Branching Networks I

Complex Networks
CSYS/MATH 303, Spring, 2011

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Introduction

Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

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The University of Vermont

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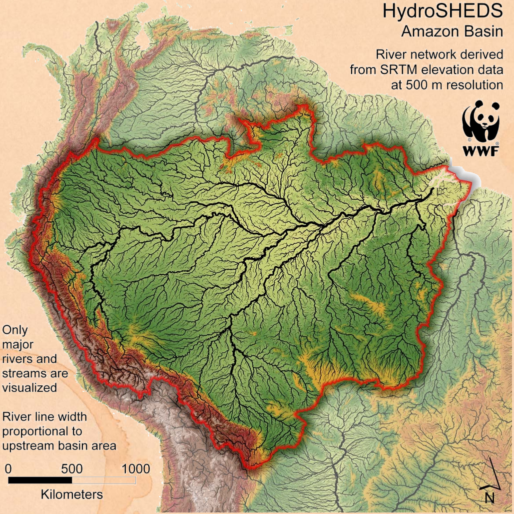

http://hydrosheds.cr.usgs.gov/ (⊞)

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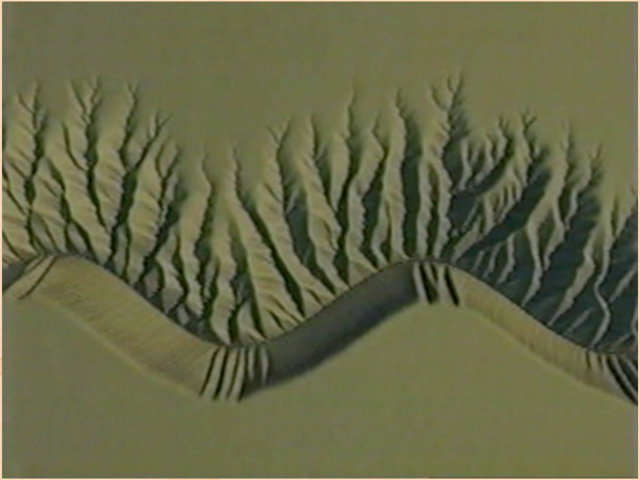
http://en.wikipedia.org/wiki/Image:Applebox.JPG (⊞)

A beautiful simulation of erosion:



Bruce Shaw (LDEO, Columbia) and Marcelo Magnasco (Rockefeller)

Geomorphological networks

Definitions

- Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
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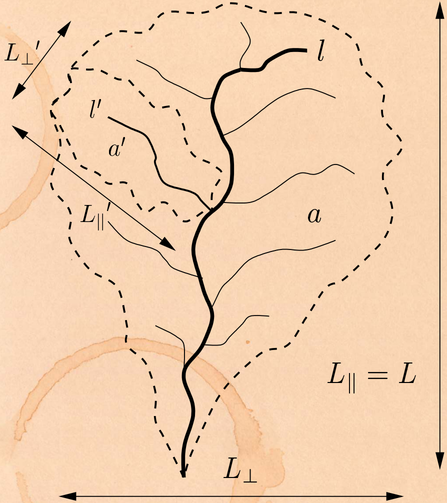
Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:



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- ℓ = length of longest (main) stream (which may be fractal)
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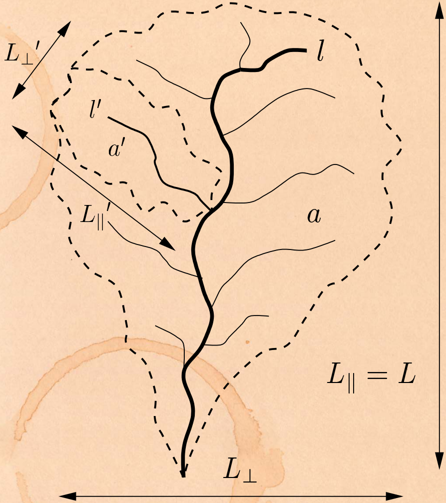

