# Power Law Size Distributions

## Principles of Complex Systems
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Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont







# Outline



Power Law Size Distributions

Overview
Introduction
Examples
Wild vs. Mild
CCDFs
Zipf's law
Zipf ⇔ CCDF
References



Add section on stable distributions
Add an assignment question or two
convolve distributions
Cauchy

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Inverse Gaussian

# Size distributions—Assignment 1 recap:

The sizes of many systems' elements appear to obey an inverse power-law size distribution:

$$P(\text{size} = x) \sim c\, x^{-\gamma}$$

$$\text{where} \quad x_{\min} < x < x_{\max}$$

$$\text{and} \quad \gamma > 1$$

- $x_{\min}$ = lower cutoff
- $x_{\max}$ = upper cutoff
- Negative linear relationship in log-log space:

$$\log P(x) = \log c - \gamma \log x$$

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# Size distributions

- Usually, only the tail of the distribution obeys a power law:

$$P(x) \sim c\, x^{-\gamma} \text{ for } x \text{ large.}$$

- Still use term 'power law distribution'

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# Size distributions

Many systems have discrete sizes $k$:

- Word frequency
- Node degree (as we have seen): # hyperlinks, etc.
- number of citations for articles, court decisions, etc.

$$P(k) \sim c\, k^{-\gamma}$$

$$\text{where} \quad k_{\min} \le k \le k_{\max}$$

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# Size distributions



Power law size distributions are sometimes called Pareto distributions (⊞) after Italian scholar Vilfredo Pareto.

- Pareto noted wealth in Italy was distributed unevenly (80–20 rule).
- Term used especially by economists



# Devilish power law distribution details:



## From assignment 1, we know many nasty things.

Exhibit A:
Given $P(x) = cx^{-\gamma}$ with $0 < x_{\rm min} < x < x_{\rm max}$, the mean is:

$$\langle x \rangle = \frac{c}{2-\gamma}\left(x_{\rm max}^{2-\gamma} - x_{\rm min}^{2-\gamma}\right).$$

- Mean 'blows up' with upper cutoff if $\gamma < 2$.
- Mean depends on lower cutoff if $\gamma > 2$.
- $\gamma < 2$: Typical sample is large.
- $\gamma > 2$: Typical sample is small.



# And in general...



## Moments:

- All moments depend only on cutoffs.
- No internal scale that dominates/matters.
- Compare to a Gaussian, exponential, etc.

## For many real size distributions: $2 < \gamma < 3$

- mean is finite (depends on lower cutoff)
- $\sigma^2$ = variance is 'infinite' (depends on upper cutoff)
- Width of distribution is 'infinite'
- If $\gamma > 3$, distribution is less terrifying and may be easily confused with other kinds of distributions.



# Moments



## Standard deviation is a mathematical convenience!:

- Variance is nice analytically...
- Another measure of distribution width:

$$\text{Mean average deviation (MAD)} = \langle |x - \langle x \rangle| \rangle$$

- For a pure power law with $2 < \gamma < 3$:

$$\langle |x - \langle x \rangle| \rangle \text{ is finite.}$$

- But MAD is unpleasant analytically...
- We still speak of infinite 'width' if $\gamma < 3$.



# How sample sizes grow...



## Given $P(x) \sim cx^{-\gamma}$:

- We can show that after $n$ samples, we expect the largest sample to be

$$x_1 \gtrsim c' n^{1/(\gamma-1)}$$

- Sampling from a finite-variance distribution gives a much slower growth with $n$.
- e.g., for $P(x) = \lambda e^{-\lambda x}$, we find

$$x_1 \gtrsim \frac{1}{\lambda} \ln n.$$



# Size distributions



## Examples:

- Earthquake magnitude (Gutenberg Richter law): $P(M) \propto M^{-3}$
- Number of war deaths: $P(d) \propto d^{-1.8}$
- Sizes of forest fires
- Sizes of cities: $P(n) \propto n^{-2.1}$
- Number of links to and from websites

# Size distributions



## Examples:

- Number of citations to papers: $P(k) \propto k^{-3}$.
- Individual wealth (maybe): $P(W) \propto W^{-2}$.
- Distributions of tree trunk diameters: $P(d) \propto d^{-2}$.
- The gravitational force at a random point in the universe: $P(F) \propto F^{-5/2}$.
- Diameter of moon craters: $P(d) \propto d^{-3}$.
- Word frequency: e.g., $P(k) \propto k^{-2.2}$ (variable)

Note: Exponents range in error; see M.E.J. Newman arxiv.org/cond-mat/0412004v3 (⊞)



# Size distributions



## Power-law distributions are..

- often called 'heavy-tailed'
- or said to have 'fat tails'

## Important!:

- Inverse power laws aren't the only ones:
  - lognormals, stretched exponentials, ...



