Branching Networks II

Complex Networks CSYS/MATH 303, Spring, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics Center for Complex Systems Vermont Advanced Computing Center University of Vermont

















Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

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Nutshell







Outline

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Horton and Tokunaga seem different:

- In terms of network achitecture. Horton's laws appear to contain less detailed information than Tokunaga's law.
 - Oddly, Horton's laws have four parameters and
 Tokunaga has two parameters.
- R_0 , R_0 , R_0 , and R_0 versus T_1 and R_T . One simple redundancy: $R_1 = R_0$. Insert question 2, assignment

 To make a connection, clearest approach is to star with Tokunaga's law...

► Known result: Takunaga — Horton [18, 19, 20, 9, 2]

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We need one more ingredient:

Space-fillingness

- ► A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks: Drainage density $\rho_{\rm dd}$ = inverse of typical distance between channels in a landscape.
- ► In terms of basin characteristics:

$$ho_{
m dd} \simeq rac{\sum {
m stream \ segment \ lengths}}{{
m basin \ area}} = rac{\sum_{\omega=1}^\Omega n_\omega ar{s}}{a_\Omega}$$

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Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law
 - $n_{\omega}/n_{\omega+1}=R_n$
- Estimate n_ω, the number of streams of order ω in terms of other n_ω, ω' > ω.
- Observe that each stream of order ω terminates by either:
 - 1. Running into another stream of order ω and generating a stream of order $\omega+1...$
 - ▶ $2n_{\omega+1}$ streams of order ω do this
 - 2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...
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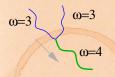
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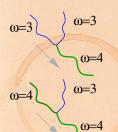
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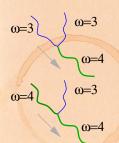
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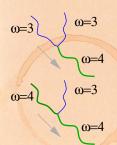
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Putting things together:

$$n_{\omega} = \underbrace{2n_{\omega+1}}_{\text{generation}} + \underbrace{7}_{\text{generation}}$$

- Use Tokunaga's law and manipulate expression to create R.s.
- ► Insert question 3, assignment 2 (⊞

Solution

 $\frac{(2+R_1+T_1)\pm\sqrt{(2+R_1+T_1)^2-8R_1}}{2}$

(The larger value is the one we want)

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Putting things together:

$$n_{\omega} = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{M} \frac{T_{\omega'-\omega}n_{\omega}}{\text{absorption}}$$

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- ► Use Tokunaga's law and manipulate expression to create R_n's.
- ► Insert question 3, assignment 2 (⊞)
- ► Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

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Nutshell





Connect Tokunaga to R_s

Now use uniform drainage density ρ_{dd} .

Assume side streams are roughly separated by

distance $1/
ho_{
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For an order ω stream segment, expected length is

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Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd} .
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- \blacktriangleright For an order ω stream segment, expected length is

$$\bar{s}_{\omega} \simeq \rho_{\mathrm{dd}}^{-1} \left(1 + \sum_{k=1}^{\omega - 1} T_k \right)$$

Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_{\omega} \simeq \rho_{\mathrm{dd}}^{-1} \left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1} \right)$$

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Altogether then:

$$\Rightarrow \bar{s}_{\omega}/\bar{s}_{\omega-1} = R_T - R_s - R_s$$

ightharpoonup Recall $R_{\ell}=R_{s}$ so

$$R_{\ell} = R_{s} = R_{T}$$

And from before:

$$R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_1}}{2}$$

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Some observations:

- $ightharpoonup R_n$ and R_ℓ depend on T_1 and R_T .
- Seems that R_a must as well
- Suggests Horton's laws must contain some redundance
- \blacktriangleright We'll in fact see that $R_s = R_s$
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]

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The other way round

Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.

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Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.

$$R_T = R_\ell$$

$$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n$$
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$$R_T = R_\ell$$

$$T_1=R_n-R_\ell-2+2R_\ell/R_n.$$

Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform)...

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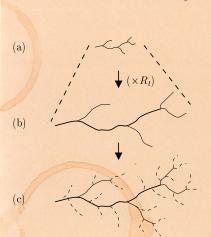
Models

Nutshell





From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
 - Start with picture showing an order ω stream and order ω generating and side streams
- Scale up by a factor of R_ℓ, orders increment to ω + 1 and ω.
 - density by adding new order == 1 streams

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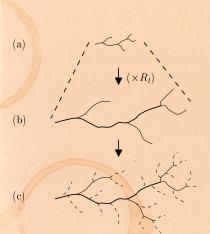
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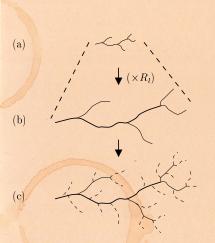
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Fluctuations

Models

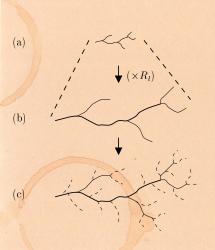
Nutshell







From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with picture showing an order ω stream and order $\omega-1$ generating and side streams.
- Scale up by a factor of R_{ℓ} , orders increment to $\omega + 1$ and ω .

