Generalized Contagion

Complex Networks
CSYS/MATH 303, Spring, 2011

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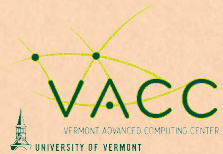
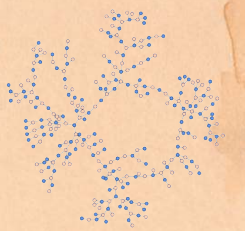
Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont





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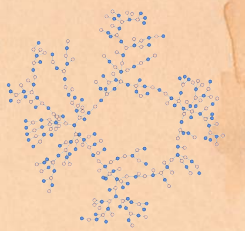

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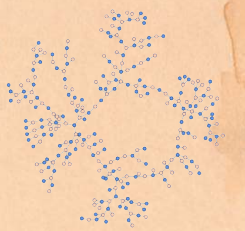
Basic questions about contagion

- How many types of contagion are there?
- How can we categorize real-world contagions?
- Can we connect models of disease-like and social contagion?
- Focus: mean field models.



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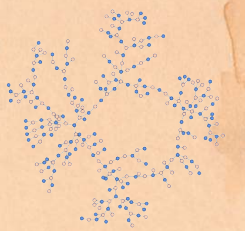

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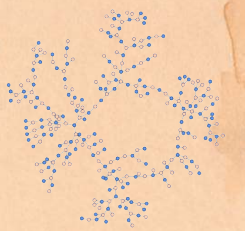

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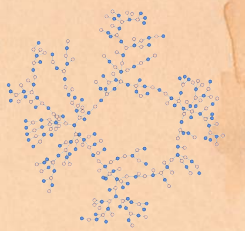

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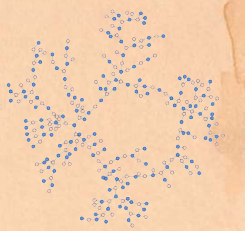

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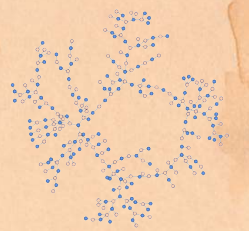
- Presumes random interactions (mass-action principle)
- Interactions are independent (no memory)
- Discrete and continuous time versions



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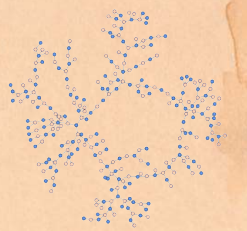

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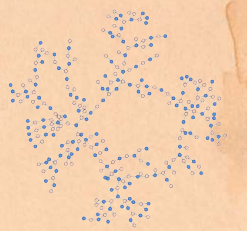

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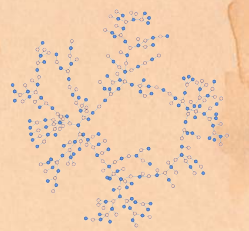

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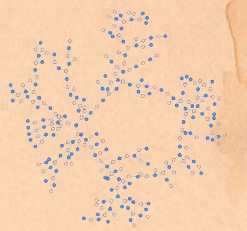

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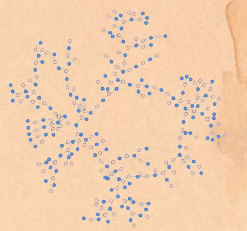

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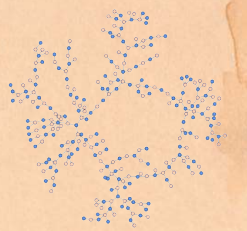

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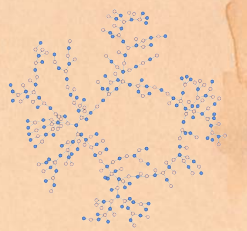

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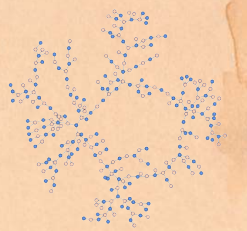

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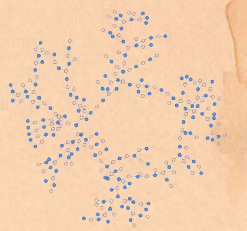

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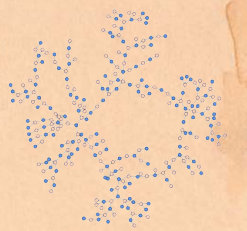

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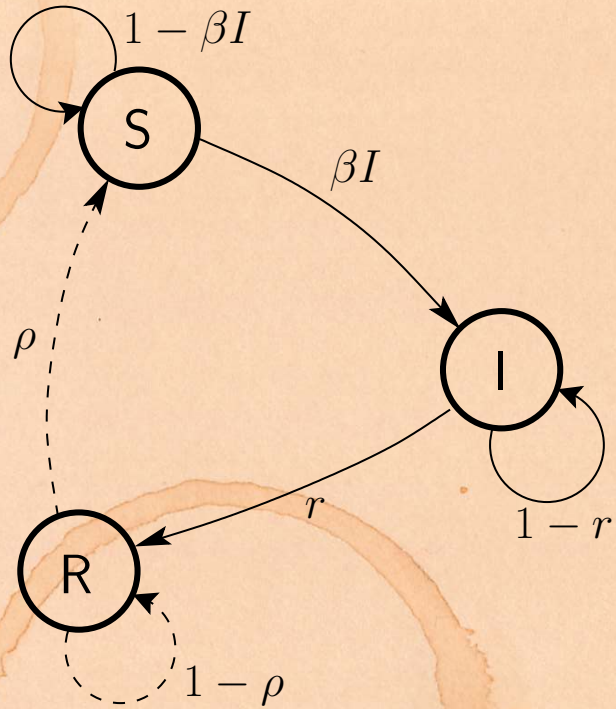

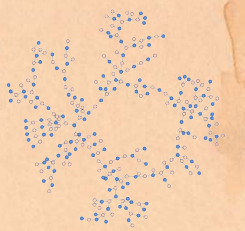

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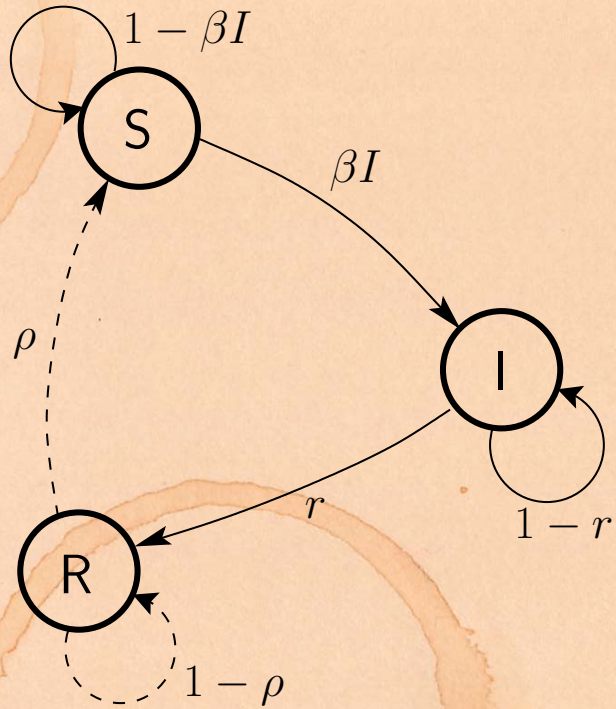

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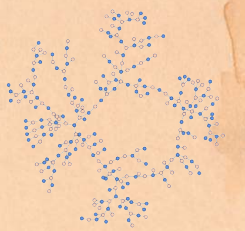

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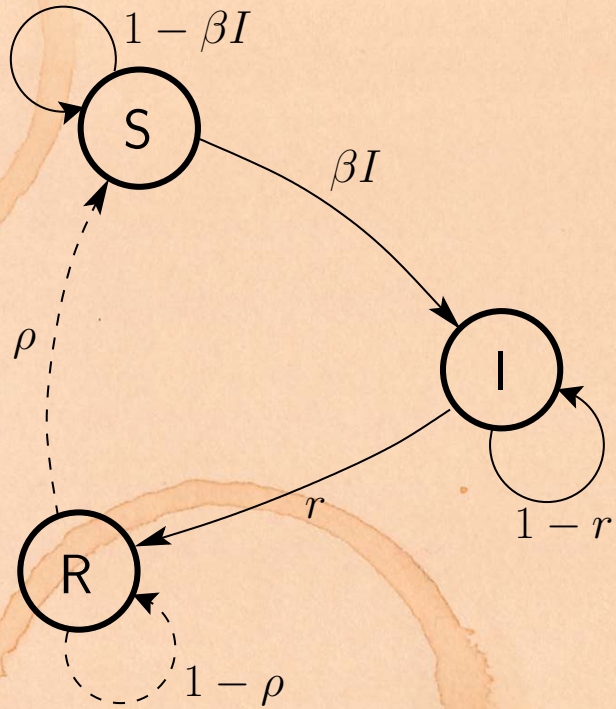
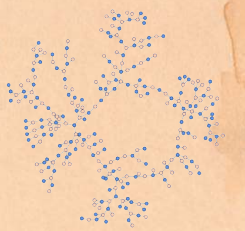


Transition Probabilities:

β for being infected given contact with infected



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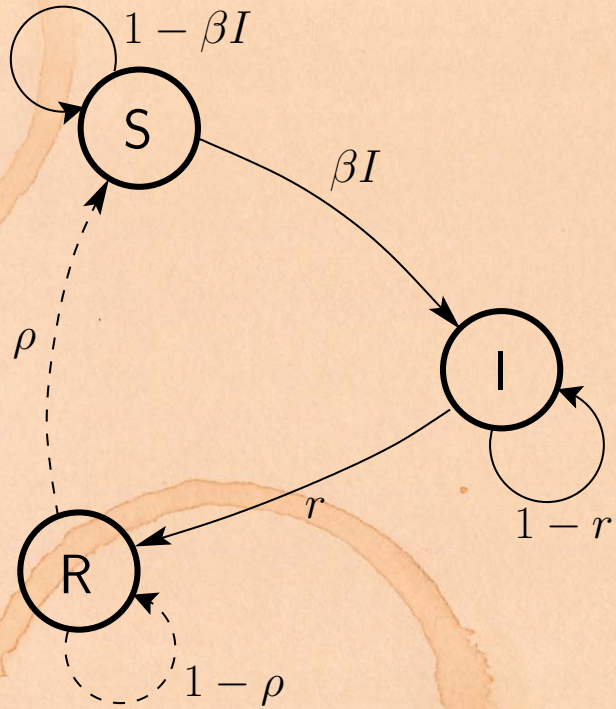

Transition Probabilities:

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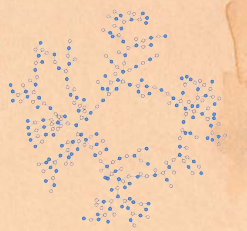
r for recovery



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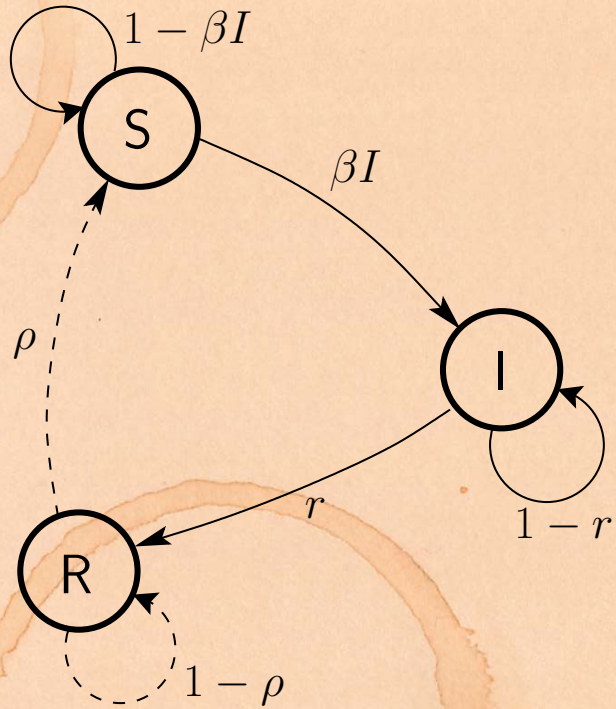
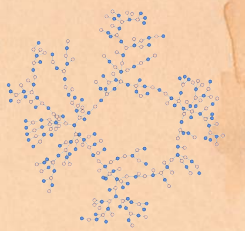


Transition Probabilities:

β for being infected given contact with infected
r for recovery
ρ for loss of immunity



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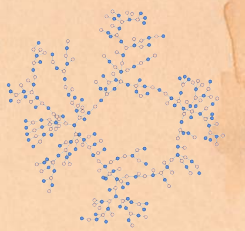
- 1920's/1930's: Kermack and McKendrick [7, 9, 8]
- Coupled differential equations with a mass-action principle



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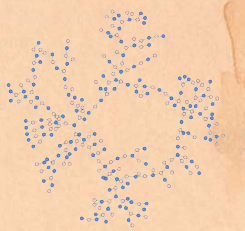

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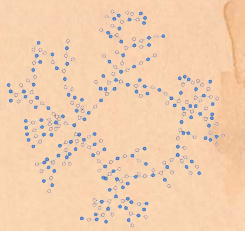

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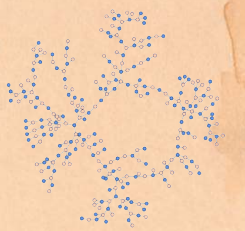

Independent Interaction models

Differential equations for continuous model

$$\frac{\mathrm{d}}{\mathrm{d}t}S = -\beta IS + \rho R$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I = \beta IS - rI$$

$$\frac{\mathrm{d}}{\mathrm{d}t}R = rI - \rho R$$

β, r, and ρ are now rates.

Reproduction Number R_0:

- R_0 = expected number of infected individuals resulting from a single initial infective
- Epidemic threshold: If $R_0 > 1$, 'epidemic' occurs.

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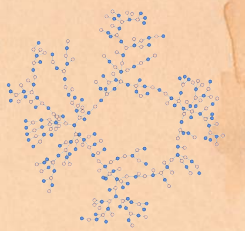

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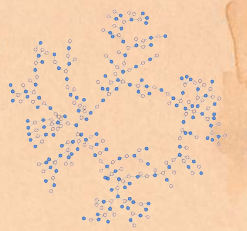

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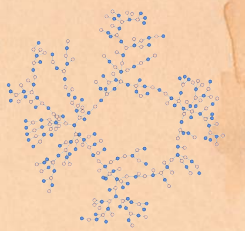

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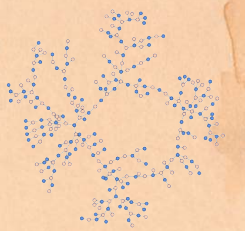

Reproduction Number R_0

Discrete version:

- Set up: One Infective in a randomly mixing population of Susceptibles
- At time $t = 0$, single infective random bumps into a Susceptible
- Probability of transmission = β
- At time $t = 1$, single Infective remains infected with probability $1 - r$
- At time $t = k$, single Infective remains infected with probability $(1 - r)^k$

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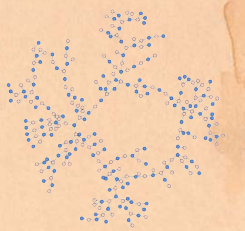

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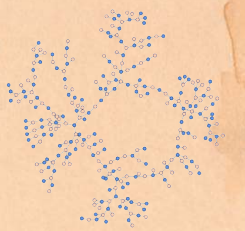

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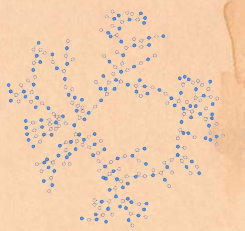

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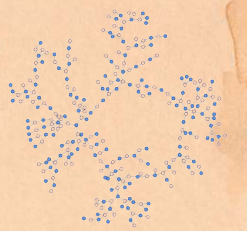

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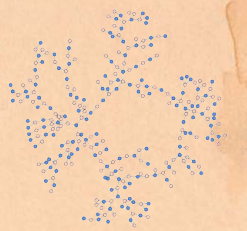

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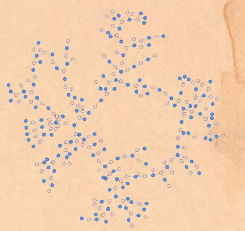
- Expected number infected by original Infective:

$$R_0 = \beta + (1-r)\beta + (1-r)^2\beta + (1-r)^3\beta + \ldots$$

$$= \beta\left(1 + (1-r) + (1-r)^2 + (1-r)^3 + \ldots\right)$$

$$= \beta\frac{1}{1-(1-r)} = \beta/r$$

- Similar story for continuous model.