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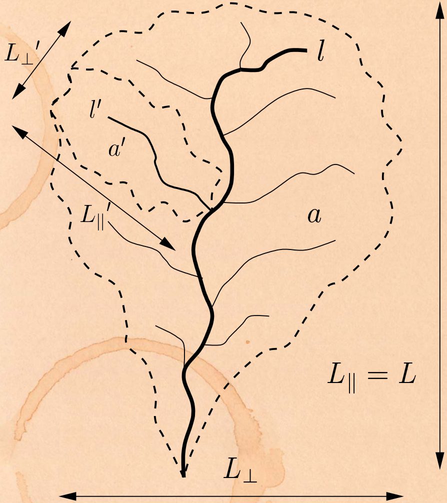

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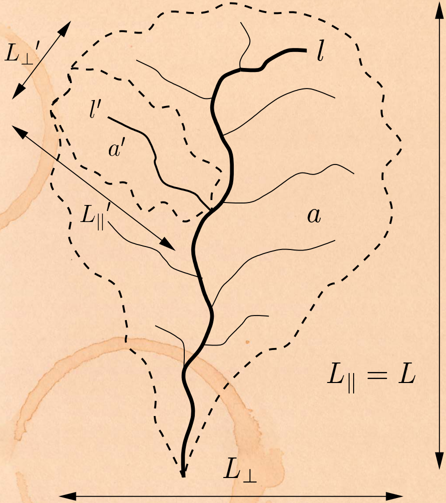

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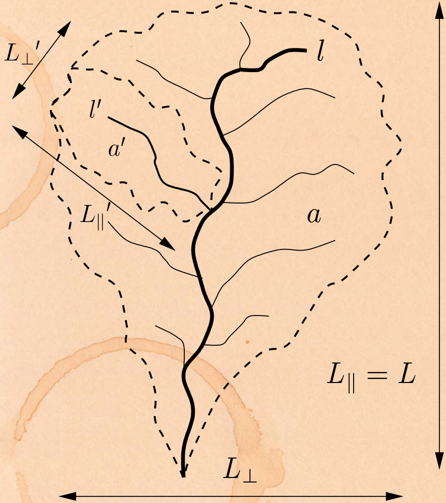

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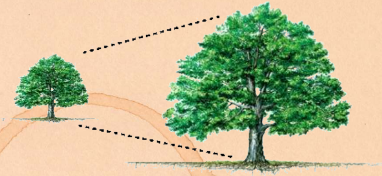
  dimensions scale linearly with each other.



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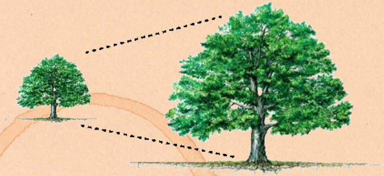

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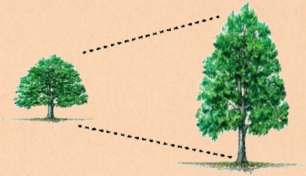

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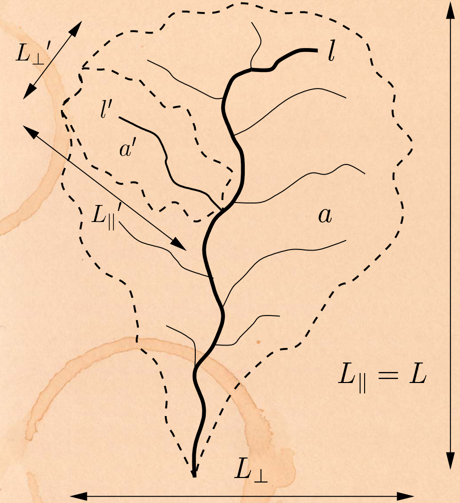

Allometric relationships:

- $$\ell \propto a^h$$
- $$\ell \propto L^d$$
- Combine above:
$$a \propto L^{d/h} \equiv L^D$$

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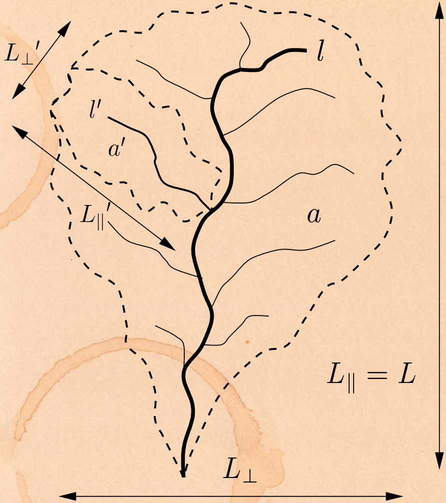