Branching Networks II

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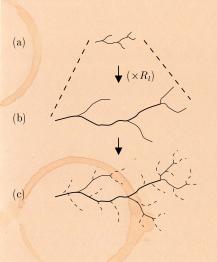
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Nutshell





From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with picture showing an order ω stream and order $\omega-1$ generating and side streams.
- Scale up by a factor of R_{ℓ} , orders increment to $\omega + 1$ and ω .
- Maintain drainage density by adding new order ω – 1 streams

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... and in detail:

- Must retain same drainage density.
- Add an extra $(R_t 1)$ first order streams for each
- Since by definition, order ω + 1 stream segment has
 T₁ order 4 side streams, we have:

$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right)$$

► For large w. Tokunaga's law is the solution—let solution.

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Just checking:

Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{i-1}$ into

$$T_k = (R_\ell - 1) \left(1 + \sum_{i=1}^{k-1} T_i \right)$$

$$(R_{\ell} - 1) \left(1 + T_1 \frac{R_{\ell}^{k-1} - 1}{R_{\ell} - 1} \right)$$

$$(R_{\ell} - 1) T_1 \frac{R_{\ell}^{k-1}}{R_{\ell}} = T_1 R_{\ell}^{k-1}$$

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$$= (R_{\ell} - 1) \left(1 + T_1 \frac{R_{\ell}^{k-1} - 1}{R_{\ell} - 1} \right)$$

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Nutshell







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$$= (R_{\ell} - 1) \left(1 + T_{1} \frac{R_{\ell}^{k-1} - 1}{R_{\ell} - 1} \right)$$

$$\simeq (R_{\ell} - 1) T_{1} \frac{R_{\ell}^{k-1}}{R_{\ell} - 1}$$

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$$= (R_{\ell} - 1) \left(1 + T_{1} \frac{R_{\ell}^{k-1} - 1}{R_{\ell} - 1} \right)$$

$$\simeq (R_{\ell} - 1) T_{1} \frac{R_{\ell}^{k-1}}{R_{\ell} - 1} = T_{1} R_{\ell}^{k-1} \quad ... \text{ yep.}$$

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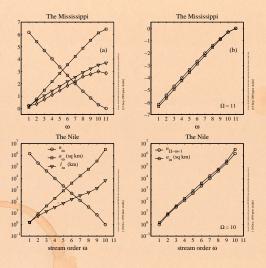
Nutshell







Horton's laws of area and number:



In right plots, stream number graph has been flipped vertically.

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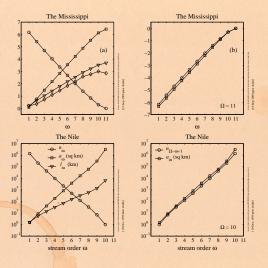
Nutshell







Horton's laws of area and number:



- In right plots, stream number graph has been flipped vertically.
- ▶ Highly suggestive that $R_n \equiv R_a$...

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Measuring Horton ratios is tricky:

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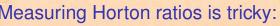
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How robust are our estimates of ratios?

Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

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Mississippi:

ω range	R_n	R_a	R_{ℓ}	$R_{\rm s}$	R_a/R_n
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3,8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7, 8]	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024

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Amazon:

ω range	R_n	Ra	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

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Rough first effort to show $R_n \equiv R_a$:

 a₀ > sum of all stream segment lengths in a order basin (assuming uniform drainage density)

$$a_\Omega \simeq \sum^\Omega n_\omega ar{s}_\omega/
ho_{
m dd}$$

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot 1}_{n_{\omega}} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_{\omega}}$$

$$= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega}$$

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Rough first effort to show $R_n \equiv R_a$:

 $a_{\Omega} \propto$ sum of all stream segment lengths in a order Ω basin (assuming uniform drainage density)

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot 1}_{n_{\omega}} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_{\omega}}$$

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$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot 1}_{n_\omega}$$

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$$=\frac{R_n^{\Omega}}{R_s}\bar{s}_1\sum_{n=1}^{\Omega}\left(\frac{R_s}{R_n}\right)^{\omega}$$

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Continued ...

$$a_{\Omega} \propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega}$$

 $\frac{R_n^{\Omega}}{R_s}\tilde{\mathsf{s}}_1\frac{R_s}{R_n}\frac{1-(R_s/R_n)^{\Omega}}{1-(R_s/R_n)}$

 $R_n^{\Omega-1} \bar{\mathbf{s}}_1 \frac{1}{1 - (R_s/R_n)}$ as Ω

 $R_0 = R_0$

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Continued ...

$$egin{aligned} oldsymbol{a}_\Omega &\propto rac{R_n^\Omega}{R_s}ar{s}_1\sum_{\omega=1}^\Omega \left(rac{R_s}{R_n}
ight)^\omega \ &= rac{R_n^\Omega}{R_s}ar{s}_1rac{R_s}{R_n}rac{1-(R_s/R_n)^\Omega}{1-(R_s/R_n)} \end{aligned}$$

 $R_n^{\Omega-1} \bar{\mathbf{s}}_1 \frac{1}{1 - (R_s/R_n)}$ as

 $R_n \equiv R_a$

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Continued ...

$$\mathbf{a}_{\Omega} \propto \frac{R_n^{\Omega}}{R_s} \bar{\mathbf{s}}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega}$$

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$$\sim R_n^{\Omega-1} \bar{\mathbf{s}}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \nearrow$$

 $R_n \equiv R_a$

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Continued ...