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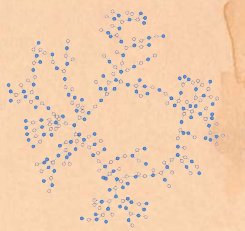

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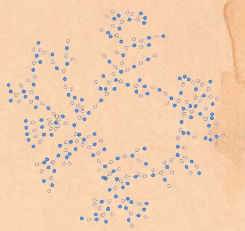

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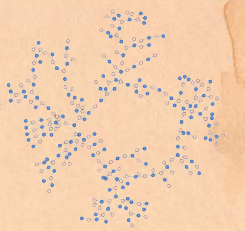

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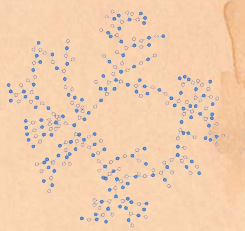

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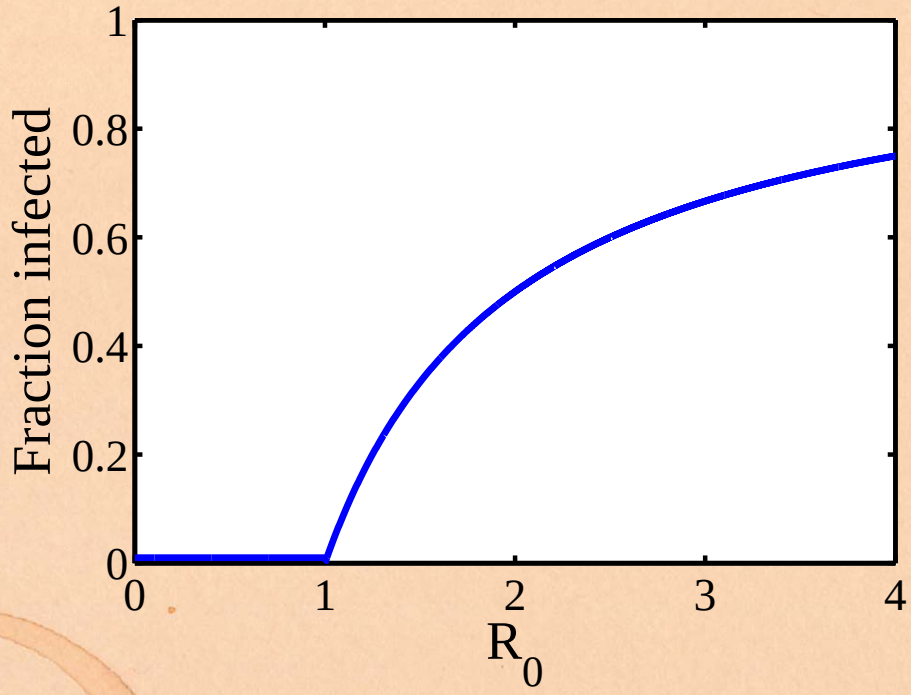

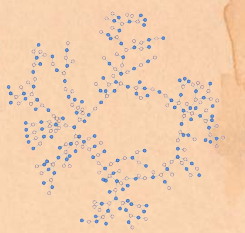

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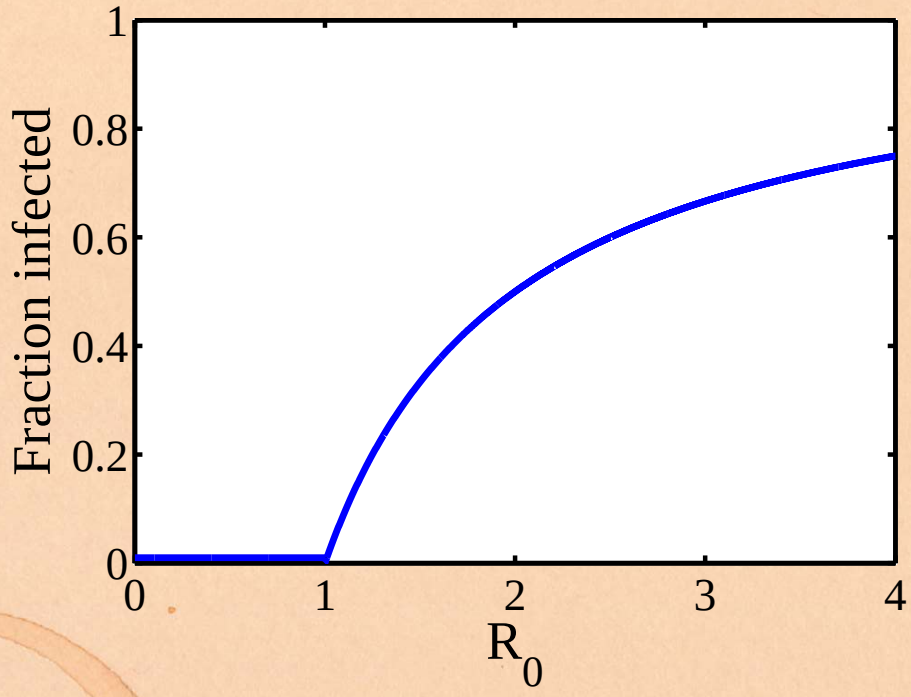
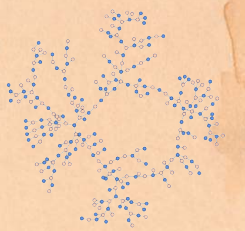


- Continuous phase transition.



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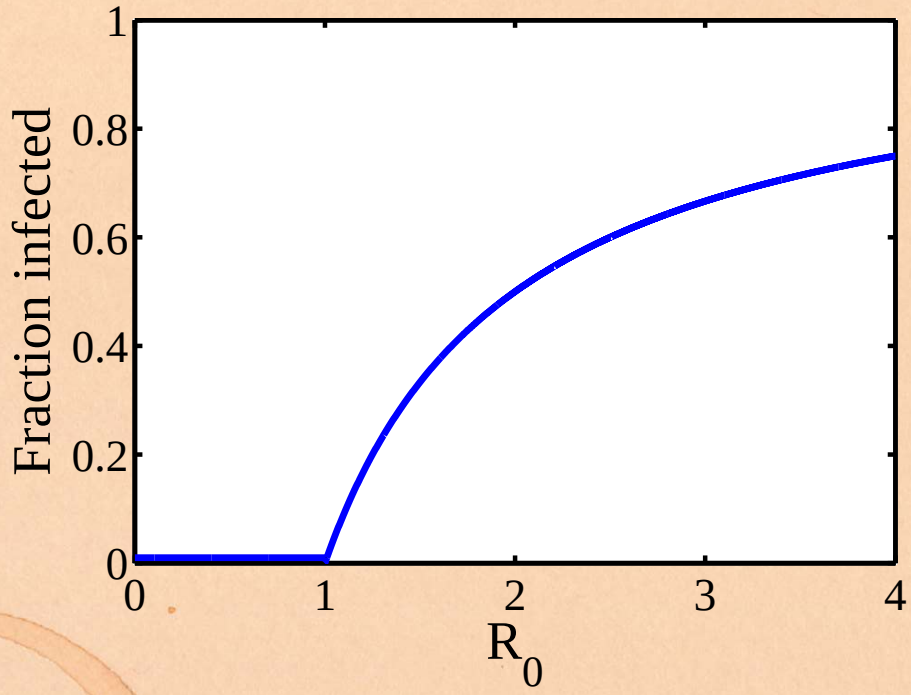
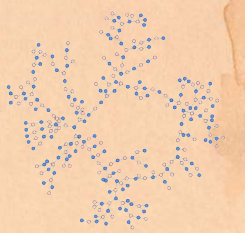


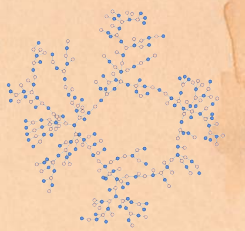
- Continuous phase transition.
- Fine idea from a simple model.



Simple disease spreading models

Valiant attempts to use SIR and co. elsewhere:

- Adoption of ideas/beliefs (Goffman & Newell, 1964) [6]
- Spread of rumors (Daley & Kendall, 1964, 1965) [2, 3]
- Diffusion of innovations (Bass, 1969) [1]
- Spread of fanatical behavior (Castillo-Chávez & Song, 2003)



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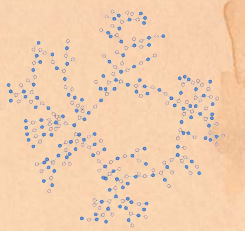

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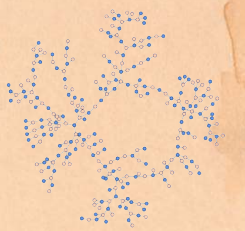

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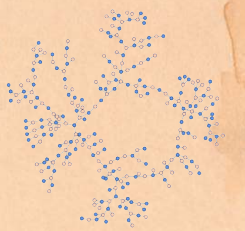

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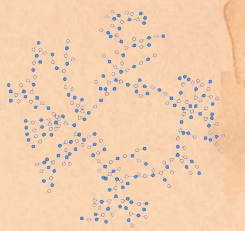
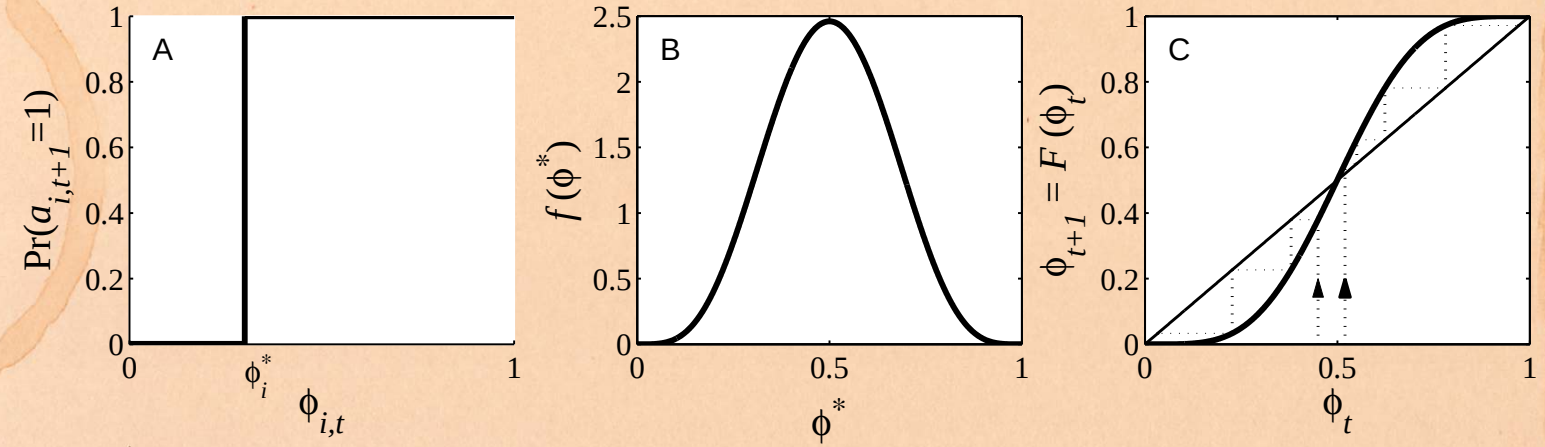
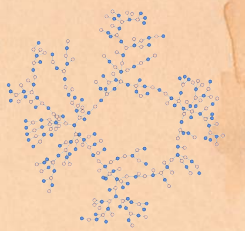


Granovetter's model (recap of recap)

- Action based on perceived behavior of others.



- Two states: S and I.
- Recovery now possible (SIS).
- ϕ = fraction of contacts 'on' (e.g., rioting).
- Discrete time, synchronous update.
- This is a Critical mass model.
- Interdependent interaction model.



Some (of many) issues

- Disease models assume independence of infectious events.
- Threshold models only involve proportions: $3/10 = 30/100$.
- Threshold models ignore exact sequence of influences
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- Mean-field models neglect network structure
- Network effects only part of story: media, advertising, direct marketing.

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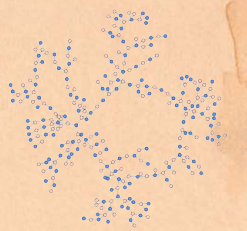

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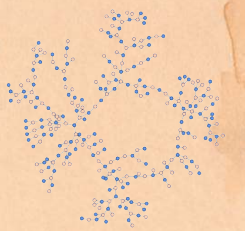

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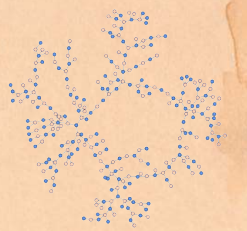

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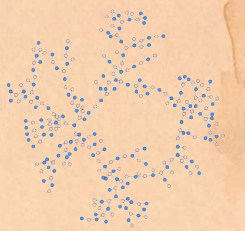

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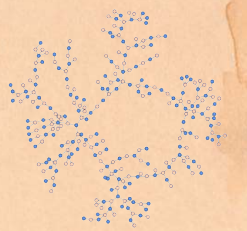

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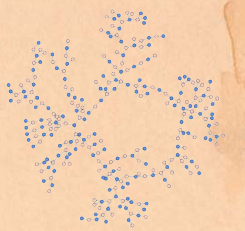

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Basic ingredients:

- Incorporate memory of a contagious element [4, 5]
- Population of N individuals, each in state S, I, or R.
- Each individual randomly contacts another at each time step.
- ϕ_t = fraction infected at time t
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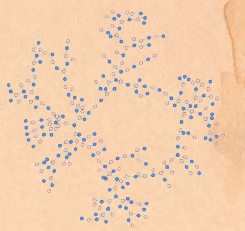
- With probability p, contact with infective leads to an exposure.
- If exposed, individual receives a dose of size d drawn from distribution f. Otherwise $d = 0$.



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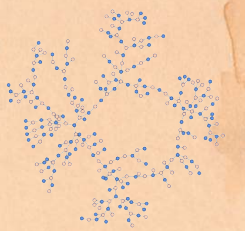

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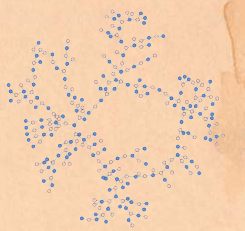

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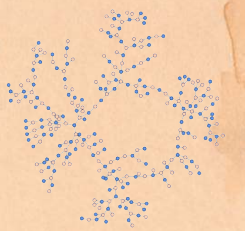

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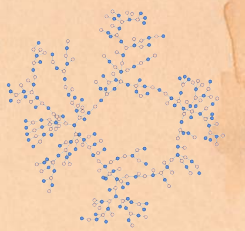

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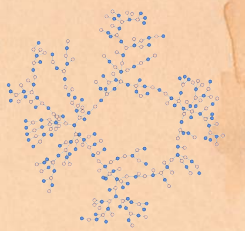
Basic ingredients:

- Incorporate memory of a contagious element [4, 5]
- Population of N individuals, each in state S, I, or R.
- Each individual randomly contacts another at each time step.
- ϕ_t = fraction infected at time t
  = probability of contact with infected individual
- With probability p, contact with infective leads to an exposure.
- If exposed, individual receives a dose of size d drawn from distribution f. Otherwise $d = 0$.



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$S \Rightarrow I$

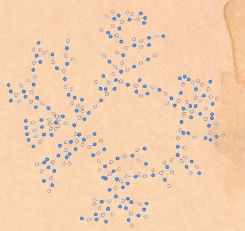
- Individuals 'remember' last T contacts:

$$D_{t,i} = \sum_{t'=t-T+1}^{t} d_i(t')$$

- Infection occurs if individual i's 'threshold' is exceeded:

$$D_{t,i} \geq d_i^*$$

- Threshold d_i^* drawn from arbitrary distribution g at $t = 0$.