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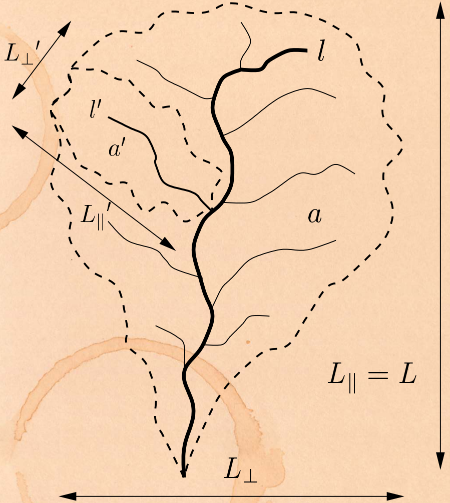

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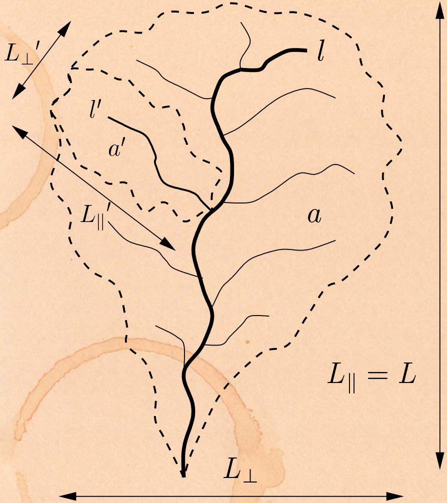

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‘Laws’

- Hack’s law (1957) [2]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$

- Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^{d}$$

reportedly $1.0 < d < 1.1$

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$D < 2 \rightarrow$ basins elongate.

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There are a few more 'laws':[1]

Relation:	Name or description:
$T_k = T_1(R_T)^k$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law

Reported parameter values: [1]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05

Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

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For (3): Many attempts: not yet sorted out...

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Method for describing network architecture:

- Introduced by Horton (1945) [3]
- Modified by Strahler (1957) [6]
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Stream Ordering:

Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.

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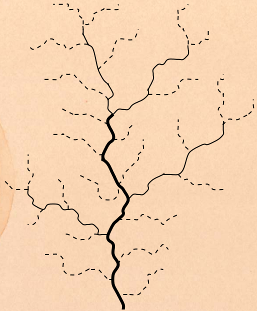

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2. Label all new source streams as order $\omega = 2$ and remove.
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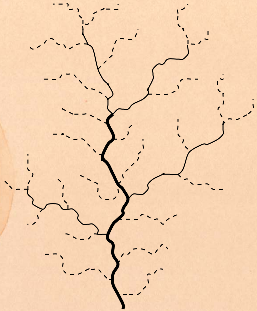

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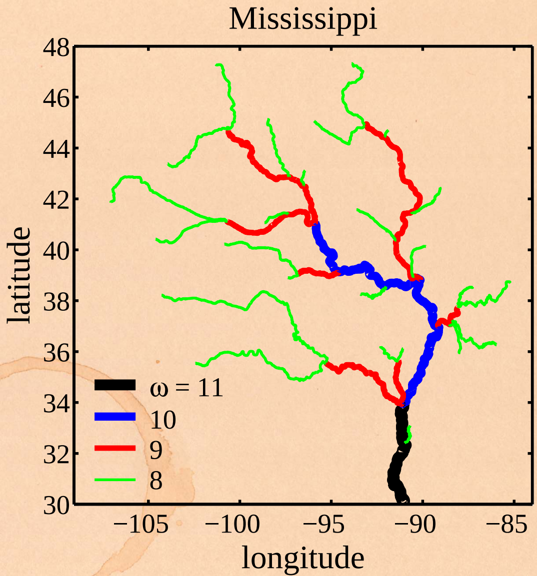
Stream Ordering—A large example:



[source=/data6/dodds/work/rivers/dems/mississippi/figures/figorder_paths_mispi10.ps]

[21−Mar−2000 peter dodds]

Stream Ordering:

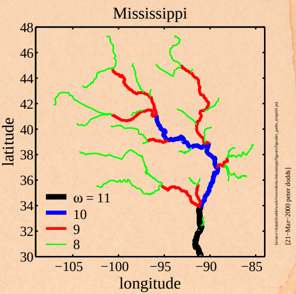
Another way to define ordering:

- As before, label all source streams as order $\omega = 1$.
- Follow all labelled streams downstream
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.



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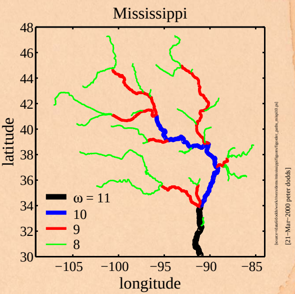

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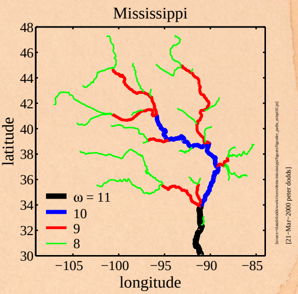

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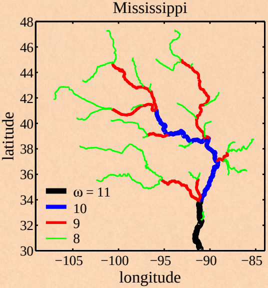

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Another way to define ordering:

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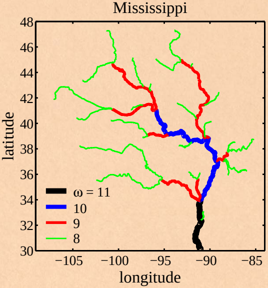

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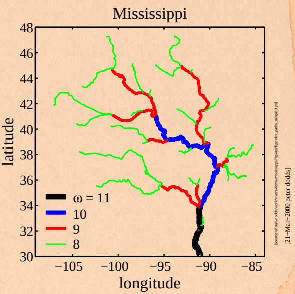

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One problem:

- Resolution of data messes with ordering
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Resultant definitions:

- A basin of order Ω has n_ω streams (or sub-basins) of order ω.
  - $n_\omega > n_{\omega+1}$
- An order ω basin has area a_ω.
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Self-similarity of river networks

- First quantified by Horton (1945) [3], expanded by Schumm (1956) [5]

Three laws:

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- Horton's law of stream segment lengths:

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- Can show that $R_s \equiv R_\ell$.
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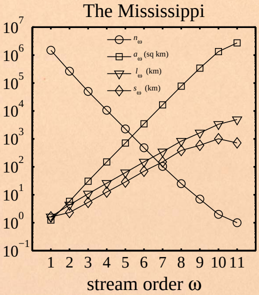

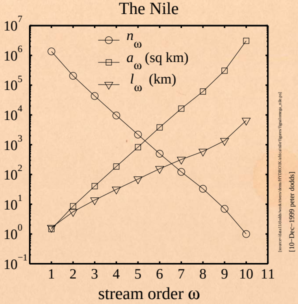

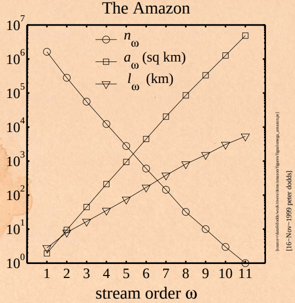

Horton's laws-at-large

Blood networks:

- Horton's laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
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Data from real blood networks

Network	R_n	R_r^{-1}	R_ℓ^{-1}	$-\frac{\ln R_r}{\ln R_n}$	$-\frac{\ln R_\ell}{\ln R_n}$	α
West *et al.*	–	–	–	1/2	1/3	3/4
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73
cat (PAT) (Turcotte *et al.* [10])	3.67	1.71	1.78	0.41	0.44	0.79
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90
pig (LCX)	3.57	1.89	2.20	0.50	0.62	0.62
pig (RCA)	3.50	1.81	2.12	0.47	0.60	0.65
pig (LAD)	3.51	1.84	2.02	0.49	0.56	0.65
human (PAT)	3.03	1.60	1.49	0.42	0.36	0.83
human (PAT)	3.36	1.56	1.49	0.37	0.33	0.94