 $\mathbf{a}_{\Omega} \propto \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega}$ $= \frac{R_n^{\Omega}}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^{\Omega}}{1 - (R_s/R_n)}$ $\sim R_n^{\Omega - 1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \nearrow$

So, a_{Ω} is growing like R_n^{Ω} and therefore:

$$R_n \equiv R_a$$

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Not quite:

- ... But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranchi
- ▶ Insert question 4, assignment 2 (⊞)

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Models Nutshell





Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω .
- Not obvious, basins of low orders not necessarily contained in basis on higher orders.
- ▶ Stor

- $R_n = R_a \Rightarrow |n_{\omega}\bar{a}_{\omega}| = \text{const.}$
- ► Reason

 $n_{\rm o} \propto (R_{\rm o})^2$

 $a_\omega \propto (R_a)^\omega \propto n_\omega$

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 $\propto (R_a)^{\omega} \propto n_{\omega}^{-1}$

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$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

► Reason:

$$n_{\omega} \propto (R_n)^{-\omega}$$
 $ar{a}_{\omega} \propto (R_a)^{\omega} \propto n_{\omega}^{-1}$

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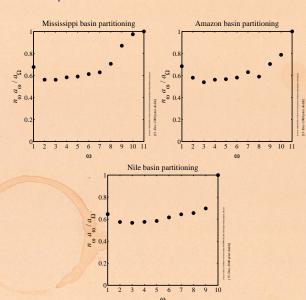
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Equipartitioning: Some examples:



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The story so far:

- Natural branching networks are hierarchical self-similar structures
- ► Hierarchy is mixed
- ▶ Tokunaga's law describes detailed architecture T_k = T₁R_k^{k-1}
- We have connected Tokunaga's and Horton's laws
 - Only two Horton laws are independent $(R_n = R_a)$
- Only two parameters are independent
- $\mathbb{E}\left(T_{1},R_{T}\right)\Leftrightarrow\left(R_{n},R_{s}\right)$

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The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- ► Tokunaga's law describes detailed architecture
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 - Only two parameters are independent
 - $(T_1,R_T)\Leftrightarrow (R_n,R_s)$

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We have connected Tokunaga's and Horton's lav
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The story so far:

- Natural branching networks are hierarchical, self-similar structures
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- ► Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.

We have connected lokunaga's and Horton's laws

Only two parameters are independent

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The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
- We have connected Tokunaga's and Horton's laws
- Only two Horton laws are independent $(R_n = R_a)$
- Only two parameters are independent: $(T_1, R_T) \Leftrightarrow (R_n, R_s)$

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A little further...

- ▶ Ignore stream ordering for the momen
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a? $P(a) \propto a^{-\tau}$ for large a
- Q: What is probability that the longest stream from ρ has length $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
- ▶ Roughly observed: 1.3 $\leq \tau \leq$ 1.5 and 1.7 $\leq \gamma \leq$ 2.0

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- ▶ Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \pi \lesssim 2$.

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A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- ➤ Q: What is probability that the p's drainage basin has
 - $P(a) \propto a^{-\tau}$ for large a
- has length \setminus $P(\ell) \propto \ell^{-\gamma}$ for large
- ► Roughly observed: 1.3 < + < 1.5 and

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Nutshell Reference

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Elustrations

Nutshell

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Models

Nutshell

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- Q: What is probability that the longest stream from p has length ℓ ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
- ▶ Roughly observed: 1.3 $\lesssim \tau \lesssim$ 1.5 and 1.7 $\lesssim \gamma \lesssim$ 2.0

Probability distributions with power-law decays

- We see them everywhere
 - Earthquake magnitudes (Gutenberg-Richter law
 - City sizes (Zipf's law)
 - ► Word frequency (Zipf's law) [21]
 - Wealth (maybe not—at least heavy tailed)
 - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
 - Arise from mechanisms: growth, randomness, optimization. ...
- ➤ Our task is always to illuminate the mechanism...

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Branching Networks II

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Nutshell





Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- ▶ Plan: Derive $P(a) \times a^{-r}$ and $P(\ell) \times \ell^{-r}$ starting with Tokunaga/Herton story [17, 1, 2]
- ▶ Let's work on $P(\ell)$
- Our first fudge: assume Horton's laws hold
 Ibroughout a basin of order Ω.
- (We know they deviate from strict laws for low ω and high ω but not too much.)
- Next: place stick between teeth. Bite stick. Proceed.

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Nutshell





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Tokunaga

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Models Nutshell





Finding γ :

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions
- The complementary cumulative distribution turns out to be most useful:

$$P_S(\ell_k) = P(\ell > \ell_k) = \int P(\ell) d$$

 $P_{>}(\ell_*) = 1 - P(\ell < \ell_*)$

Also known as the exceedance probability

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Nutshell





Finding γ :

The connection between P(x) and $P_{>}(x)$ when P(x) has a power law tail is simple:

 ℓ large ℓ then for large enough

$$\sim \int_{\ell=\ell_*}^{\ell_{\sf max}} \ell^{-\gamma} {
m d}\ell$$

$$\left. \cdot \frac{\ell^{-\gamma+1}}{-\gamma+1} \right|_{\ell=\ell_*}^{\ell_{\mathsf{max}}}$$

$$\propto \ell^{-\gamma+1}$$

for $\ell_{\mathsf{max}} \gg \ell_*$

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Finding γ :