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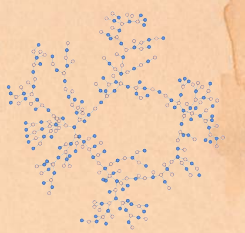

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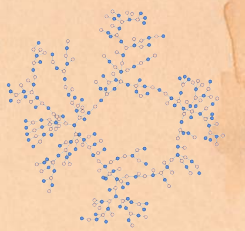

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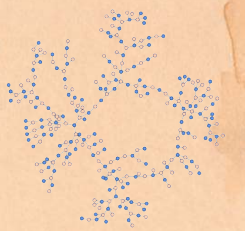


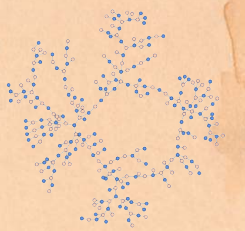
Generalized model—ingredients

I ⇒ R

When $D_{t,i} < d_i^*$,
individual i recovers to state R with probability r.

R ⇒ S

Once in state R, individuals become susceptible again with probability ρ.



Generalized model—ingredients

$\boxed{\mathrm{I} \Rightarrow \mathrm{R}}$

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$\boxed{\mathrm{R} \Rightarrow \mathrm{S}}$

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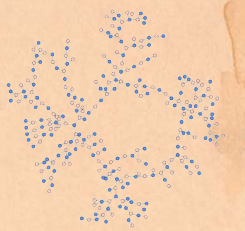

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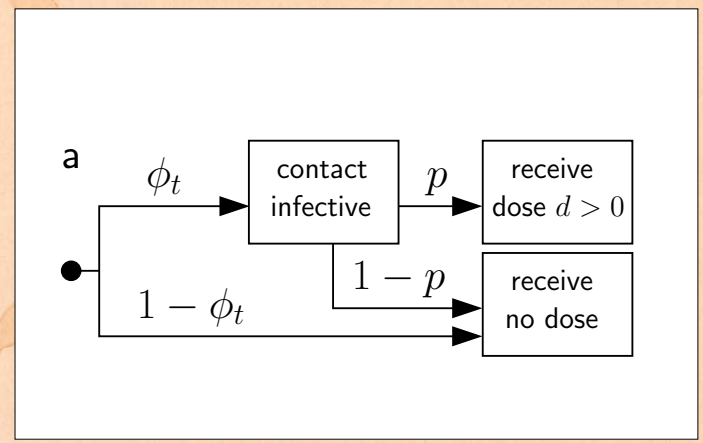

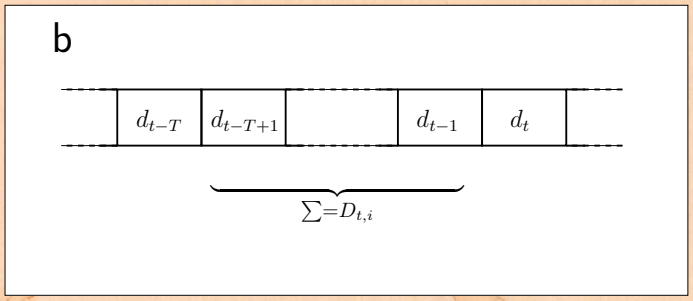

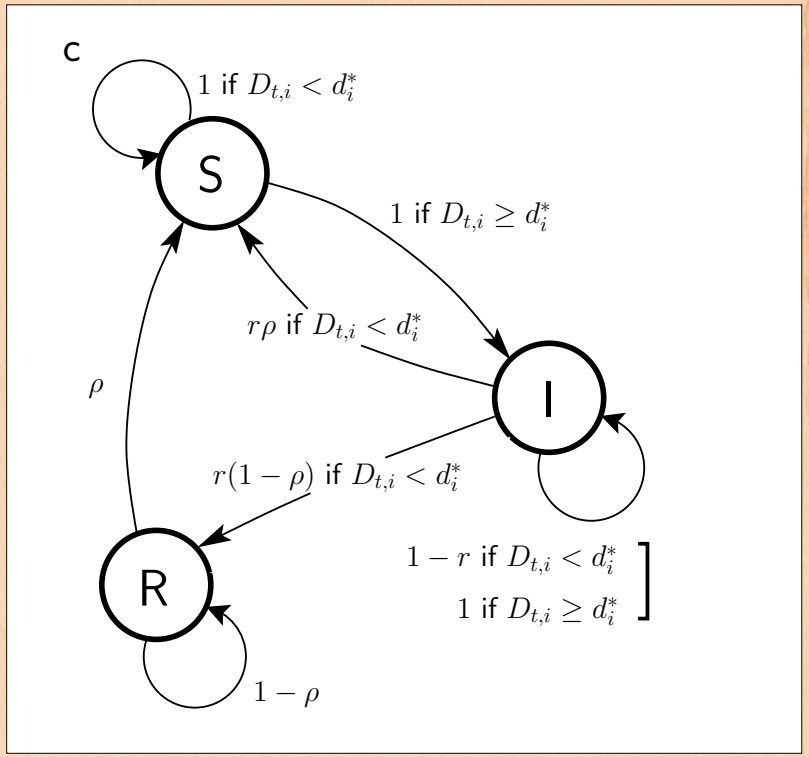

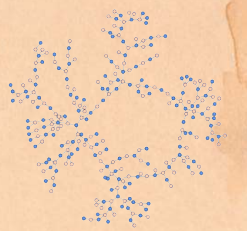

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Homogeneous version:

- All individuals have threshold d^*
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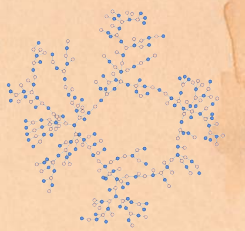

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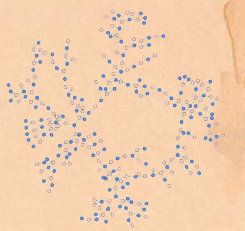

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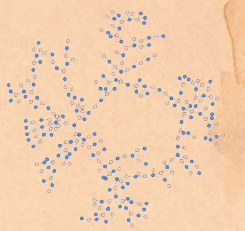

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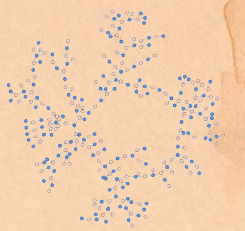

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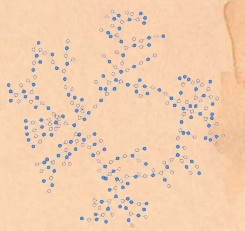

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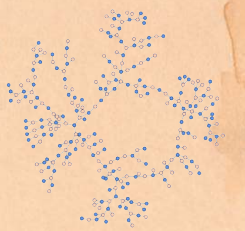

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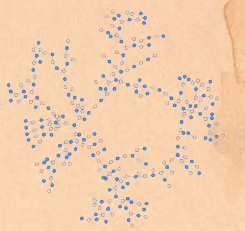

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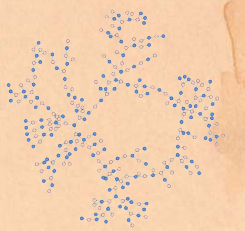

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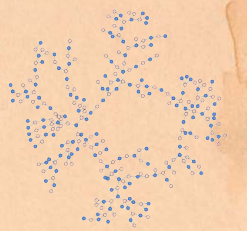

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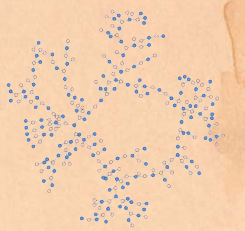

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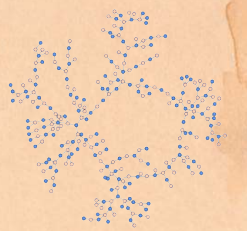

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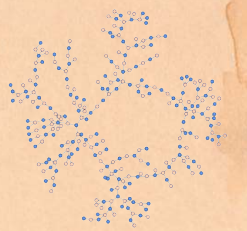

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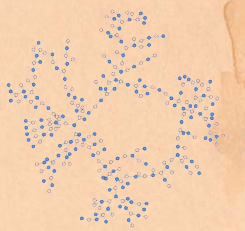

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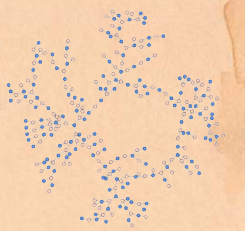

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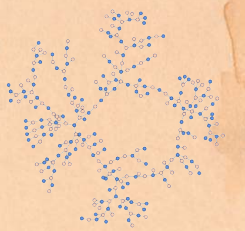

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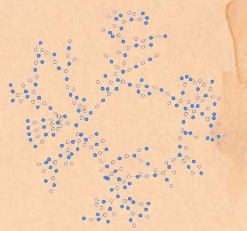

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$$\phi^* = p\phi^* + (1 - p\phi^*)\phi^*(1 - r)$$

$$\Rightarrow 1 = p + (1 - p\phi^*)(1 - r), \quad \phi^* \neq 0,$$

$$\Rightarrow \phi^* = \frac{1 - r/p}{1 - r} \quad \text{and} \quad \phi^* = 0.$$

- Critical point at $p = p_c = r$.
- Spreading takes off if $p/r > 1$
- Find continuous phase transition as for SIR model.
- Goodness: Matches $R_0 = \beta/\gamma > 1$ condition.

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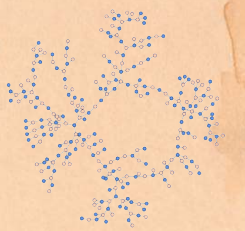

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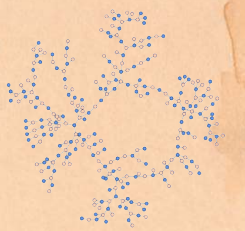

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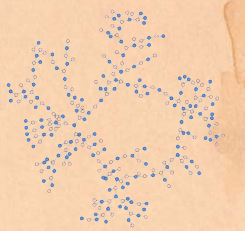

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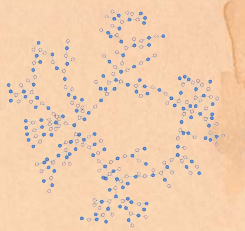

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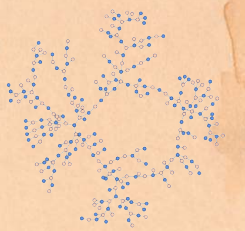

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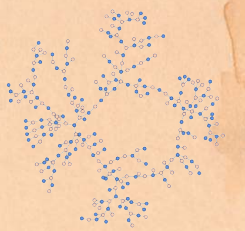

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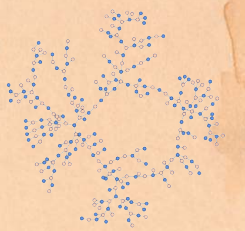

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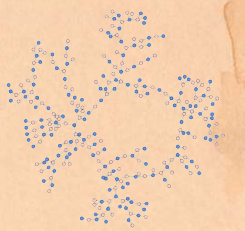

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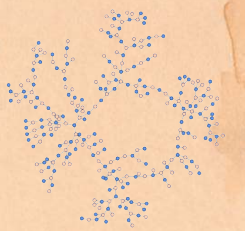
- Call ϕ^* the steady state level of infection.
- Pr(infected) = 1 - Pr(uninfected):

$$\phi^* = 1 - (1 - p\phi^*)^T.$$



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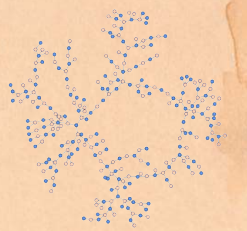

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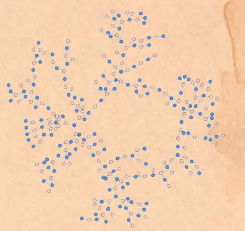

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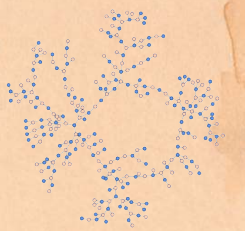

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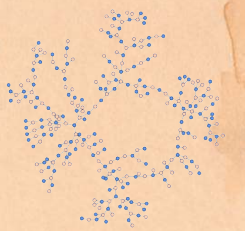

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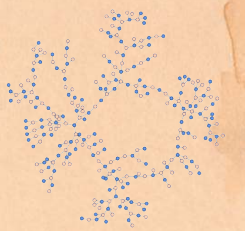

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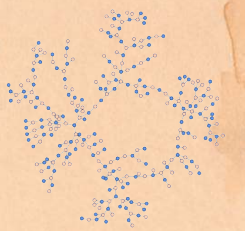

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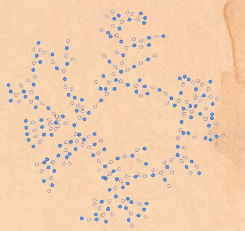

Homogeneous, one hit models:

Fixed points for $r = 1$, $d^* = 1$, and $T > 1$

- Closed form expression for ϕ^*:

$$\phi^* = 1 - (1 - p\phi^*)^T.$$

- Look for critical infection probability p_c.
- As $\phi^* \to 0$, we see

$$\phi^* \simeq pT\phi^* \quad \Rightarrow p_c = 1/T.$$

- Again find continuous phase transition...
- Note: we can solve for p but not ϕ^*:

$$p = (\phi^*)^{-1}[1 - (1 - \phi^*)^{1/T}].$$

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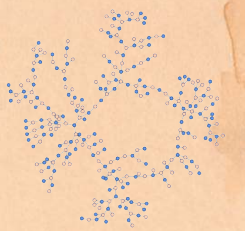

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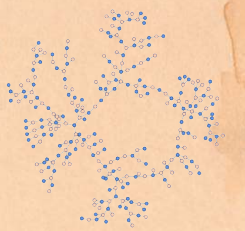

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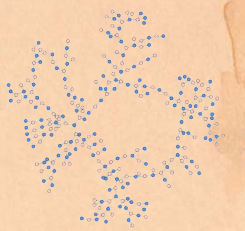

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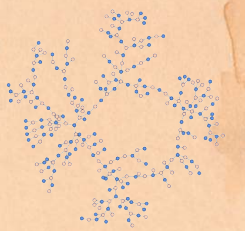

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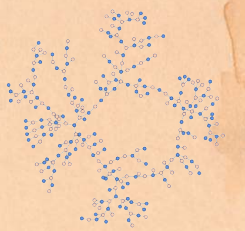

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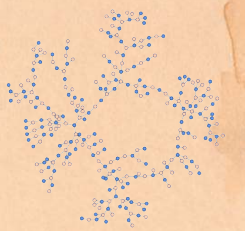

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- Start with $r = 1$, $d^* = 1$, and $T \geq 1$ case we have just examined:

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- For $r < 1$, add to right hand side fraction who:
  1. Did not receive any infections in last T time steps,
  2. And did not recover from a previous infection.
- Define corresponding dose histories. Example:

$$H_1 = \{\ldots, d_{t-T-2}, d_{t-T-1}, 1, \underbrace{0, 0, \ldots, 0, 0}_{T \text{ 0's}}\},$$

- With history H_1, probability of being infected (not recovering in one time step) is $1 - r$.

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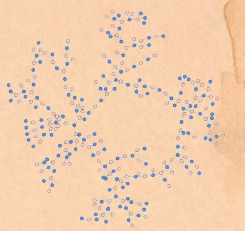

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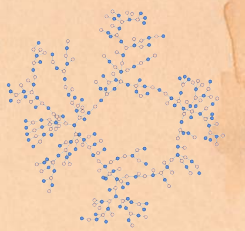

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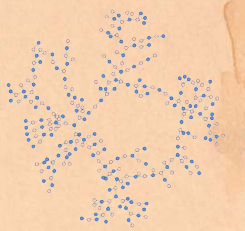

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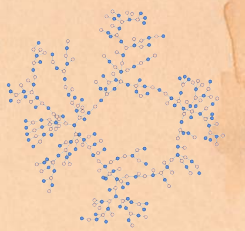

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- Start with $r = 1$, $d^* = 1$, and $T \geq 1$ case we have just examined:

$$\phi^* = 1 - (1 - p\phi^*)^T.$$

- For $r < 1$, add to right hand side fraction who:
  1. Did not receive any infections in last T time steps,
  2. And did not recover from a previous infection.
- Define corresponding dose histories. Example:

$$H_1 = \{\ldots, d_{t-T-2}, d_{t-T-1}, 1, \underbrace{0, 0, \ldots, 0, 0}_{T \text{ 0's}}\}.$$

- With history H_1, probability of being infected (not recovering in one time step) is $1 - r$.

