real and fixed (rivers),
  - ▶ real and dynamic (airline routes),
  - ▶ abstract with physical impact (hyperlinks),
  - ▶ or purely abstract (semantic connections between concepts).
- ▶ **Links** may be directed or undirected.
- ▶ **Links** may be binary or weighted.



# Basic definitions:

**Node degree** = Number of links per node

- ▶ Notation: Node  $i$ 's degree =  $k_i$ .
- ▶  $k_i = 0, 1, 2, \dots$
- ▶ Notation: the average degree of a network =  $\langle k \rangle$  (and sometimes as  $z$ )
- ▶ For undirected networks, connection between number of edges  $m$  and average degree:

$$\langle k \rangle = \frac{2m}{N}$$

- ▶ For directed networks,

$$\langle k_{\text{out}} \rangle = \langle k_{\text{in}} \rangle = \frac{m}{N}$$

- ▶ **Defn:**  $\mathcal{N}_i$  = the set of  $i$ 's  $k_i$  neighbors

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References





# Basic definitions:

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

## Adjacency matrix:

- ▶ We represent a graph or network by a matrix  $A$  with link weight  $a_{ij}$  for nodes  $i$  and  $j$  in entry  $(i, j)$ .
- ▶ e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- ▶ (n.b., for numerical work, we always use sparse matrices.)



## What passes for a complex network?

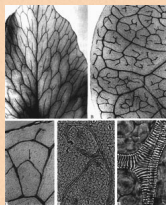
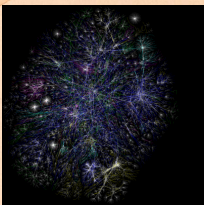
- ▶ Complex networks are **large** (in node number)
- ▶ Complex networks are **sparse** (low edge to node ratio)
- ▶ Complex networks are usually **dynamic** and **evolving**
- ▶ Complex networks can be social, economic, natural, informational, abstract, ...



# Examples

## Physical networks

- ▶ River networks
- ▶ Neural networks
- ▶ Trees and leaves
- ▶ Blood networks
- ▶ The Internet
- ▶ Road networks
- ▶ Power grids



- ▶ **Distribution** (branching) versus **redistribution** (cyclical)

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

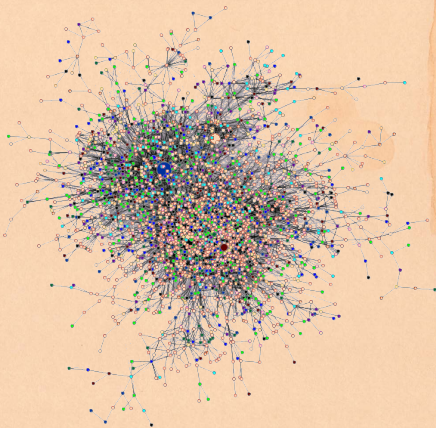


# Examples

Overview

## Interaction networks

- ▶ The Blogosphere
- ▶ Biochemical networks
- ▶ Gene-protein networks
- ▶ Food webs: who eats whom
- ▶ The World Wide Web (?)
- ▶ Airline networks
- ▶ Call networks (AT&T)
- ▶ The Media



[datamining.typepad.com](http://datamining.typepad.com) (田)

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Examples

Overview

## Interaction networks: social networks

- ▶ Snogging
- ▶ Friendships
- ▶ Acquaintances
- ▶ Boards and directors
- ▶ Organizations

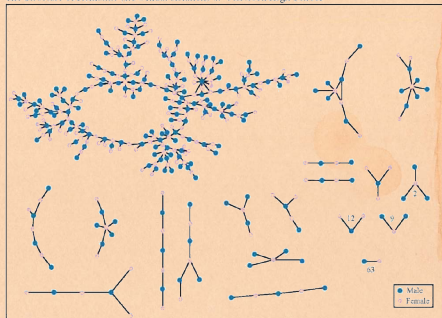
▶ [twitter.com](https://twitter.com) (田)

[facebook.com](https://facebook.com) (田),

(Bearman *et al.*, 2004)

- ▶ 'Remotely sensed' by: tweets (open), instant messaging, Facebook posts, emails, phone logs (\*cough\*).