- The connection between P(x) and $P_{>}(x)$ when P(x) has a power law tail is simple:
- ▶ Given $P(\ell) \sim \ell^{-\gamma}$ large ℓ then for large enough ℓ_*

$$P_{>}(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \,\mathrm{d}\ell$$

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$$P_{>}(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \,\mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} \frac{\ell^{-\gamma}}{\ell} \mathrm{d}\ell$$

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$$\sim \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} \ell^{-\gamma} \mathrm{d}\ell$$

$$= \left. \frac{\ell^{-\gamma+1}}{-\gamma+1} \right|_{\ell=\ell_*}^{\ell_{\max}}$$

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$$P_{>}(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} P(\ell) \,\mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_*}^{\ell_{\mathsf{max}}} \frac{\ell^{-\gamma}}{d\ell} \mathrm{d}\ell$$

$$= \frac{\ell^{-\gamma+1}}{-\gamma+1} \Big|_{\ell=\ell_*}^{\ell_{\max}}$$

$$\propto \ell_*^{-\gamma+1}$$
 for $\ell_{\text{max}} \gg \ell_*$

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Models Nutshell





Finding γ :

Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$

Assume some spatial sampling resolution in

Landscape is broken up into grid of $\Delta \times \Delta$ sites

▶ Approximate P_¬(ℓ_¬) as

 $P_{\beta}(\ell_{c}) = \frac{N_{\beta}(\ell_{c}, \Delta)}{N_{\beta}(0, \Delta)}.$

where $N_s(X, \Delta)$ is the number of sites with main stream length Y

► Use Horton's law of stream segments:

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Finding γ :

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ

 \triangleright Landscape is proken up into grid of $\Delta \times \Delta$ sites

Approximate P_¬(ℓ_x) as

 $P_{\sigma}(\ell_{\sigma}) = \frac{N_{\sigma}(\ell_{\sigma}, \Delta)}{N_{\sigma}(0, \Delta)}$

✓ where N_s(X \(\Delta\)) is the number of sites with mail stream length > I

▶ Use Horton's law of stream segments:

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Finding γ :

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- ▶ Landscape is broken up into grid of $\Delta \times \Delta$ sites
- ► Approximate P₁ (f₁) as
 - $P_{>}(\ell_{>}) = rac{N_{>}(\ell_{>}, \Delta)}{N_{>}(\ell_{>}, \Delta)}$
- where $N_s(\langle \cdot, \Delta \rangle)$ is the number of sites with main
- ► Use Horton's law of stream segments:

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Finding γ :

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- ▶ Approximate $P_>(\ell_*)$ as

$$P_{>}(\ell_*) = \frac{N_{>}(\ell_*; \Delta)}{N_{>}(0; \Delta)}.$$

where $N_{>}(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.

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Nutshell





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where $N_{>}(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.

Use Horton's law of stream segments: $s_{\omega}/s_{\omega-1} = R_s...$

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Finding γ :

▶ Set $\ell_* = \ell_\omega$ for some $1 \ll \omega \ll \Omega$.

$$P_{>}(\ell_{\omega}) = rac{N_{>}(\ell_{\omega}, \Delta)}{N_{>}(0, \Delta)} \simeq rac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- ▶ ∆'s cancel
- ▶ Denominator is ao p_{da}, a constant.
- using Horton's laws...

$$P_{\omega'(\omega)} = \sum_{\omega'=\omega+1}^{\Omega} a_{\omega} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'}) (\bar{s}_1 \cdot R_s^{\omega'-1})$$

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$$T_n \simeq \sum_{n'=n+1}^M (1{\cdot}R_n^{\Omega-\omega'})(ar{s}_1{\cdot}R_s^{\omega'-1})$$

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- Δ's cancel
- ▶ Denominator is ao p_{da}, a constant.
- using Horton's laws...

$$\simeq \sum_{\omega'=\omega+1}^{\scriptscriptstyle M} (1\!\cdot\! R_n^{\Omega-\omega'}) (ar{s}_1\!\cdot\! R_s^{\omega'-1})$$

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- Δ's cancel
- ▶ Denominator is $a_{\Omega}\rho_{\rm dd}$, a constant.
- using Horton's laws...

$$\simeq \sum_{\omega'=\omega+1}^{M} (1{\cdot}R_n^{\Omega-\omega'})(ar{s}_1{\cdot}R_s^{\omega'-1})$$

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- ▶ ∆'s cancel
- ▶ Denominator is $a_{\Omega}\rho_{\rm dd}$, a constant.
- So. Harro Horton's laws...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}$$
 $(1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$

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$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 R_{n}^{\Omega-\omega'}) (\bar{s}_{1} R_{s}^{\omega})$$

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Nutshell





Finding γ :

▶ Set $\ell_* = \ell_\omega$ for some $1 \ll \omega \ll \Omega$.

.

$$P_{>}(\ell_{\omega}) = rac{N_{>}(\ell_{\omega};\Delta)}{N_{>}(0;\Delta)} \simeq rac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'}/\Delta}$$

- Δ's cancel
- ▶ Denominator is $a_{\Omega}\rho_{\rm dd}$, a constant.
- So... using Horton's laws...