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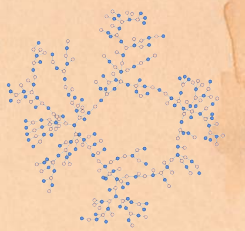

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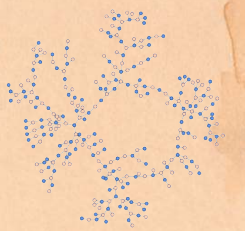

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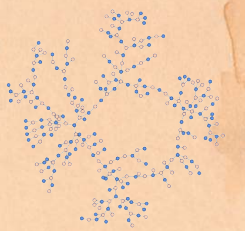

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- In general, relevant dose histories are:

$$H_{m+1} = \{\ldots, d_{t-T-m-1}, 1, \underbrace{0, 0, \ldots, 0, 0}_{m\ 0\text{'s}}, \underbrace{0, 0, \ldots, 0, 0}_{T\ 0\text{'s}}\}.$$

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$$P(H_1) = p\phi^*(1 - p\phi^*)^T(1 - r),$$

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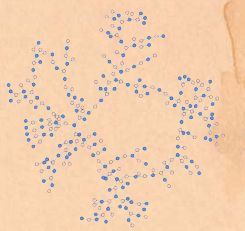

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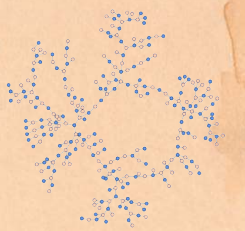

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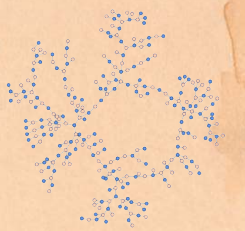

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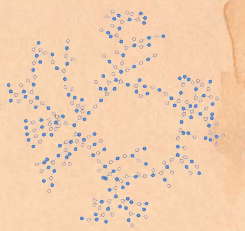

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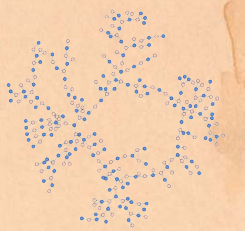

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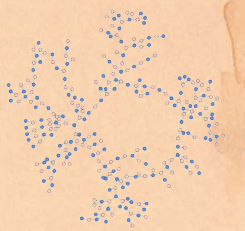

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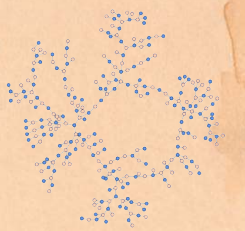

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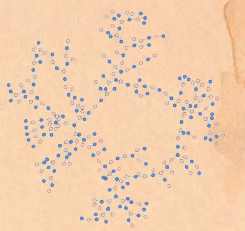

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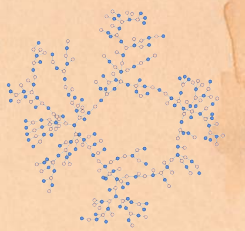

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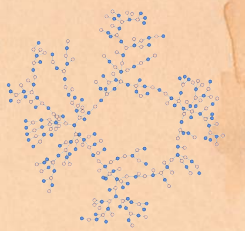

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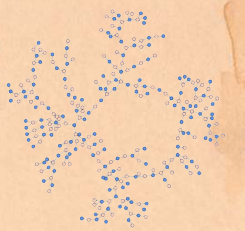

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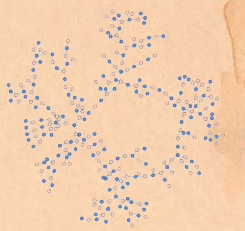
- Find critical exposure probability by examining above as $\phi^* \to 0$.
- $$\Rightarrow \quad p_c = \frac{1}{T + 1/r - 1} = \frac{1}{T + \tau}.$$

  where τ = mean recovery time for simple relaxation process.
- Decreasing r keeps individuals infected for longer and decreases p_c.



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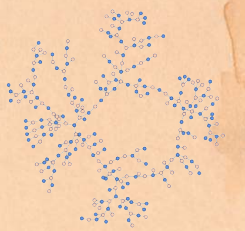

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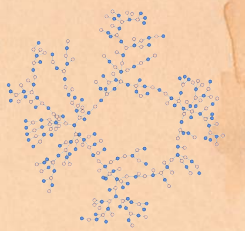

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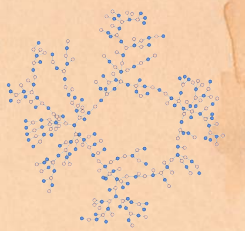
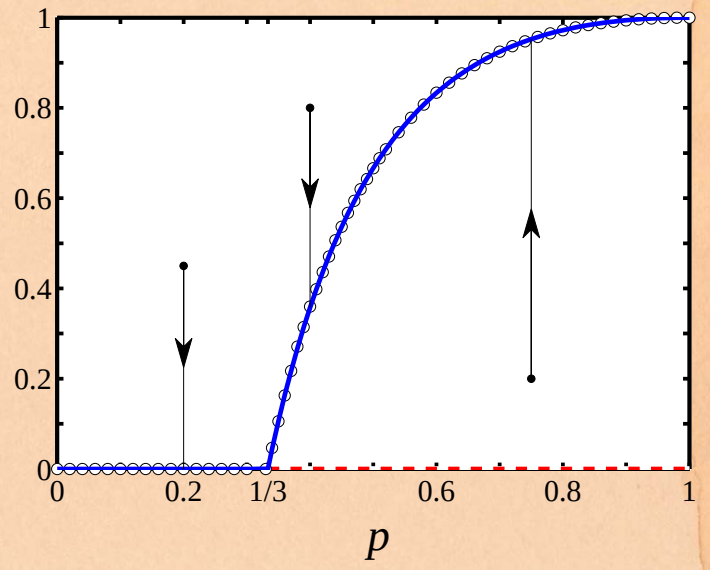


Epidemic threshold:

Fixed points for $d^* = 1$, $r \leq 1$, and $T \geq 1$

- $\phi^* = 1 - \frac{r(1-p\phi^*)^T}{1-(1-p\phi^*)(1-r)}$
- $\phi^* = 0$
- $p_c = 1/(T+\tau)$



- Example details: $T = 2$ & $r = 1/2 \Rightarrow p_c = 1/3$.
- Blue = stable, red = unstable, fixed points.
- $\tau = 1/r - 1$ = characteristic recovery time = 1.
- $T + \tau \simeq$ average memory in system = 3.
- Phase transition can be seen as a transcritical bifurcation. [11]

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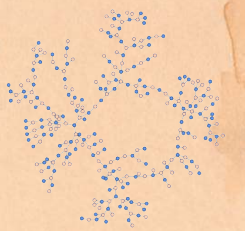

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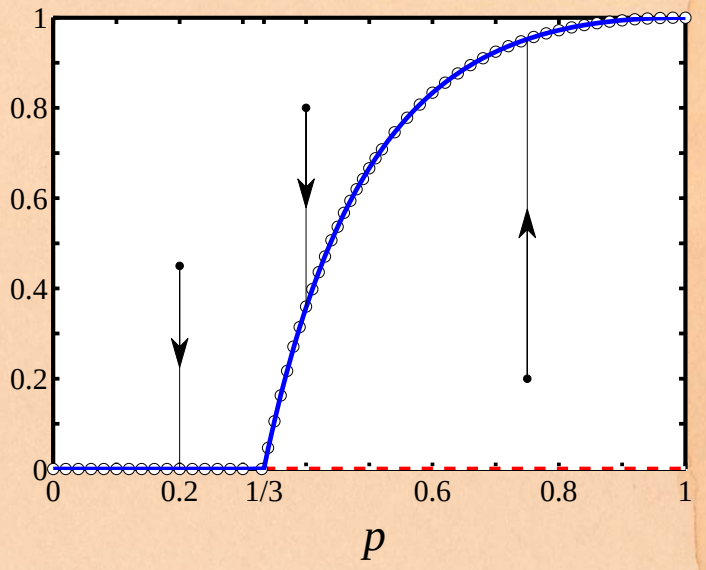

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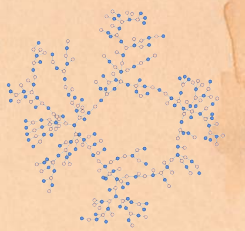

Homogeneous, multi-hit models:

- All right: $d^* = 1$ models correspond to simple disease spreading models.
- What if we allow $d^* \geq 2$?
- Again first consider SIS with immediate recovery ($r = 1$)
- Also continue to assume unit dose sizes ($f(d) = \delta(d-1)$).
- To be infected, must have at least d^* exposures in last T time steps.
- Fixed point equation:

$$\phi^* = \sum_{i=d^*}^{T} \binom{T}{i} (p\phi^*)^i (1-p\phi^*)^{T-i}.$$

- As always, $\phi^* = 0$ works too.

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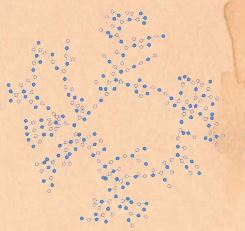

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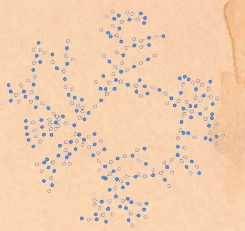

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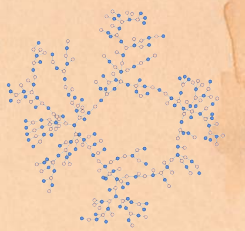

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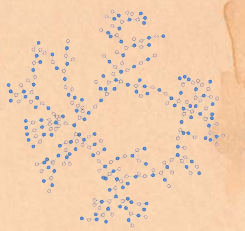

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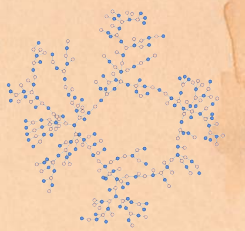

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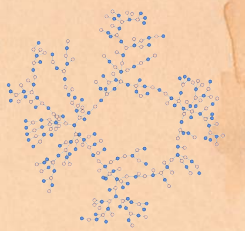

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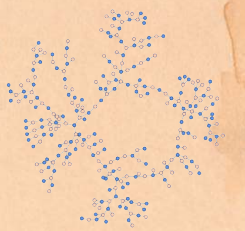

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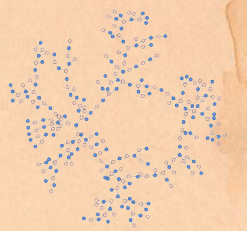

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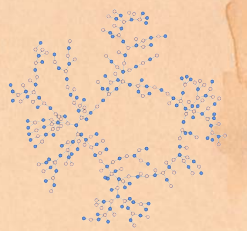

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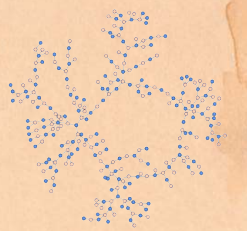

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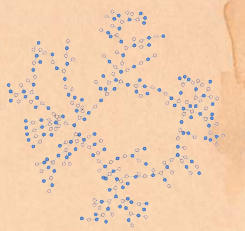

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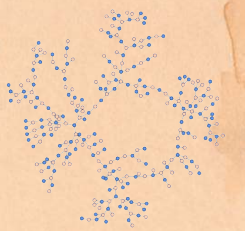

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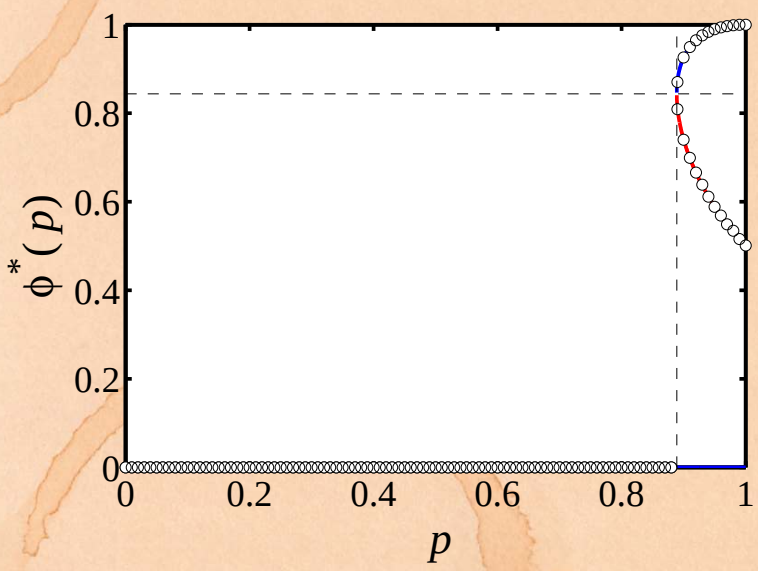

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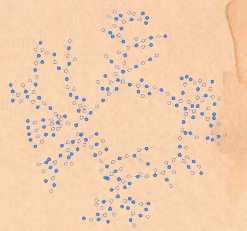

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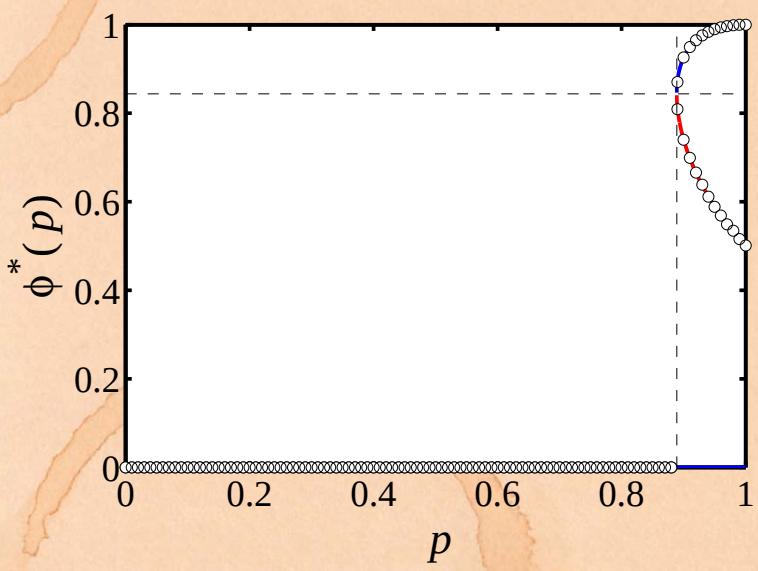

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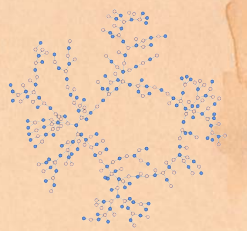