The Structure of Romantic and Sexual Relations at "Jefferson High School"



Each circle represents a student and lines connecting students represent romantic relations occurring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else)

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



# Examples

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

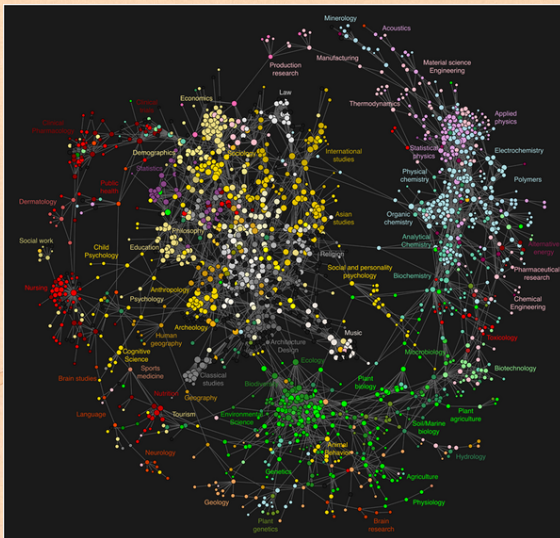
## Relational networks

- ▶ Consumer purchases  
(Wal-Mart:  $\approx 2.5$  petabyte =  $2.5 \times 10^{15}$  bytes) (田)
- ▶ Thesauri: Networks of words generated by meanings
- ▶ Knowledge/Databases/Ideas
- ▶ Metadata—Tagging: delicious (田), flickr (田)



# Clickworthy Science:

Overview



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

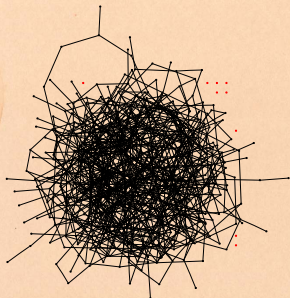
References

Bollen et al. [6]; a higher resolution figure is here (田)



## A notable feature of large-scale networks:

- ▶ Graphical renderings are often just a big mess.



⇐ Typical hairball

- ▶ number of nodes  $N = 500$
- ▶ number of edges  $m = 1000$
- ▶ average degree  $\langle k \rangle = 4$

- ▶ And even when renderings somehow look good:  
 “That is a very graphic analogy which aids understanding wonderfully while being, strictly speaking, wrong in every possible way”  
 said Ponder [Stibbons] —*Making Money*, T. Pratchett.
- ▶ We need to extract **digestible, meaningful aspects**.

Class admin

Basic definitions

Popularity

Examples of  
Complex NetworksProperties of  
Complex NetworksModelling Complex  
Networks

Nutshell

References





## Some key aspects of real complex networks:

- ▶ degree distribution\*
  - ▶ assortativity
  - ▶ homophily
  - ▶ clustering
  - ▶ motifs
  - ▶ modularity
  - ▶ concurrency
  - ▶ hierarchical scaling
  - ▶ network distances
  - ▶ centrality
  - ▶ efficiency
  - ▶ robustness
- ▶ Plus coevolution of network structure and processes on networks.
- \* Degree distribution is the elephant in the room that we are now all very aware of...

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



## 1. degree distribution $P_k$

- ▶  $P_k$  is the probability that a randomly selected node has degree  $k$
- ▶  $k$  = node degree = number of connections
- ▶ **ex 1:** Erdős-Rényi random networks:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- ▶ Distribution is Poisson



## 1. degree distribution $P_k$

- ▶ ex 2: “Scale-free” networks:  $P_k \propto k^{-\gamma} \Rightarrow$  ‘hubs’
- ▶ link cost controls skew
- ▶ hubs may facilitate or impede contagion



## Note:

- ▶ Erdős-Rényi random networks are a *mathematical construct*.
- ▶ 'Scale-free' networks are **growing networks** that form according to a **plausible mechanism**.
- ▶ Randomness is out there, just not to the degree of a completely random network.