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) \bar{s}_{+} R_{s}^{\Omega}$$

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Finding γ :

We are here:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_{n}^{\Omega-\omega'}) (\bar{s}_{1} \cdot R_{s}^{\omega'-1})$$

Gleaning up irrelevant constants

► Change summation order by substituting

(equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)

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Cleaning up irrelevant constants:

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega'}$$

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Change summation order by substituting $\omega'' = \Omega - \omega'$.

(equivalent to $\omega'=\Omega$ down to $\omega'=\omega+1$)

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Finding γ :

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(rac{R_{s}}{R_{n}}
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▶ Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$.

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
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again using $\sum_{i=0}^{n-1} a^{i} = (a^{n} - 1)/(a - 1)$

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Finding γ :

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$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

▶ Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,

$$\left(\frac{R_n}{R_e}\right)$$

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Finding γ :

$$P_{>}(\ell_{\omega}) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

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Finding γ :

Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
ight)^{-\omega}$$

- Need to express right hand side in terms of
- ► Recall that $\ell \simeq \overline{\ell} R^s$

 $R_{\ell}^{\omega} = R_{\epsilon}^{\omega} = e^{\omega \ln R}$

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Finding γ :

Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
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▶ Need to express right hand side in terms of ℓ_{ω} .

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Finding γ :

Nearly there:

$$P_{>}(\ell_{\omega}) \propto \left(rac{R_n}{R_s}
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- ▶ Need to express right hand side in terms of ℓ_{ω} .
- ▶ Recall that $\ell_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$.

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- ▶ Need to express right hand side in terms of ℓ_{ω} .
- ▶ Recall that $\ell_{\omega} \simeq \bar{\ell}_1 R_{\ell}^{\omega-1}$.

$$\ell_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$

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Finding γ :

▶ Therefore:

$$P_{>}(\ell_{\omega}) \propto e^{-\omega \ln(R_n/R_s)}$$

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Finding γ :

Therefore:

$$P_>(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}
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$$\propto \ell_{\omega} - \ln(R_n/R_s) / \ln R_s$$

 $= f - (\ln R_n - \ln R_s) / \ln R$

 $\rho = \ln R_n / \ln R_s + 1$

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$$=\ell_{\omega}^{-\gamma+1}$$

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Finding γ :

And so we have:

$$\gamma = \ln R_n / \ln R_s$$

Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert question 5, assignment 2 (⊞)

Such connections between exponents are called scaling relations

▶ Let's connect to one last relationship: Hack's law

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Hack's law: [6]



$$e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}\right)^{\ln R_s/\ln R_n}$$

$$\propto (R_n^{\omega})^{\ln R_s/\ln R_n} \propto a_{\omega}^{\ln R_s/\ln R_n} \Rightarrow \boxed{h = \ln R_s/\ln R_n}$$

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Hack's law: [6]



Typically observed that $0.5 \lesssim h \lesssim 0.7$.

Use Horton laws to connect h to Horton ratios:

$$\propto R_s^{\omega}$$
 and $a_{\omega} \propto R_n^{\omega}$

win
$$R_s \propto \left(e^{\omega \ln R_n}
ight)^{\ln R_s/\ln R_n}$$

$$\propto (R_n^{\,\omega})^{\ln R_s/\ln R_n} \propto a_\omega^{\,\ln R_s/\ln R_n} \Rightarrow \boxed{h = \ln R_s/\ln R_n}$$

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Hack's law: [6]

$\ell \propto a^h$

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$$\ell_\omega \propto R_s^\omega$$
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Observe:

$$\ell_{\omega} \propto e^{\omega \ln R_s}$$

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$$\propto (R_n^{\omega})^{\ln R_s/\ln R_n}$$

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Connecting exponents Only 3 parameters are independent: e.g., take *d*, *R*_n, and *R*_s

relation:	scaling relation/parameter: [2]
$\ell \sim {\sf L}^{\sf d}$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_{\omega}/n_{\omega+1}=R_n$	R_n
$ar{a}_{\omega+1}/ar{a}_{\omega}=R_a$	$R_a = \frac{R_n}{R_n}$
$ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}$	$R_\ell = R_{ m s}$
$\ell \sim \pmb{a^h}$	$h = \log R_s / \log R_n$
$a\sim L^D$	D = d/h
${\it L}_{\perp} \sim {\it L}^{\it H}$	H = d/h - 1
$P(a) \sim a^{- au}$	au = 2 - h
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^eta$	$\beta = 1 + h$
$\lambda \sim {\cal L}^{arphi}$	$arphi= extsf{d}$

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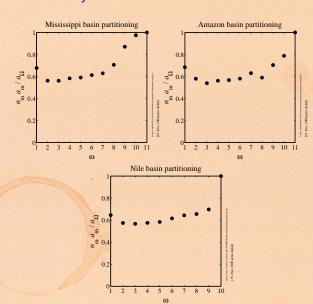
Fluctuation

Models Nutshell





Equipartitioning reexamined: Recall this story:



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Equipartitioning

What about

$$P(a) \sim a^{-\tau}$$

 $aP(a) \sim a^{-\tau+1} \neq \text{const}$

- ▶ P(a) overcounts basins within basins
 - while stream ordering separates basins

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Equipartitioning

What about

$$P(a) \sim a^{-\tau}$$

Since $\tau > 1$, suggests no equipartitioning:

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Equipartitioning

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- while stream ordering separates basins...