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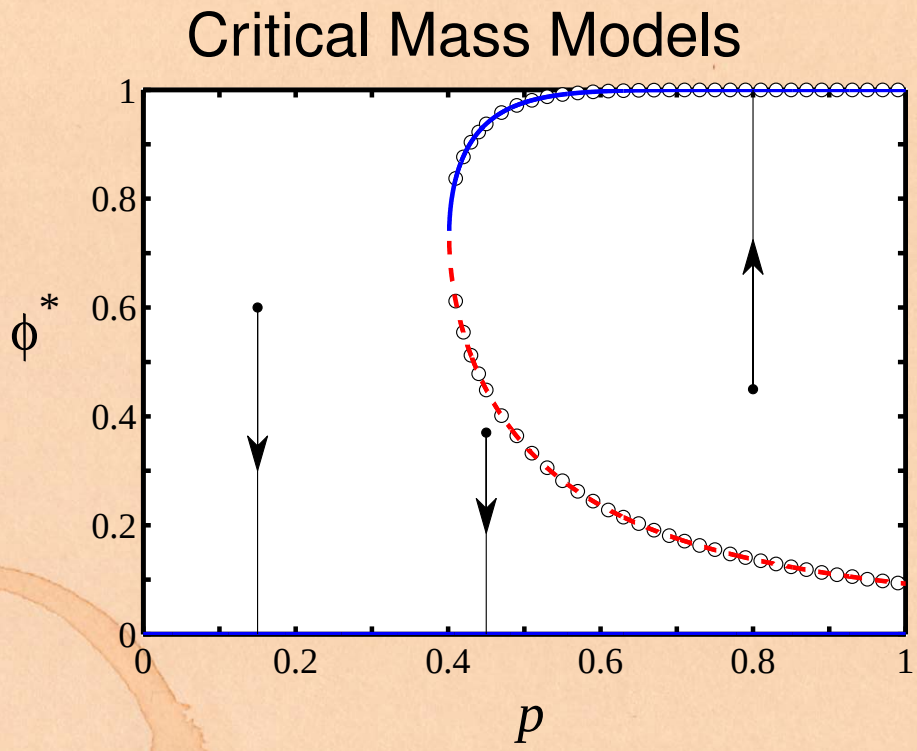
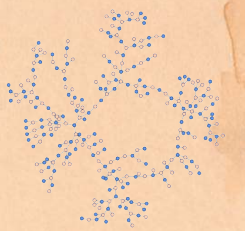
- Another example:

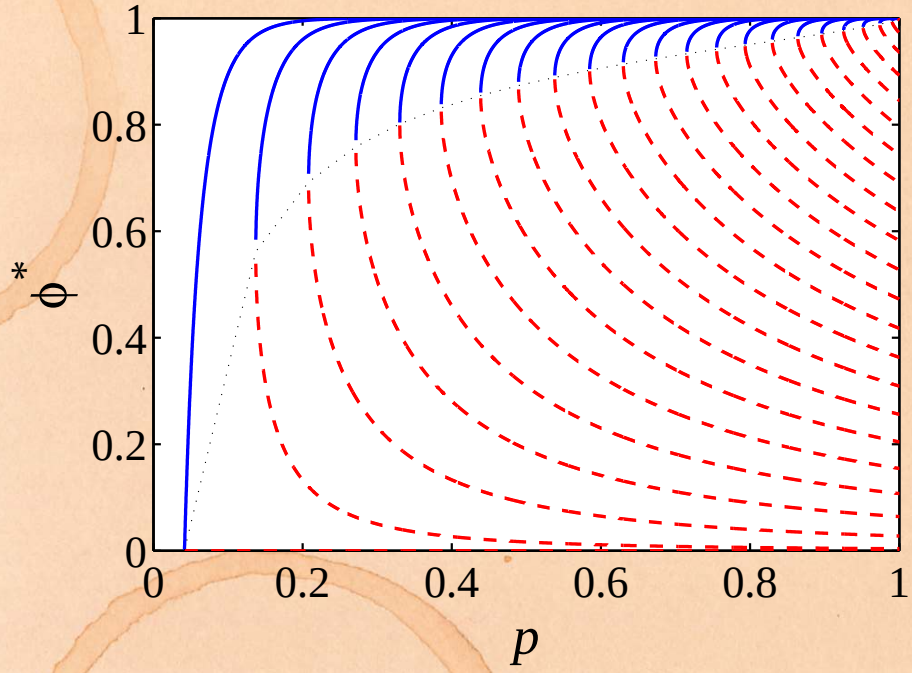


- $r = 1$, $d^* = 3$, $T = 12$ Saddle-node bifurcation.

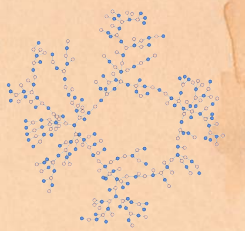


Fixed points for $r = 1$, $d^* > 1$, and $T \geq 1$

- $T = 24$, $d^* = 1, 2, \ldots 23$.



- $d^* = 1 \rightarrow d^* > 1$: jump between continuous phase transition and pure critical mass model.
- Unstable curve for $d^* = 2$ does not hit $\phi^* = 0$.
- See either simple phase transition or saddle-node bifurcation, nothing in between.



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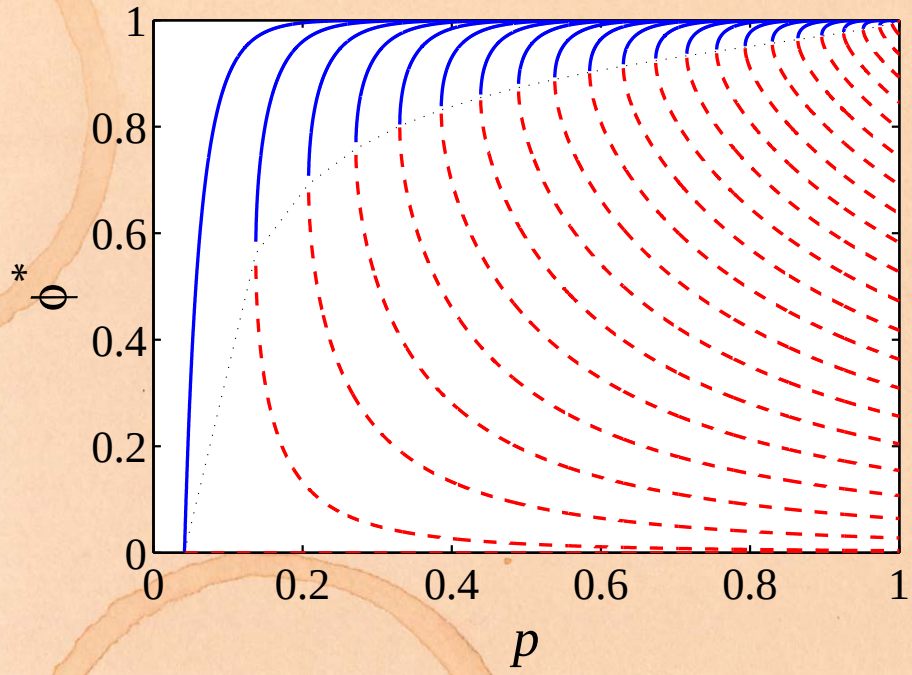

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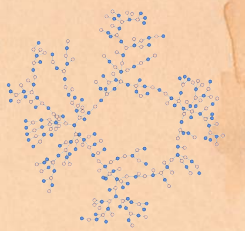

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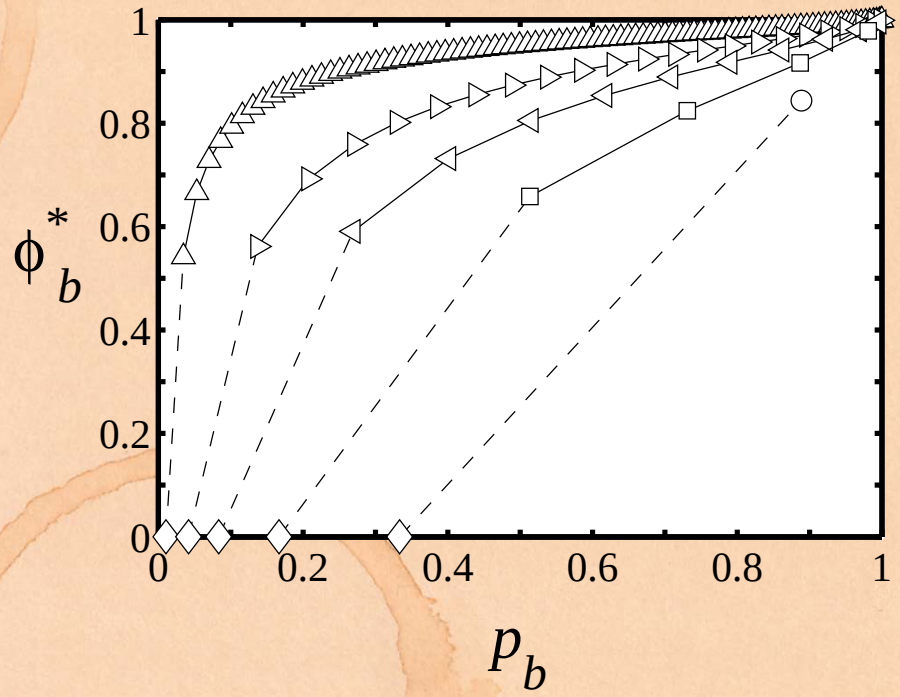
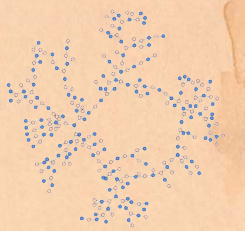
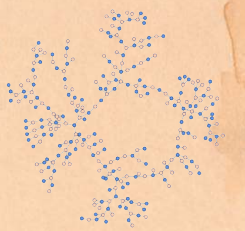
- Bifurcation points for example fixed T, varying d^*:



- $T = 96$ ($\triangle$).
- $T = 24$ ($\triangleright$),
- $T = 12$ ($\triangleleft$),
- $T = 6$ ($\square$),
- $T = 3$ ($\bigcirc$),



Fun with amazon's recommender system (⊞).
[amaznode.fladdict.net]



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- For $r < 1$, need to determine probability of recovering as a function of time since dose load last dropped below threshold.
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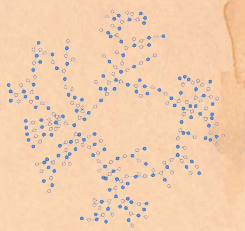
$$D_i(t) = \sum_{t'=t-T+1}^{t} d_i(t')$$

- Example for $T - 24$, $d^* - 14$:



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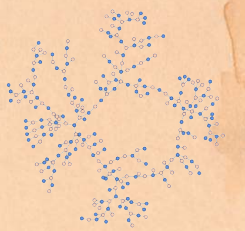

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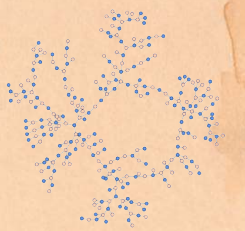

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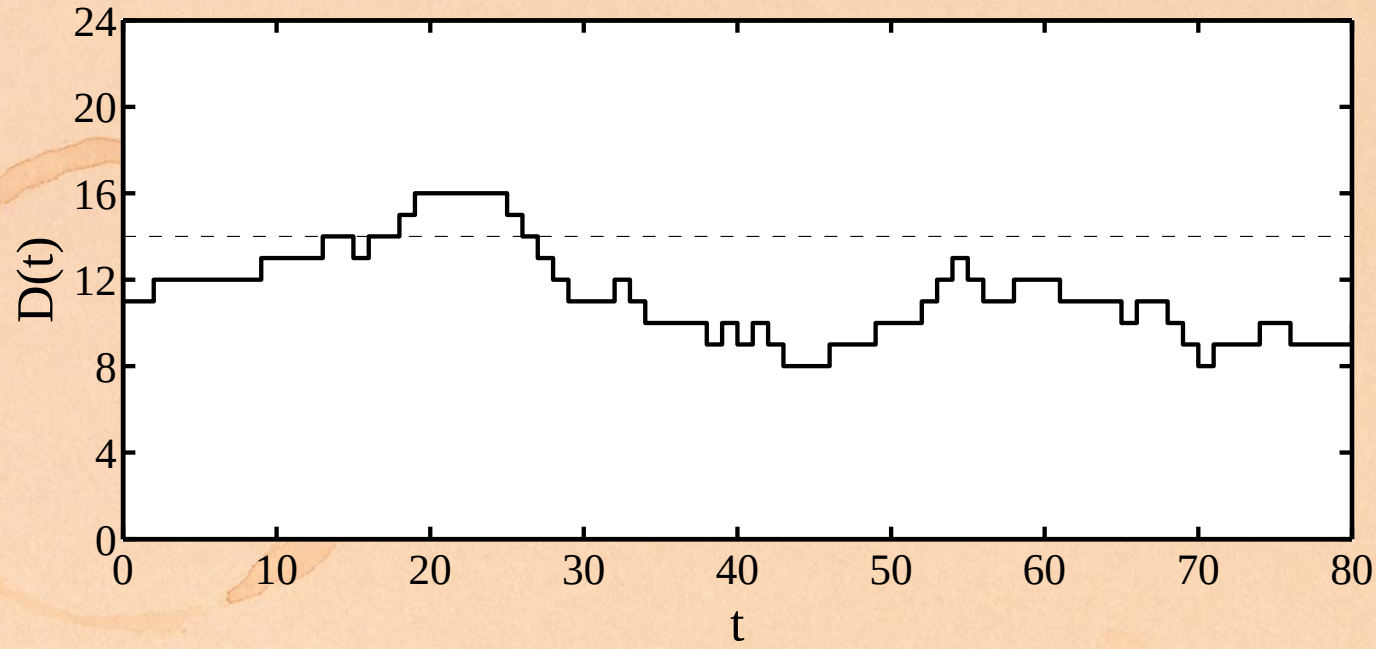

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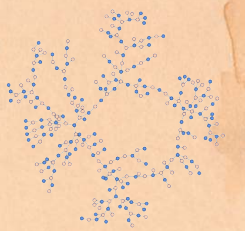

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- Fixed point equation:

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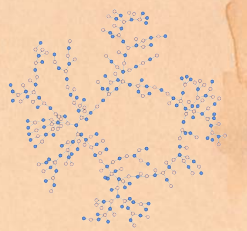

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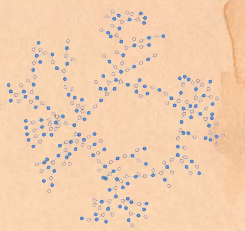

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$$\Gamma(p, \phi^*; r) = \sum_{m=1}^{\infty} (1 - r)^m \gamma_m(p, \phi^*).$$

- Fixed point equation:

$$\phi^* = \Gamma(p, \phi^*; r) + \sum_{i=d^*}^{T} \binom{T}{i} (p\phi^*)^i (1 - p\phi^*)^{T-i}.$$

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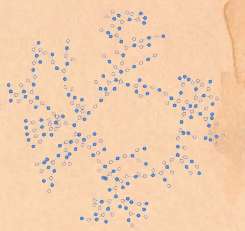

Fixed points for $r < 1$, $d^* > 1$, and $T \geq 1$

Example: $T = 3$, $d^* = 2$

- Want to examine how dose load can drop below threshold of $d^* = 2$:

$$D_n - 2 \to D_{n+1} - 1$$

- Two subsequences do this:
  $\{d_{n-2}, d_{n-1}, d_n, d_{n+1}\} = \{1, 1, 0, 0\}$
  and $\{d_{n-2}, d_{n-1}, d_n, d_{n+1}, d_{n+2}\} = \{1, 0, 1, 0, 0\}$.
- Note: second sequence includes an extra 0 since this is necessary to stay below $d^* = 2$.
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$$a - \{0\} \quad \text{and} \quad b - \{1, 0, 0\}.$$

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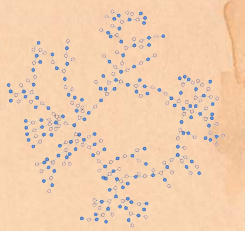

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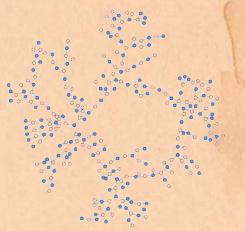

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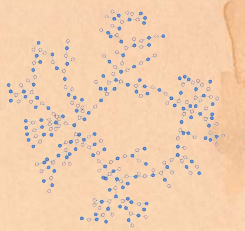

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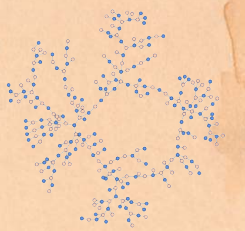

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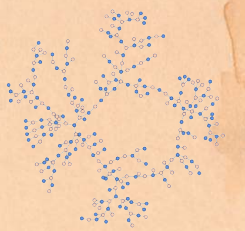

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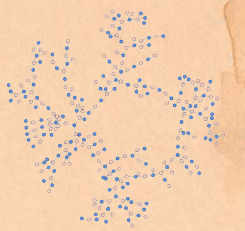

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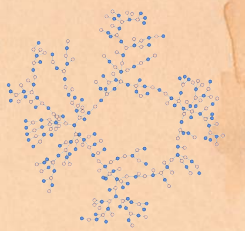

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- Determine number of sequences of length m that keep dose load below $d^* = 2$.
- N_a = number of $a = \{0\}$ subsequences.
- N_b = number of $b = \{1, 0, 0\}$ subsequences.

$$m = N_a \cdot 1 + N_b \cdot 3$$

Possible values for N_b:

$$0, 1, 2, \ldots, \left\lfloor \frac{m}{3} \right\rfloor.$$

where $\lfloor \cdot \rfloor$ means floor.