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Observations:

- Horton's ratios vary:

R_n	3.0–5.0
R_a	3.0–6.0
R_ℓ	1.5–3.0

- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...
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Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure [7, 8, 9]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
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Definition:

- $T_{\mu,\nu}$ = the average number of side streams of order ν that enter as tributaries to streams of order μ
- $\mu, \nu = 1, 2, 3, \ldots$
- $\mu \geq \nu + 1$
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- Property 1: Scale independence—depends only on difference between orders:

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$$\boxed{T_k = T_1 (R_T)^{k-1}} \quad \text{where } R_T \simeq 2$$

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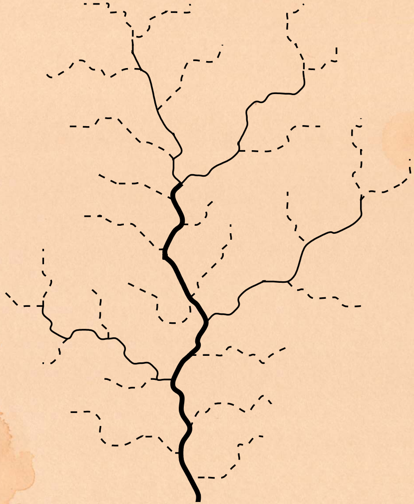
Tokunaga's law—an example:

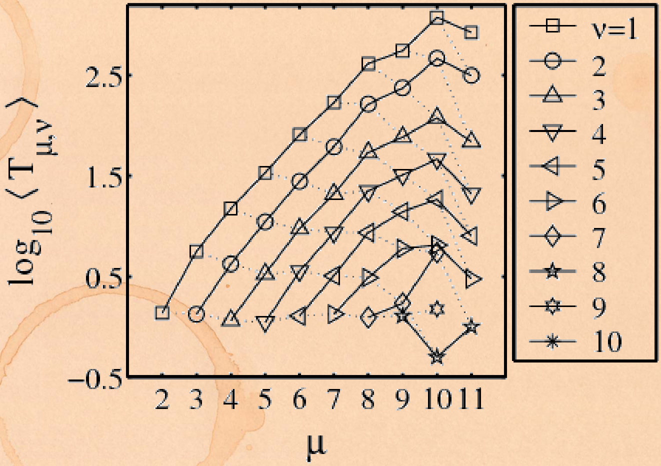
$T_1 \simeq 2$

$R_T \simeq 4$



The Mississippi

A Tokunaga graph:



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- Show remarkable self-similarity over many scales.
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References I

[1] P. S. Dodds and D. H. Rothman.
Unified view of scaling laws for river networks.
Physical Review E, 59(5):4865–4877, 1999. pdf (⊞)

[2] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957.

[3] R. E. Horton.
Erosional development of streams and their drainage basins; hydrophysical approach to quatitative morphology.
Bulletin of the Geological Society of America, 56(3):275–370, 1945.

References II

[4] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.

[5] S. A. Schumm.
Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersey.
Bulletin of the Geological Society of America, 67:597–646, 1956.

[6] A. N. Strahler.
Hypsometric (area altitude) analysis of erosional topography.
Bulletin of the Geological Society of America, 63:1117–1142, 1952.

References III

[7] E. Tokunaga.
The composition of drainage network in Toyohira River Basin and the valuation of Horton's first law.
Geophysical Bulletin of Hokkaido University, 15:1–19, 1966.

[8] E. Tokunaga.
Consideration on the composition of drainage networks and their evolution.
Geographical Reports of Tokyo Metropolitan University, 13:G1–27, 1978.

[9] E. Tokunaga.
Ordering of divide segments and law of divide segment numbers.
Transactions of the Japanese Geomorphological Union, 5(2):71–77, 1984.

References IV

[10] D. L. Turcotte, J. D. Pelletier, and W. I. Newman.
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998.
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