## 2. Assortativity/3. Homophily:

- ▶ Social networks: Homophily (☒) = birds of a feather
- ▶ e.g., degree is standard property for sorting: measure degree-degree correlations.
- ▶ **Assortative** network: <sup>[15]</sup> similar degree nodes connecting to each other.  
*Often **social**: company directors, coauthors, actors.*
- ▶ **Disassortative** network: high degree nodes connecting to low degree nodes.  
*Often **techological** or **biological**: Internet, WWW, protein interactions, neural networks, food webs.*



## 4. Clustering:

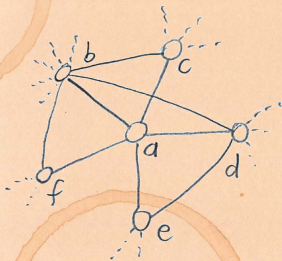
- ▶ Your friends tend to know each other.
- ▶ Two measures (explained on following slides):

### 1. Watts & Strogatz [23]

$$C_1 = \left\langle \frac{\sum_{j_1, j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

### 2. Newman [16]

$$C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$$



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

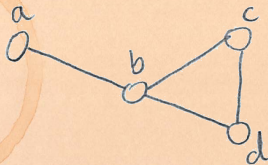
Nutshell

References

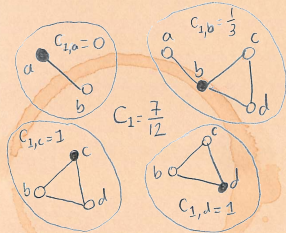


# First clustering measure:

Example network:



Calculation of  $C_1$ :



- ▶  $C_1$  is the **average fraction of pairs of neighbors who are connected**.
- ▶ Fraction of pairs of neighbors who are connected is

$$\frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

where  $k_i$  is node  $i$ 's degree, and  $\mathcal{N}_i$  is the set of  $i$ 's neighbors.

- ▶ Averaging over all nodes, we have:

$$C_1 = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

Class admin

Basic definitions

Popularity

Examples of  
Complex NetworksProperties of  
Complex NetworksModelling Complex  
Networks

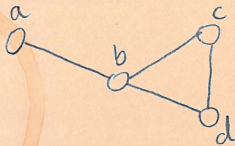
Nutshell

References

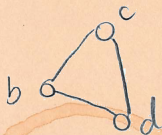


# Triples and triangles

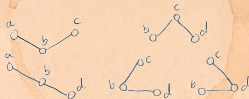
Example network:



Triangles:



Triples:



- ▶ Nodes  $i_1$ ,  $i_2$ , and  $i_3$  form a **triple** around  $i_1$  if  $i_1$  is connected to  $i_2$  and  $i_3$ .
- ▶ Nodes  $i_1$ ,  $i_2$ , and  $i_3$  form a **triangle** if each pair of nodes is connected
- ▶ The definition  $C_2 = \frac{3 \times \# \text{triangles}}{\# \text{triples}}$  measures the fraction of **closed triples**
- ▶ The '3' appears because for each triangle, we have 3 closed triples.
- ▶ Social Network Analysis (SNA): fraction of **transitive triples**.

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



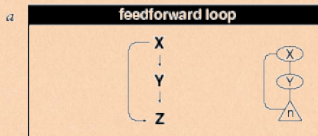


- ▶ For sparse networks,  $C_1$  tends to discount highly connected nodes.
- ▶  $C_2$  is a useful and often preferred variant
- ▶ In general,  $C_1 \neq C_2$ .
- ▶  $C_1$  is a global average of a local ratio.
- ▶  $C_2$  is a ratio of two global quantities.



## 5. motifs:

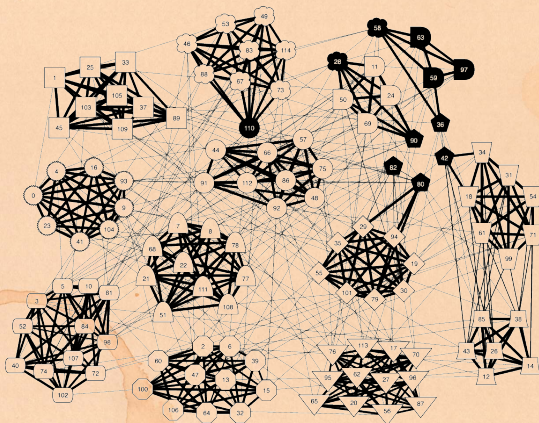
- ▶ small, recurring functional subnetworks
- ▶ e.g., Feed Forward Loop:



Shen-Orr, Uri Alon, *et al.* [18]



## 6. modularity and structure/community detection:



Clauset *et al.*, 2006 <sup>[9]</sup>: NCAA football

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

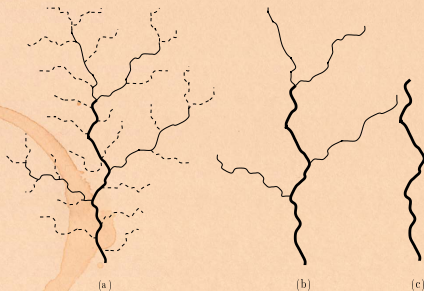
## 7. concurrency:

- ▶ transmission of a contagious element only occurs during contact
- ▶ rather obvious but easily missed in a simple model
- ▶ dynamic property—static networks are not enough
- ▶ knowledge of previous contacts crucial
- ▶ beware cumulated network data
- ▶ Kretzschmar and Morris, 1996<sup>[14]</sup>



## 8. Horton-Strahler ratios:

- ▶ Metrics for branching networks:
  - ▶ Method for ordering streams hierarchically
  - ▶ Number:  $R_n = N_\omega / N_{\omega+1}$
  - ▶ Segment length:  $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
  - ▶ Area/Volume:  $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



## 9. network distances:

### (a) shortest path length $d_{ij}$ :

- ▶ Fewest number of steps between nodes  $i$  and  $j$ .
- ▶ (Also called the chemical distance between  $i$  and  $j$ .)

### (b) average path length $\langle d_{ij} \rangle$ :

- ▶ Average shortest path length in whole network.
- ▶ Good algorithms exist for calculation.
- ▶ Weighted links can be accommodated.



## 9. network distances:

- ▶ **network diameter  $d_{\max}$ :**

Maximum shortest path length between any two nodes.

- ▶ **closeness  $d_{cl} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$ :**

Average 'distance' between any two nodes.

- ▶ Closeness handles disconnected networks ( $d_{ij} = \infty$ )

- ▶  $d_{cl} = \infty$  only when all nodes are isolated.

- ▶ Closeness perhaps compresses too much into one number



## 10. centrality:

- ▶ Many such measures of a node's 'importance.'
- ▶ **ex 1:** Degree centrality:  $k_i$ .
- ▶ **ex 2:** Node  $i$ 's betweenness  
= fraction of shortest paths that pass through  $i$ .
- ▶ **ex 3:** Edge  $\ell$ 's betweenness  
= fraction of shortest paths that travel along  $\ell$ .
- ▶ **ex 4:** Recursive centrality: Hubs and Authorities (Jon Kleinberg<sup>[13]</sup>)





## Some important models:

1. generalized random networks (touched on in 300)
2. scale-free networks (田) (covered in 300)
3. small-world networks (田) (covered in 300)
4. statistical generative models ( $p^*$ )
5. generalized affiliation networks (partly covered in 300)



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

## 1. generalized random networks:

- ▶ Arbitrary degree distribution  $P_k$ .
- ▶ Wire nodes together randomly.
- ▶ Create ensemble to test deviations from randomness.
- ▶ Interesting, applicable, rich mathematically.
- ▶ We will have fun with these guys...



## 2. 'scale-free networks':



- ▶ Introduced by Barabasi and Albert [3]
- ▶ Generative model
- ▶ Preferential attachment model with growth:
- ▶  $P[\text{attachment to node } i] \propto k_i^\alpha$ .
- ▶ Produces  $P_k \sim k^{-\gamma}$  when  $\alpha = 1$ .
- ▶ Trickiness: other models generate skewed degree distributions.