# Power law distributions



## Gaussians versus power-law distributions:

- Example: Height versus wealth.
- Mild versus Wild (Mandelbrot)
- Mediocristan versus Extremistan
  (See "The Black Swan" by Nassim Taleb [1])



# Turkeys...



FIGURE 1: ONE THOUSAND AND ONE DAYS OF HISTORY


A turkey before and after Thanksgiving. The history of a process over a thousand days tells you nothing about what is to happen next. This naïve projection of the future from the past can be applied to anything.

From "The Black Swan" [1]



# Taleb's table [1]



## Mediocristan/Extremistan

- Most typical member is mediocre/Most typical is either giant or tiny
- Winners get a small segment/Winner take almost all effects
- When you observe for a while, you know what's going on/It takes a very long time to figure out what's going on
- Prediction is easy/Prediction is hard
- History crawls/History makes jumps
- Tyranny of the collective/Tyranny of the rare and accidental



## Complementary Cumulative Distribution Function:



## CCDF:

- $$P_{\geq}(x) = P(x' \geq x) = 1 - P(x' < x)$$
- $$= \int_{x'=x}^{\infty} P(x')\mathrm{d}x'$$
- $$\propto \int_{x'=x}^{\infty} (x')^{-\gamma}\mathrm{d}x'$$
- $$= \frac{1}{-\gamma+1}(x')^{-\gamma+1}\Big|_{x'=x}^{\infty}$$
- $$\propto x^{-\gamma+1}$$

## Complementary Cumulative Distribution Function:

### CCDF:

- $$P_{\geq}(x) \propto x^{-\gamma+1}$$
- Use when tail of $P$ follows a power law.
- Increases exponent by one.
- Useful in cleaning up data.





## Complementary Cumulative Distribution Function:

- Discrete variables:

$$P_{\geq}(k) = P(k' \geq k)$$

$$= \sum_{k'=k}^{\infty} P(k)$$

$$\propto k^{-\gamma+1}$$

- Use integrals to approximate sums.





## Zipfian rank-frequency plots

### George Kingsley Zipf:

- Noted various rank distributions followed power laws, often with exponent -1 (word frequency, city sizes...)
- Zipf's Magnum Opus: "Human Behaviour and the Principle of Least-Effort" [2] Addison-Wesley, Cambridge MA, 1949.
- We'll study Zipf's law in depth...





## Zipfian rank-frequency plots

### Zipf's way:

- $s_r$ = the size of the $r$th ranked object.
- $r = 1$ corresponds to the largest size.
- Example: $s_1$ could be the frequency of occurrence of the most common word in a text.
- Zipf's observation:

$$s_r \propto r^{-\alpha}$$

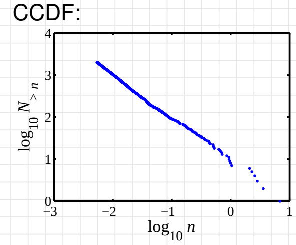




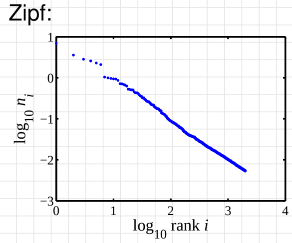
## Size distributions

### Brown Corpus (1,015,945 words):

CCDF:




Zipf:




- The, of, and, to, a, ... = 'objects'
- 'Size' = word frequency
- Beep: CCDF and Zipf plots are related...





## Size distributions

### Brown Corpus (1,015,945 words):

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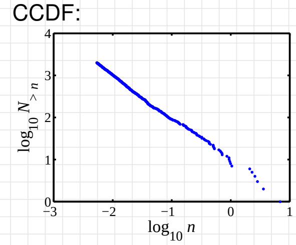
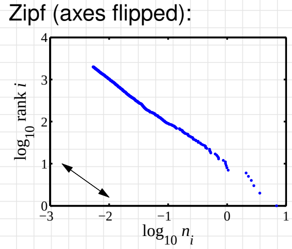



Zipf (axes flipped):




- The, of, and, to, a, ... = 'objects'
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# Size distributions



## Observe:

- $NP_{\geq}(x) =$ the number of objects with size at least $x$ where $N$ = total number of objects.
- If an object has size $x_r$, then $NP_{\geq}(x_r)$ is its rank $r$.
- So

$$x_r \propto r^{-\alpha} = (NP_{\geq}(x_r))^{-\alpha}$$

$$\propto x_r^{(-\gamma+1)(-\alpha)}$$

Since $P_{\geq}(x) \sim x^{-\gamma+1}$,

$$\boxed{\alpha = \frac{1}{\gamma - 1}}$$

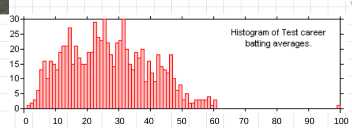
- A rank distribution exponent of $\alpha = 1$ corresponds to a size distribution exponent $\gamma = 2$.



# The Don



Extreme deviations in test cricket




Don Bradman's batting average = 166% next best.



# References I



[1] N. N. Taleb.
The Black Swan.
Random House, New York, 2007.

[2] G. K. Zipf.
Human Behaviour and the Principle of Least-Effort.
Addison-Wesley, Cambridge, MA, 1949.