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Moving beyond the mean:

▶ Both Horton's laws and Tokunaga's law relate average properties, e.g..

 $\bar{s}_{\rm o}/\bar{s}_{\rm o-1}=R_{\rm s}$

- ► Natural generalization to consideration relationships between probability distributions
- Yields rish and full description of branching network structure
- > See into the heart of randomness.

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Horton ⇔ Tokunaga

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Branching Networks II

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 Natural generalization to consideration relationships between probability distributions Scaling relations

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See into the heart of randomness



Branching Networks II

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Tokunaga

Reducing Horton

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Horton ⇔ Tokunaga

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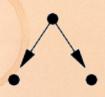
Nutshell





A toy model—Scheidegger's model

Directed random networks [11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$

- ► Flow is directed downwards
- Useful and interesting test case—more later...

Branching Networks II

Horton ⇔ Tokunaga

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$$\blacktriangleright \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega} F_{\ell}(\ell/R_{\ell}^{\omega})$$

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References





All moments grow exponentially with order



- $\blacktriangleright \bar{\ell}_{\omega} \propto (R_{\ell})^{\omega} \Rightarrow N(\ell|\omega) = (R_{n}R_{\ell})^{-\omega} F_{\ell}(\ell/R_{\ell}^{\omega})$
- $lack ar a_\omega \propto (R_a)^\omega \Rightarrow {\sf N}(a|\omega) = (R_n^2)^{-\omega} {\sf F}_a(a/R_n^\omega)$

Branching Networks II

Horton ⇔ Tokunaga

Reducing Horton

Scaling relations

Fluctuations

Models Nutshell



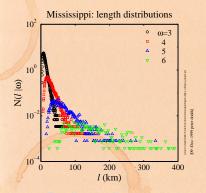
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for intermediate orders

All moments grow exponentially with order

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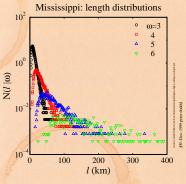
Nutshell

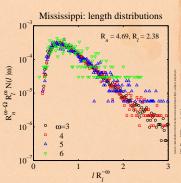




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Scaling collapse works well for intermediate orders

Branching Networks II

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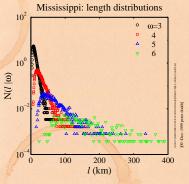


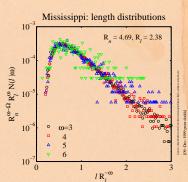




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$$ullet$$
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- Scaling collapse works well for intermediate orders
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Branching Networks II

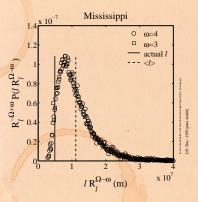
Fluctuations

Nutshell





How well does overall basin fit internal pattern?



- ➤ Actual length = 4920 kr (at 1 km res)
- ► Predicted Mean length = 11100 km
- ► Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay

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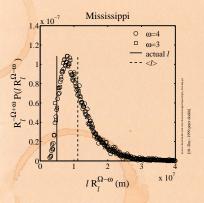
Models

Nutshell





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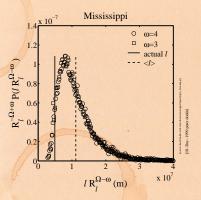
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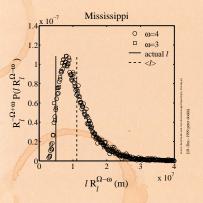
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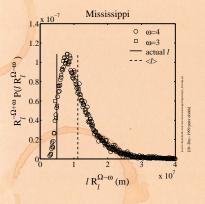
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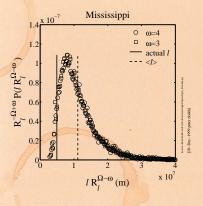
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Fluctuations

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Nutshell





hasin:

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10³ km):

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Daoiii.	~77	~72	\circ_{ℓ}	~/~22	0 (/ 252
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73
		-		/-	,-
	а	$ar{a}_\Omega$	σ_{a}	$a/ar{a}_\Omega$	$\sigma_{\pmb{a}}/ar{\pmb{a}}_{\pmb{\Omega}}$
Mississippi	2.74	a_{Ω} 7.55	σ_a 5.58	a/a_{Ω} 0.36	σ_a/a_{Ω} 0.74
Mississippi Amazon				,	'
• •	2.74	7.55	5.58	0.36	0.74
Amazon	2.74 5.40	7.55 9.07	5.58 8.04	0.36	0.74 0.89
Amazon Nile	2.74 5.40 3.08	7.55 9.07 0.96	5.58 8.04 0.79	0.36 0.60 3.19	0.74 0.89 0.82

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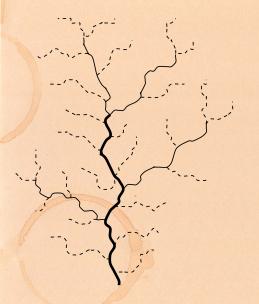
Nutshell







Combining stream segments distributions:



Stream segments sum to give main stream lengths

 $\ell_{\omega} = \sum_{j=1}^{\mu=\omega} s_{\mu}$

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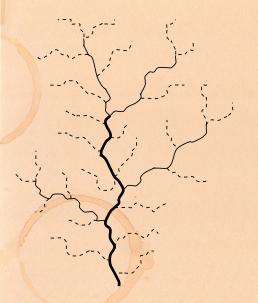
Nutshell







Combining stream segments distributions:



Stream segments sum to give main stream lengths

 $\ell_\omega = \sum s_\mu$

 $\triangleright P(\ell_{\omega})$ is a convolution of distributions for the s_{ω}

Branching Networks II

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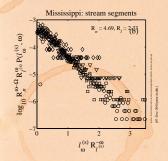






Sum of variables $\ell_{\omega} = \sum_{\mu=1}^{\mu=\omega} s_{\mu}$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



 $N(s|\omega) = \frac{1}{R_n^2 R_n^2} F(s/R_n^2)$ $F(x) = e^{-x/8}$

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Fluctuations

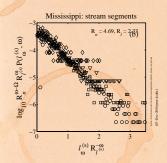
Models Nutshell





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$$N(s|\omega) = \frac{1}{R_n^{\omega} R_{\ell}^{\omega}} F(s/R_{\ell}^{\omega})$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

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Scaling relation

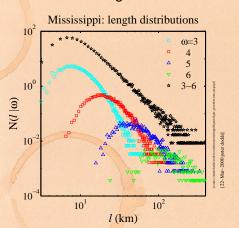
Fluctuations

Models Nutshell









 $ightharpoonup P(\ell) \sim \ell^{-\gamma}$

Scaling relations **Fluctuations**

Models

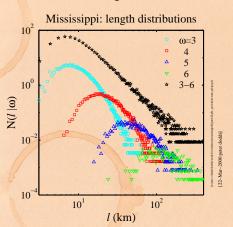
Nutshell







Next level up: Main stream length distributions must combine to give overall distribution for stream length



 $ightharpoonup P(\ell) \sim \ell^{-\gamma}$

- Another round of convolutions [3]
- Interesting...

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Reducing Horton Scaling relations

Fluctuations

Models

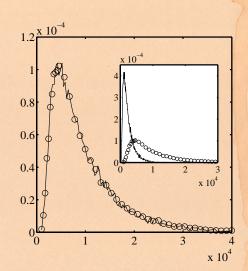
Nutshell







Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$.



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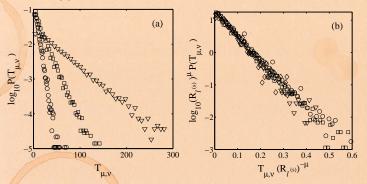
Nutshell







Scheidegger:



- ▶ Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

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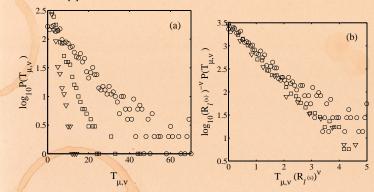
Nutshell







Mississippi:



Same data collapse for Mississippi...

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So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

$$P(s_{\mu}) \Leftrightarrow P(T_{\mu,\nu})$$

- Exponentials arise from randomness.
- Look at joint probability $P(s_{\mu}, T_{\mu,\nu})$.

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Models Nutshell





Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



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Models Nutshell





Follow streams segments down stream from their beginning

Probability (or rate) of an order μ stream segmenterminating is constant:

 $ilde{p}_n \simeq 1/(R_s)^{\mu-1} \xi$

Probability decays exponentially with stream order

Inter-tributary lengths exponentially distributed

s random spatial distribution of stream segments

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Models Nutshell





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- → random spatial distribution of stream segments

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Deferences





Joint distribution for generalized version of Tokunaga's law:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} inom{s_{\mu} - 1}{T_{\mu, \nu}}
ho_{
u}^{T_{\mu,
u}} (1 - p_{
u} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu,
u} - 1}$$

where

• $p_{\nu}=$ probability of absorbing an order ν side stream

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Joint distribution for generalized version of Tokunaga's law:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} {s_{\mu} - 1 \choose T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$

where

- $p_{\nu} =$ probability of absorbing an order ν side stream
- $m{ ilde{
 ho}}_{\mu}=$ probability of an order μ stream terminating

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$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$

where

- p_{ν} = probability of absorbing an order ν side stream
- ightharpoonup $ilde{p}_{\mu}=$ probability of an order μ stream terminating
- Approximation: depends on distance units of s_{μ}
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

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Now deal with thing:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$

• Set $(x, y) = (s_{\mu}, T_{\mu, \nu})$ and $q = 1 - \rho_{\nu} - \tilde{p}$

approximate liberally

▶ Ohtail

 $P(x, y) = Nx^{-1/2} [F(y/x)]^{x}$

 $\begin{pmatrix} -(1-v) \begin{pmatrix} v \\ - \end{pmatrix} \end{pmatrix}$

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Nutshell





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Set $(x, y) = (s_{\mu}, T_{\mu, \nu})$ and $q = 1 - p_{\nu} - \tilde{p}_{\mu}$, approximate liberally.