- Corresponding possible values for N_a:

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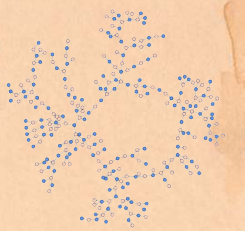

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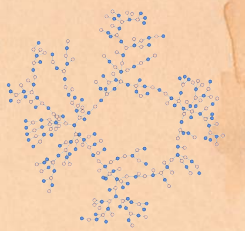

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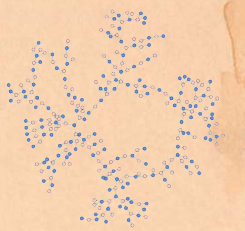

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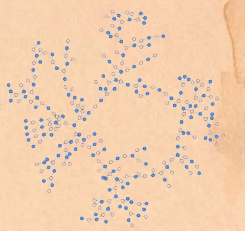

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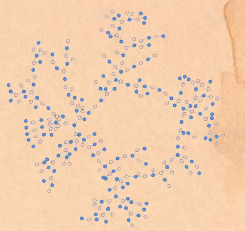

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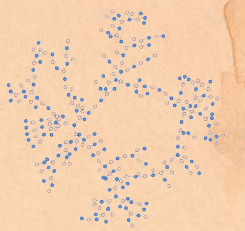


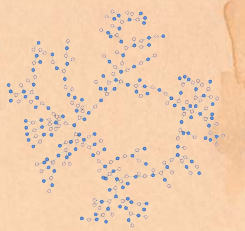
Fixed points for $r < 1$, $d^* > 1$, and $T \geq 1$

- How many ways to arrange N_a a's and N_b b's?
- Think of overall sequence in terms of subsequences:

$$\{Z_1, Z_2, \ldots, Z_{N_a+N_b}\}$$

- $N_a + N_b$ slots for subsequences.
- Choose positions of either a's or b's:

$$\binom{N_a + N_b}{N_a} = \binom{N_a + N_b}{N_b}.$$



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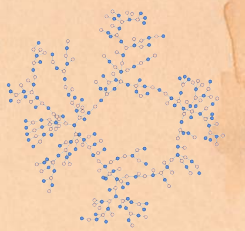

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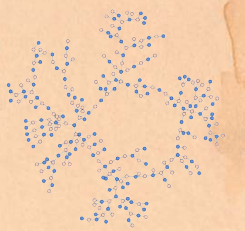

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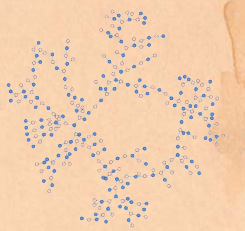

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- Total number of allowable sequences of length m:

$$\sum_{N_b=0}^{\lfloor m/3 \rfloor} \binom{N_b + N_a}{N_b} = \sum_{k=0}^{\lfloor m/3 \rfloor} \binom{m-2k}{k}$$

  where $k = N_b$ and we have used $m = N_a + 3N_b$.

- $P(a) = (1 - p\phi^*)$ and $P(b) = p\phi^*(1 - p\phi^*)^2$
- Total probability of allowable sequences of length m:

$$\chi_m(p, \phi^*) = \sum_{k=0}^{\lfloor m/3 \rfloor} \binom{m-2k}{k} (1 - p\phi^*)^{m-k} (p\phi^*)^k.$$

- Notation: Write a randomly chosen sequence of a's and b's of length m as $D_m^{a,b}$.

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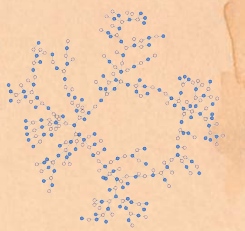

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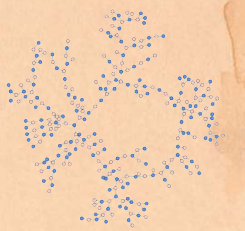

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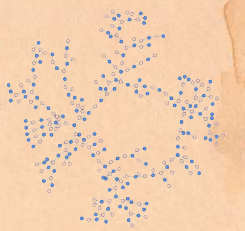

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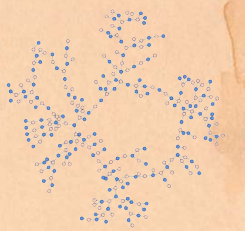

Fixed points for $r < 1$, $d^* > 1$, and $T \geq 1$

- Nearly there... must account for details of sequence endings.
- Three endings $\Rightarrow$ Six possible sequences:

$$D_1 = \{1, 1, 0, 0, D^{a,b}_{m-1}\}$$

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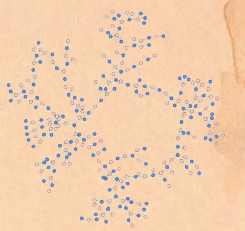

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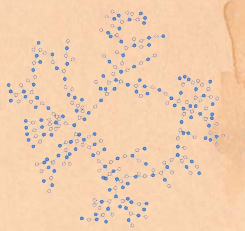

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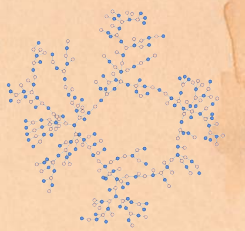


Fixed points for $r < 1$, $d^* = 2$, and $T = 3$

F.P. Eq: $\phi^* = \Gamma(p, \phi^*; r) + \sum_{i=d^*}^{T} \binom{T}{i} (p\phi^*)^i (1 - p\phi^*)^{T-i}.$

where $\Gamma(p, \phi^*; r) =$

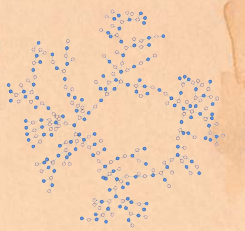
$$(1 - r)(p\phi)^2(1 - p\phi)^2 + \sum_{m=1}^{\infty}(1 - r)^m(p\phi)^2(1 - p\phi)^2 \times$$

$$\left[\chi_{m-1} + \chi_{m-2} + 2p\phi(1 - p\phi)\chi_{m-3} + p\phi(1 - p\phi)^2\chi_{m-4}\right]$$

and

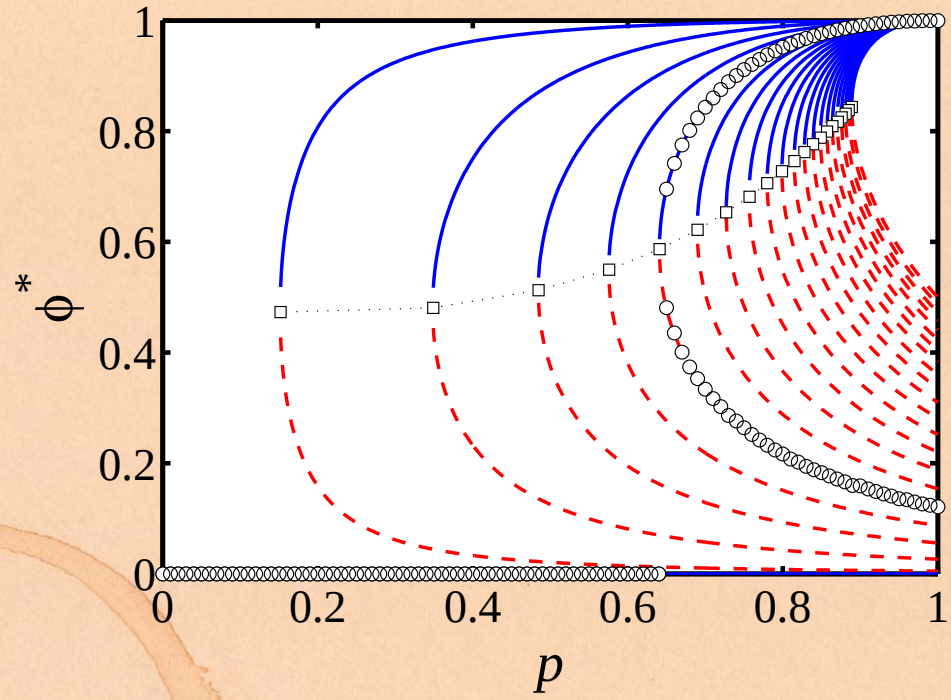
$$\chi_m(p, \phi^*) = \sum_{k=0}^{\lfloor m/3 \rfloor} \binom{m - 2k}{k} (1 - p\phi^*)^{m-k}(p\phi^*)^k.$$

Note: $(1 - r)(p\phi)^2(1 - p\phi)^2$ accounts for $\{1, 0, 1, 0\}$ sequence.

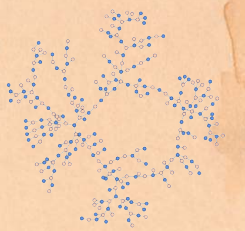


Fixed points for $r < 1$, $d^* > 1$, and $T \geq 1$

$T = 3$, $d^* = 2$



- $r = 0.01, 0.05, 0.10, 0.15, 0.20, \ldots, 1.00$.



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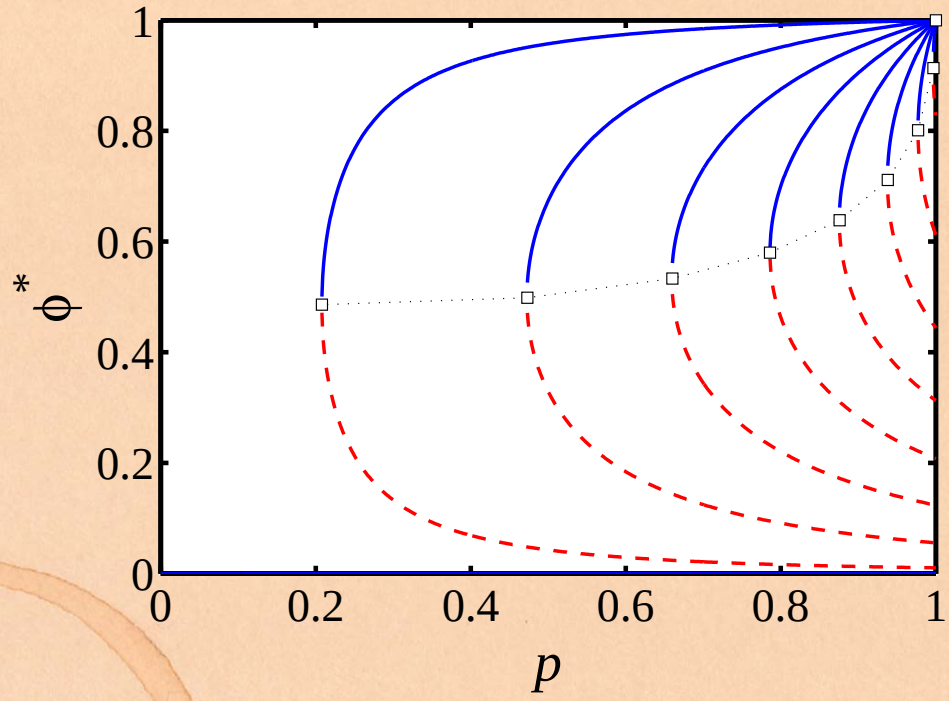

- $r = 0.01, 0.05, 0.10, \ldots, 0.3820 \pm 0.0001$.
- No spreading for $r \gtrsim 0.382$.

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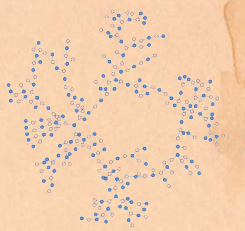

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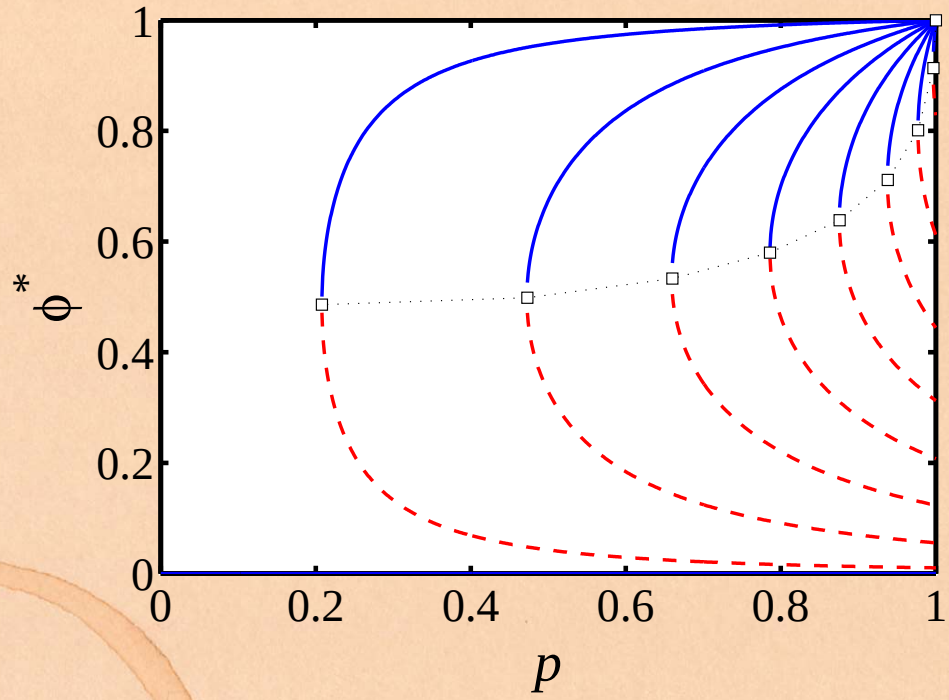

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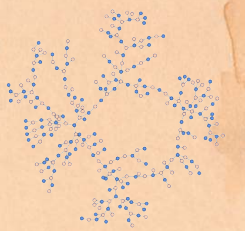

What we have now:

- Two kinds of contagion processes:
  1. Continuous phase transition: SIR-like.
  2. Saddle-node bifurcation: threshold model-like.
- $d^* = 1$: spreading from small seeds possible.
- $d^* > 1$: critical mass model.
- Are other behaviors possible?

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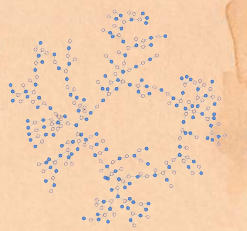

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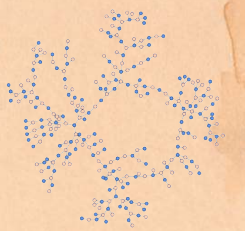

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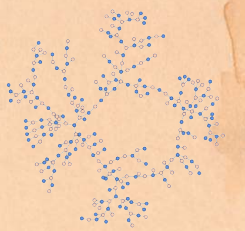

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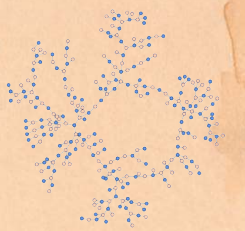

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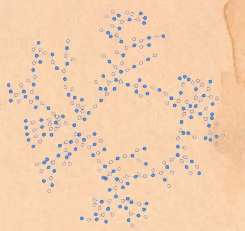

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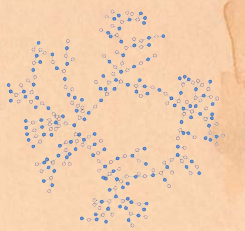

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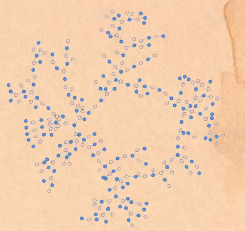
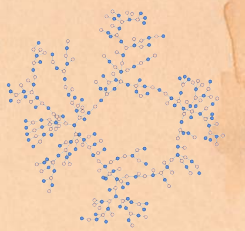


Generalized model

- Now allow for dose distributions (f) and threshold distributions (g) with width.
- Key quantities:

$$P_k = \int_0^\infty \mathrm{d}d^* \, g(d^*) P\left(\sum_{j=1}^{k} d_j \geq d^*\right) \text{ where } 1 \leq k \leq T.$$

- P_k = Probability that the threshold of a randomly selected individual will be exceeded by k doses.
- e.g.,
  P_1 = Probability that one dose will exceed the threshold of a random individual
  = Fraction of most vulnerable individuals.