$\gamma = 2.5$   
 $\langle k \rangle = 1.8$   
 $N = 150$

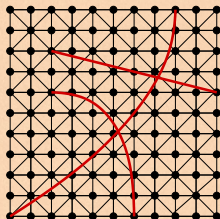


## 3. small-world networks

- ▶ Introduced by Watts and Strogatz [23]

Two scales:

- ▶ **local regularity** (an individual's friends know each other)
  - ▶ **global randomness** (shortcuts).
- 
- ▶ Shortcuts allow disease to jump
  - ▶ Number of infectives increases exponentially in time
  - ▶ Facilitates synchronization



Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

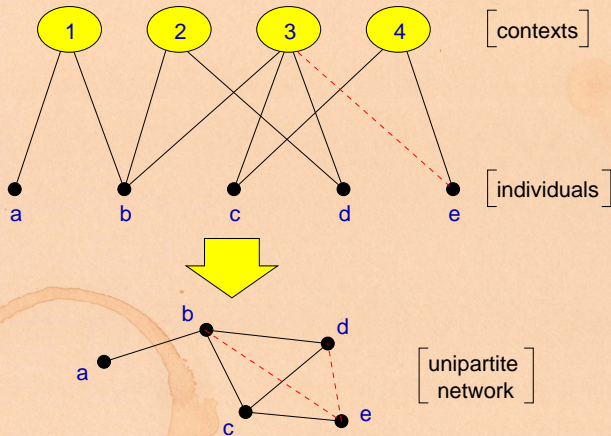
Nutshell

References



# Models

## 5. generalized affiliation networks



Bipartite affiliation networks: boards and directors, movies and actors.

Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

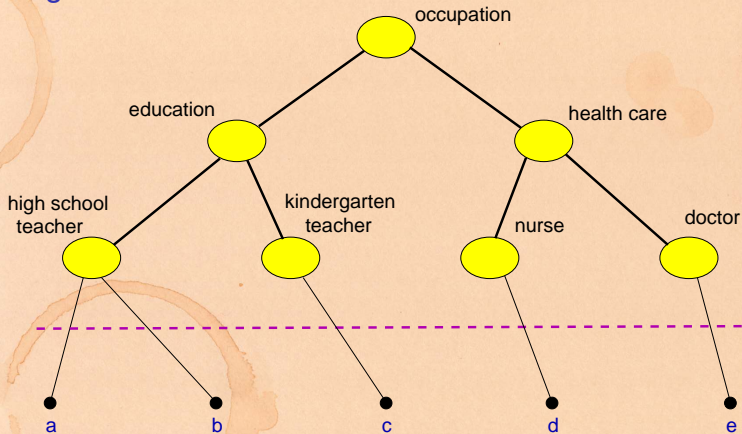
Properties of  
Complex Networks

Modelling Complex  
Networks

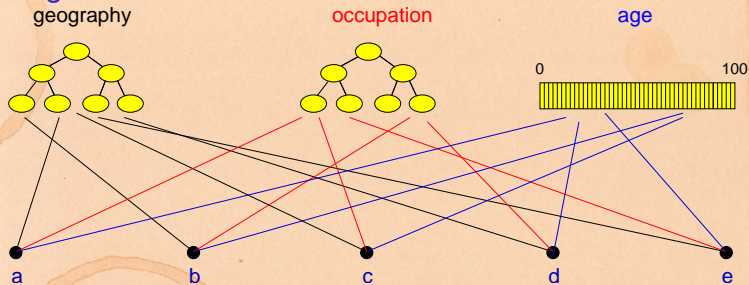
Nutshell

References

## 5. generalized affiliation networks



## 5. generalized affiliation networks



► Blau & Schwartz [4], Simmel [19], Breiger [8], Watts *et al.* [22]



## Overview Key Points:

- ▶ The field of complex networks came into existence in the late 1990s.
- ▶ Explosion of papers and interest since 1998/99.
- ▶ Hardened up much thinking about complex systems.
- ▶ Specific focus on networks that are **large-scale**, **sparse**, **natural** or **man-made**, **evolving** and **dynamic**, and (crucially) **measurable**.
- ▶ Three main (blurred) categories:
  1. **Physical** (e.g., river networks),
  2. **Interactional** (e.g., social networks),
  3. **Abstract** (e.g., thesauri).

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References





## Overview Key Points (cont.):

- ▶ Obvious connections with the vast extant field of graph theory.
- ▶ But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- ▶ Two main areas of focus:
  1. **Description:** Characterizing very large networks
  2. **Explanation:** Micro story  $\Rightarrow$  Macro features
- ▶ Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- ▶ Still much work to be done, especially with respect to dynamics... **exciting!**

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

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Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References



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Overview

Class admin

Basic definitions

Popularity

Examples of  
Complex Networks

Properties of  
Complex Networks

Modelling Complex  
Networks

Nutshell

References