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Now deal with thing:

$$P(s_{\mu}, T_{\mu, \nu}) = \tilde{p}_{\mu} \binom{s_{\mu} - 1}{T_{\mu, \nu}} p_{\nu}^{T_{\mu, \nu}} (1 - p_{\nu} - \tilde{p}_{\mu})^{s_{\mu} - T_{\mu, \nu} - 1}$$

- Set $(x, y) = (s_{\mu}, T_{\mu, \nu})$ and $q = 1 p_{\nu} \tilde{p}_{\mu}$, approximate liberally.
- ▶ Obtain

$$P(x,y) = Nx^{-1/2} [F(y/x)]^x$$

where

$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

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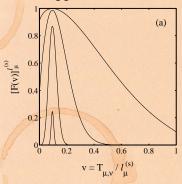
Models Nutshell

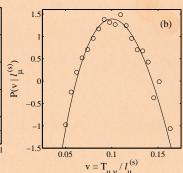




▶ Checking form of $P(s_{\mu}, T_{\mu,\nu})$ works:

Scheidegger:





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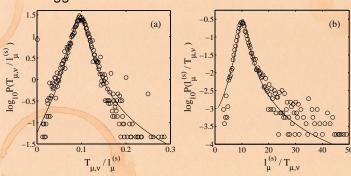






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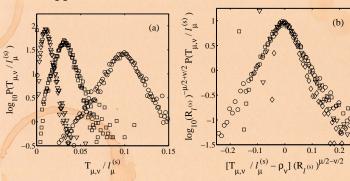
Nutshell





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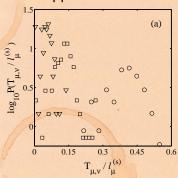
Nutshell

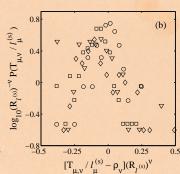




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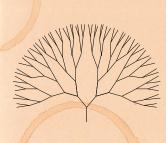
Models

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Random subnetworks on a Bethe lattice [13]



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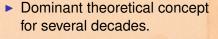
Nutshell







Random subnetworks on a Bethe lattice [13]



- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river networ statistics [7]
- But Bethe lattices unconnected with surfaces.
- ► In fact, Bethe lattices
 infinite dimensional spaces
 (oops)
- So let's move on.



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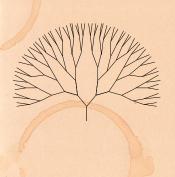
Models

Nutshell

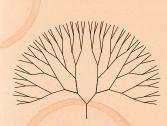








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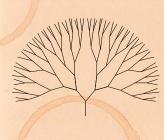
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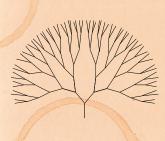
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So let's move on....

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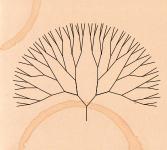
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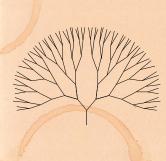
Models

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Directed random networks [11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$

[15, 16, 14]

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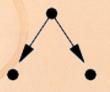
Models

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Directed random networks [11, 12]



$$P(\searrow) = P(\swarrow) = 1/2$$

 Functional form of all scaling laws exhibited but exponents differ from real world [15, 16, 14]

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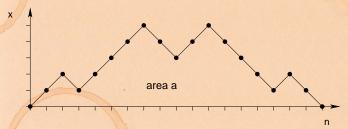




A toy model—Scheidegger's model

Random walk basins:

▶ Boundaries of basins are random walks



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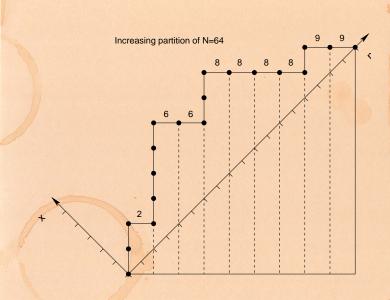
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Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

 $P(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2}$

and so $P(\ell) \propto \ell^{-3/2}$.

▶ Typical area for a walk of length n is $\propto n^{3/2}$

Find $\tau = 4/3$, h = 2/3, y = 3/2, d =

Note $\tau = 2 - h$ and $\gamma = 1/h$.

▶ R_n and R_p have not been derived analytically

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Rodríguez-Iturbe, Rinaldo, et al. [10]

Landscapes h(x) evolve such that energy dissipations is minimized, where

$$\dot{arepsilon} \propto \int \mathsf{d}ec{r} \; (\mathsf{flux}) imes (\mathsf{force}) \sim \sum_i a_i
abla h_i \sim \sum_i a_i^\gamma$$

- ▶ Landscapes obtained numerically give exponents near that of real networks.
- ▶ But: numerical method used matters
- And: Maritan et al. find basic universality classes ar that of Scheideguer self-similar, and a third kind of random rietwork [8]

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Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5-0.7	1.0–1.2

 $h \Rightarrow \ell \propto a^h$ (Hack's law). $d \Rightarrow \ell \propto L_{||}^d$ (stream self-affinity).

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Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.
- hib. for 2-d networks, these laws are 'planform' laws
 - Abundant scaling relations can be derived.
- ▶ Can take H_n, H_r, and o as three independent parameters necessary to describe all 2-d branching parameters.
- For scaling laws, only $h = \ln R_b / \ln R_h$ and d are needed.
- ▶ Laws can be extended nicely to laws of distributions
- Numérous models of branching network evolution exist: nothing rock solid vet.

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Models Nutshell





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