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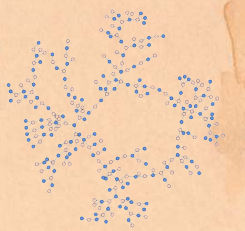

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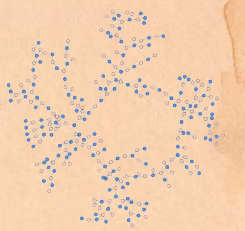

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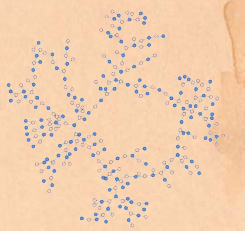

Generalized model—heterogeneity, $r = 1$

- Fixed point equation:

$$\phi^* = \sum_{k=1}^{T} \binom{T}{k} (p\phi^*)^k (1 - p\phi^*)^{T-k} \underline{P_k}$$

- Expand around $\phi^* = 0$ to find when spread from single seed is possible:

$$pP_1T \geq 1 \quad \text{or} \quad \Rightarrow p_c = 1/(TP_1)$$

- Very good:
  1. $P_1 T$ is the expected number of vulnerables the initial infected individual meets before recovering.
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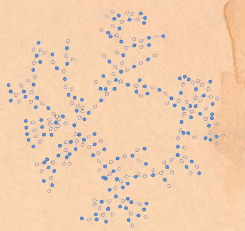

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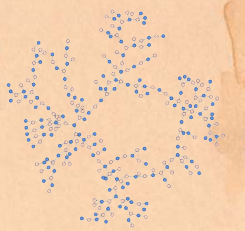

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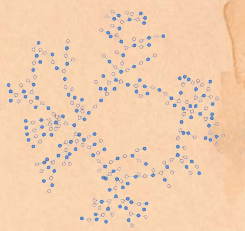

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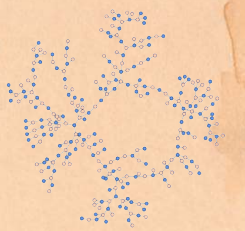

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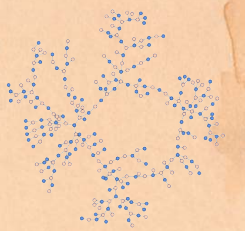

Generalized model—heterogeneity, $r = 1$

- Fixed point equation:

$$\phi^* = \sum_{k=1}^{T} \binom{T}{k} (p\phi^*)^k (1 - p\phi^*)^{T-k} \underline{P_k}$$

- Expand around $\phi^* = 0$ to find when spread from single seed is possible:

$$\boxed{pP_1T \geq 1} \quad \text{or} \quad \boxed{\Rightarrow p_c = 1/(TP_1)}$$

- Very good:
  1. $P_1 T$ is the expected number of vulnerables the initial infected individual meets before recovering.
  2. $pP_1 T$ is $\therefore$ the expected number of successful infections (equivalent to R_0).
- Observe: p_c may exceed 1 meaning no spreading from a small seed.

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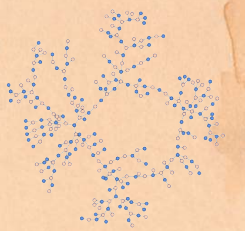

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- Behavior near fixed point depends on whether this slope is
  1. positive: $P_1 > P_2/2$ (continuous phase transition)
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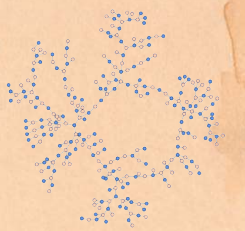

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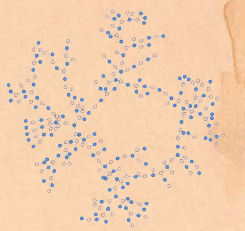

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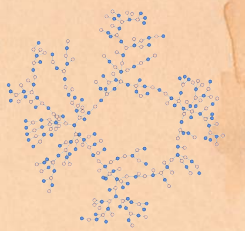

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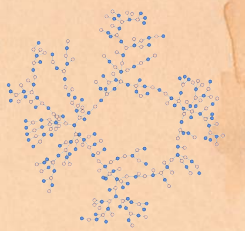

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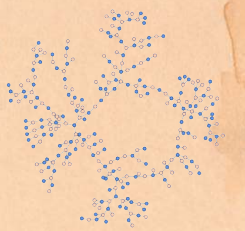

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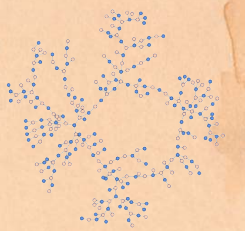

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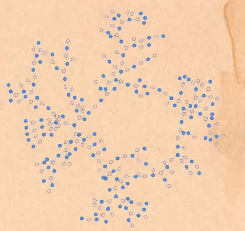

Heterogeneous case

Example configuration:

- Dose sizes are lognormally distributed with mean 1 and variance 0.433.
- Memory span: $T = 10$.
- Thresholds are uniformly set at
  1. $d_* = 0.5$
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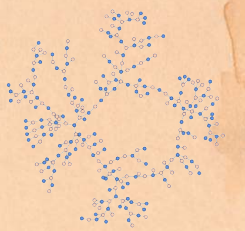
  3. $d_* = 3$
- Spread of dose sizes matters, details are not important.



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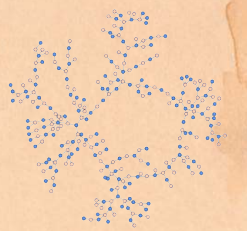

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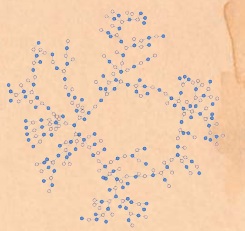

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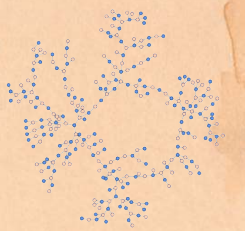
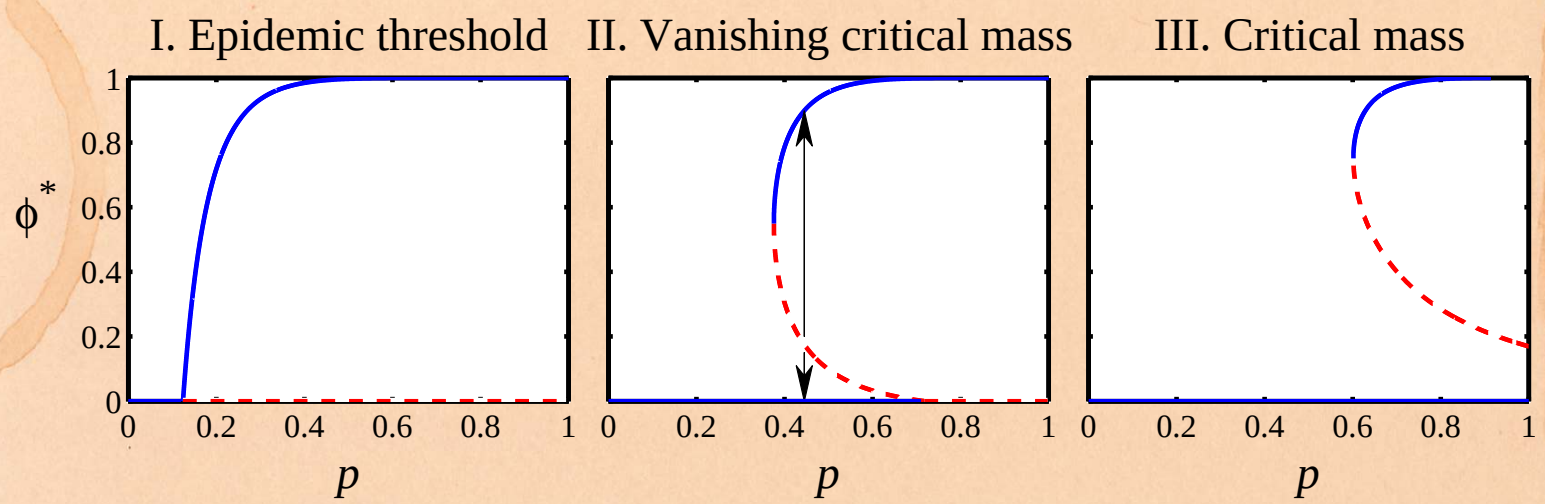


Three universal classes



- Epidemic threshold: $P_1 > P_2/2$, $p_c = 1/(TP_1) < 1$
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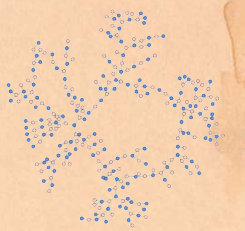

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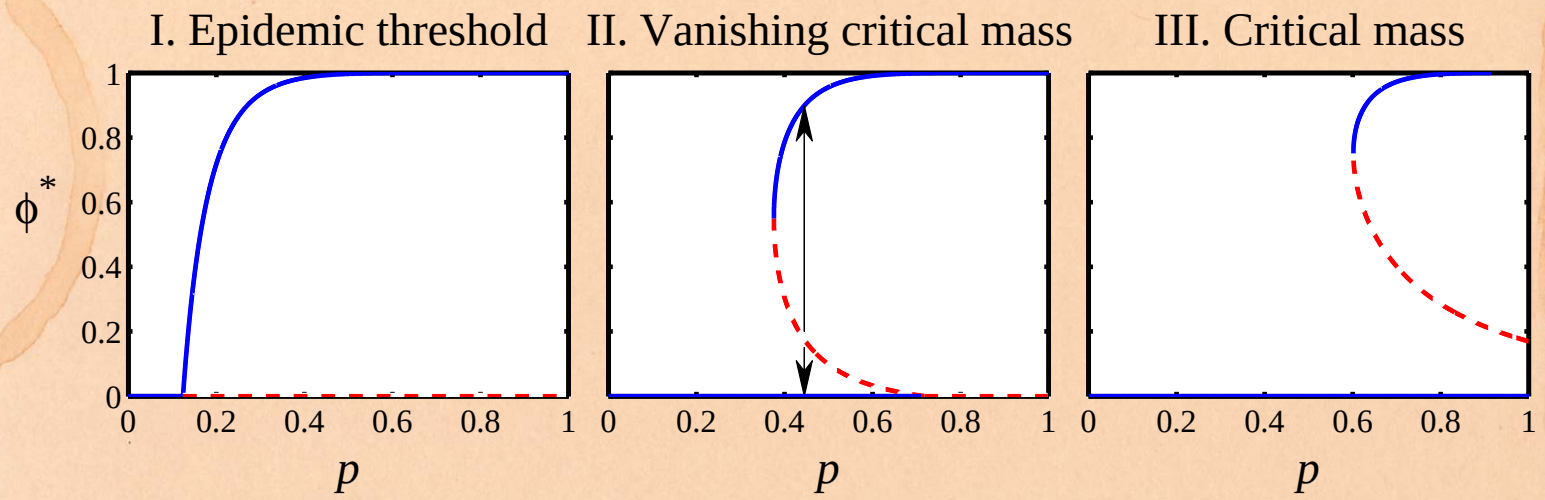

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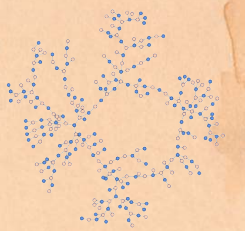

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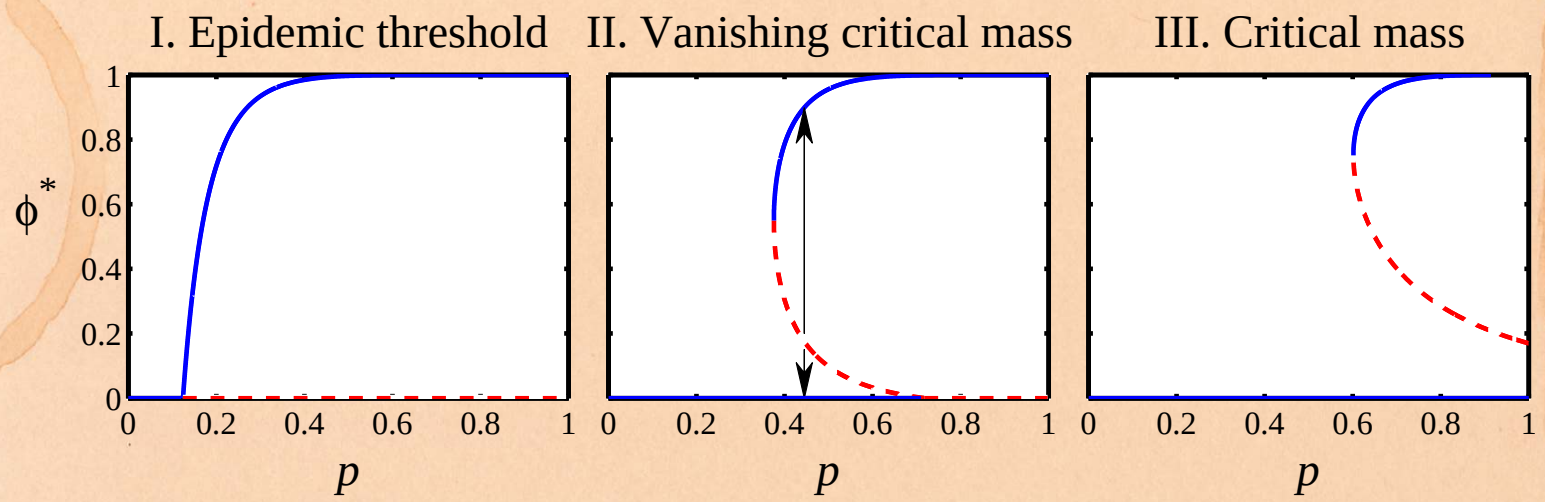

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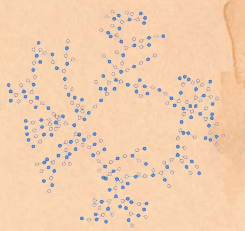

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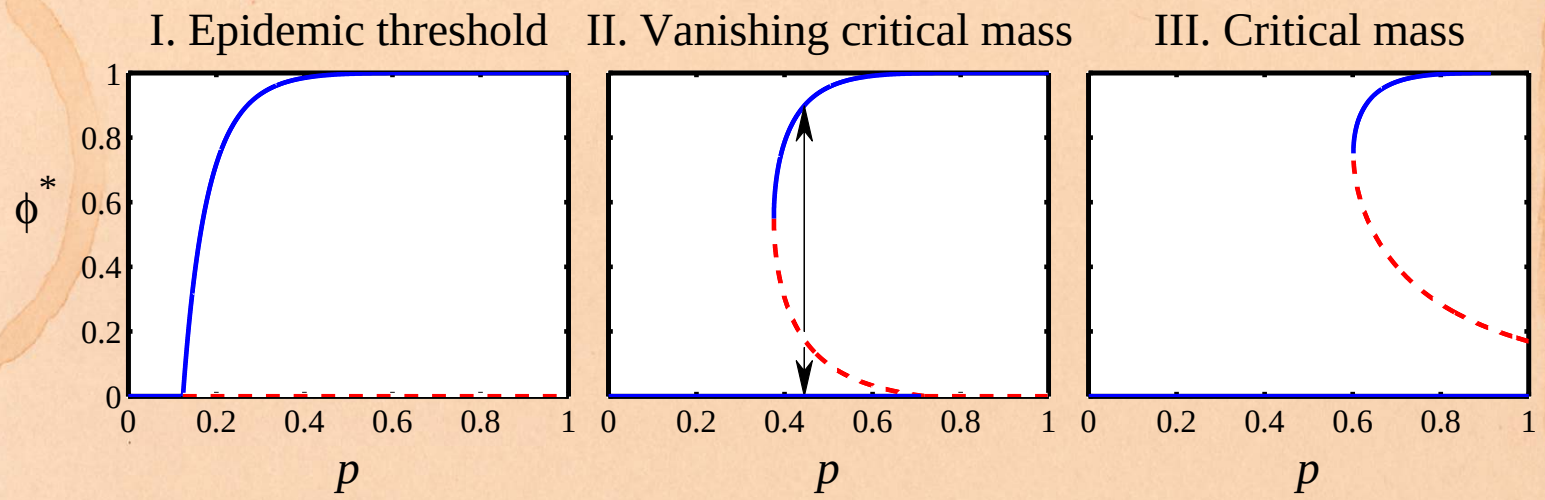

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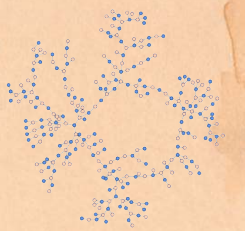
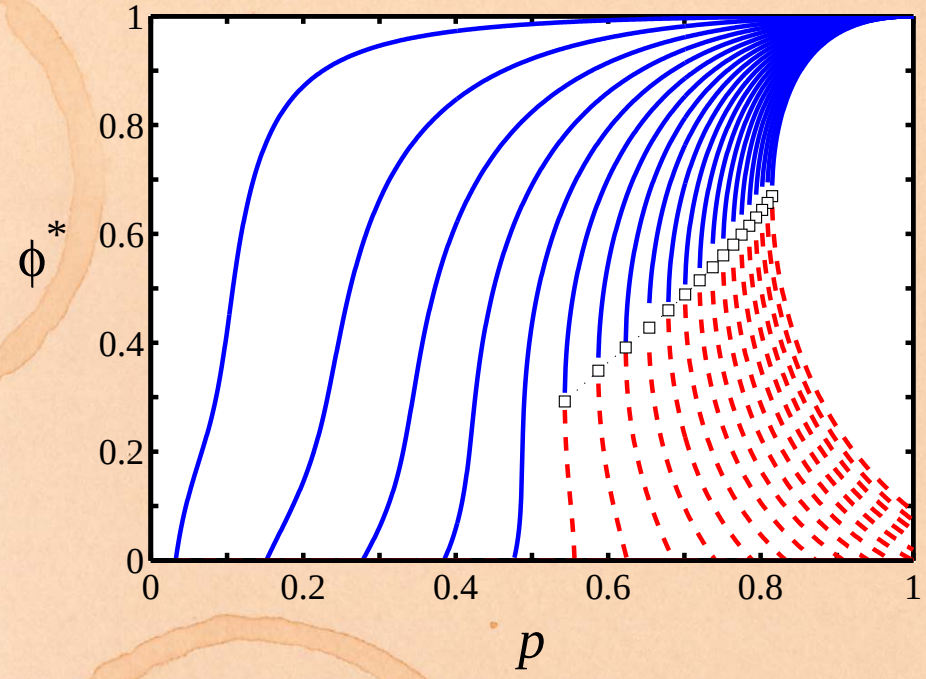
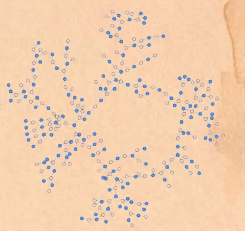


Heterogeneous case

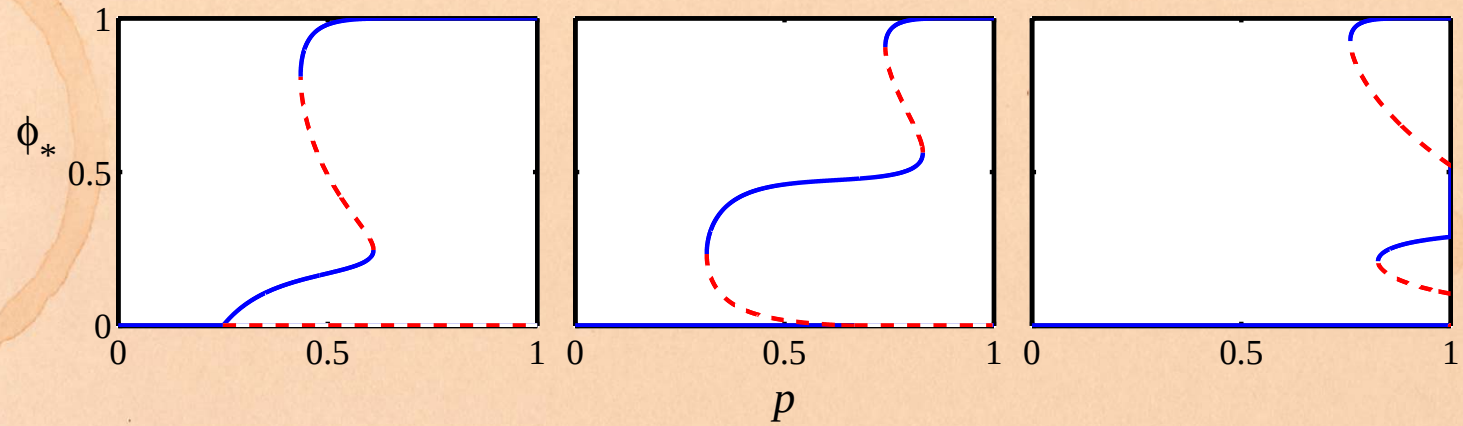
Now allow $r < 1$:



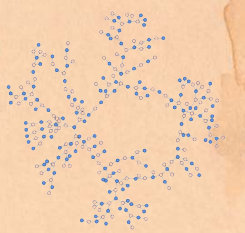
- II-III transition generalizes: $p_c = 1/[P_1(T + \tau)]$ where $\tau = 1/r - 1 =$ expected recovery time
- I-II transition less pleasant analytically.



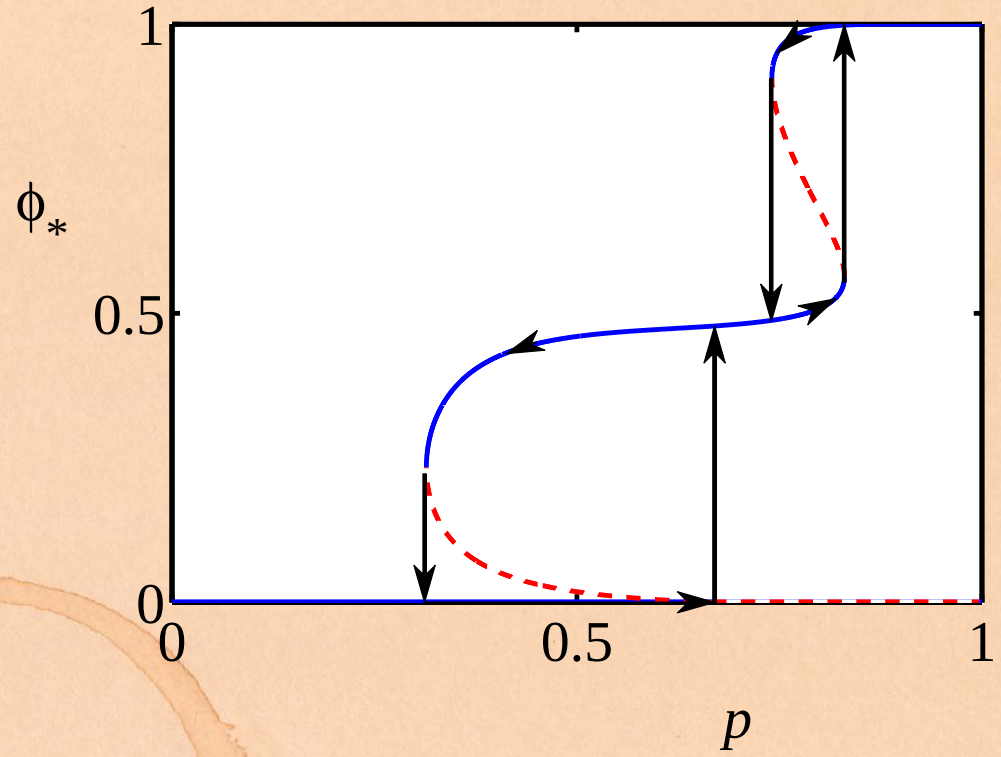
More complicated models



- Due to heterogeneity in individual thresholds.
- Three classes based on behavior for small seeds.
- Same model classification holds: I, II, and III.



Hysteresis in vanishing critical mass models



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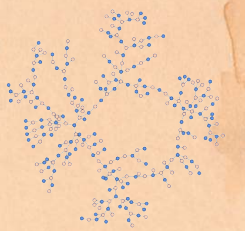

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Discussion

- Memory is a natural ingredient.
- Three universal classes of contagion processes:
  1. I. Epidemic Threshold
  2. II. Vanishing Critical Mass
  3. III. Critical Mass
- Dramatic changes in behavior possible.
- To change kind of model: 'adjust' memory, recovery, fraction of vulnerable individuals (T, r, ρ, P_1, and/or P_2).
- To change behavior given model: 'adjust' probability of exposure (p) and/or initial number infected (ϕ_0).

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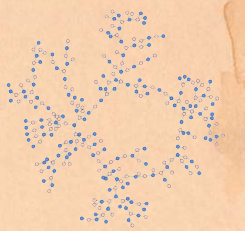

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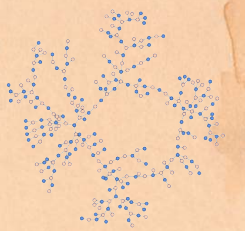

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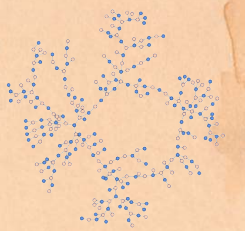

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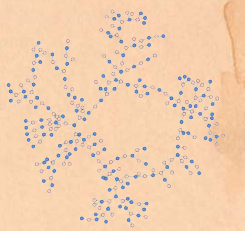

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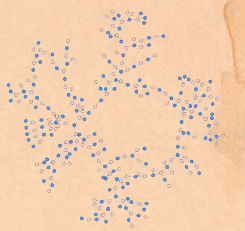

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- Single seed infects others if $pP_1(T+\tau) \geq 1$.
- Key quantity: $p_c = 1/[P_1(T+\tau)]$
- If $p_c < 1 \Rightarrow$ contagion can spread from single seed.
- Depends only on:
  1. System Memory $(T+\tau)$.
  2. Fraction of highly vulnerable individuals (P_1).
- Details unimportant: Many threshold and dose distributions give same P_k.
- Another example of a model where vulnerable/gullible population may be more important than a small group of super-spreaders or influentials.

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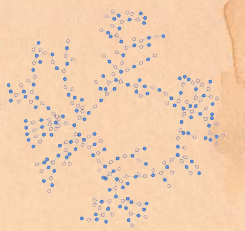

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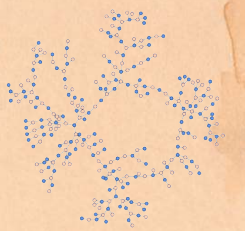

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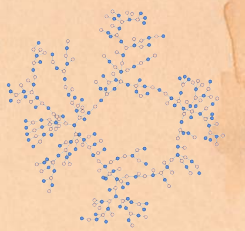

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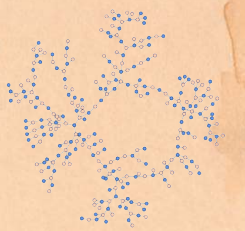

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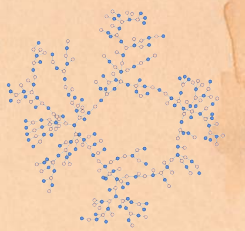

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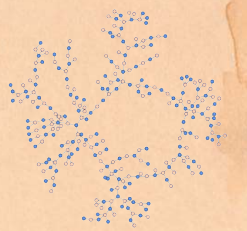

Details for Class I-II transition:

$$\phi^* = \sum_{k=1}^{T} \binom{T}{k} P_k (p\phi^*)^k (1 - p\phi^*)^{T-k},$$

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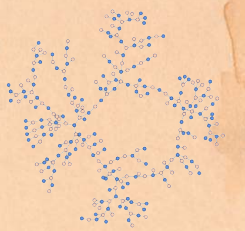

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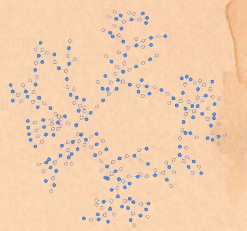

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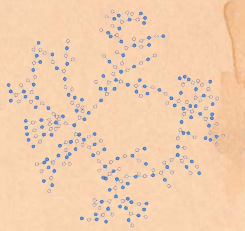

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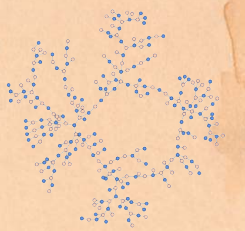

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&= \sum_{m=1}^{T} \sum_{k=1}^{m} \binom{T}{k} \binom{T-k}{m-k} P_k (-1)^{m-k} (p\phi^*)^m, \\
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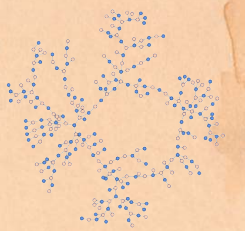

Details for Class I-II transition:

$$C_m = (-1)^m \binom{T}{m} \sum_{k=1}^{m} (-1)^k \binom{m}{k} P_k,$$

since

$$\begin{aligned}
\binom{T}{k}\binom{T-k}{m-k} &= \frac{T!}{k!(T-k)!}\frac{(T-k)!}{(m-k)!(T-m)!} \\
&= \frac{T!}{m!(T-m)!}\frac{m!}{k!(m-k)!} \\
&= \binom{T}{m}\binom{m}{k}.
\end{aligned}$$

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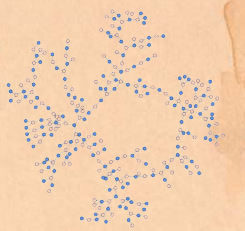

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- Linearization gives

$$\phi^* \simeq C_1 p \phi^* + C_2 p_c^2 \phi^{*2}.$$

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$$\phi^* \simeq \frac{C_1}{C_2 p_c^2}(p - p_c) = \frac{T^2 P_1^3}{(T-1)(P_1 - P_2/2)}(p - p_c).$$

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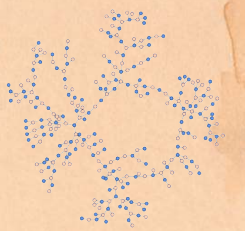

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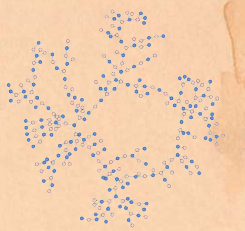

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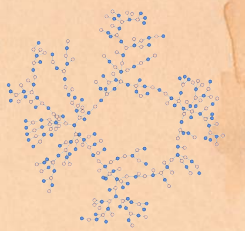

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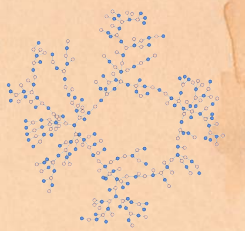

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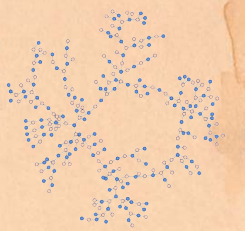

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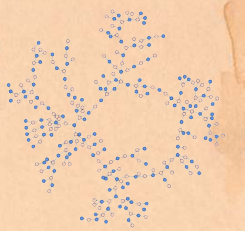

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