Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
CSYS/MATH 300, Fall, 2010

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Basics
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PLIPLO
Growth Mechanisms
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Mechanisms

A powerful story in the rise of complexity:

- structure arises out of randomness.

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- Exhibit A: Random walks... (⊞)

Random walks

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

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Variances sum: (⊞)*

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$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

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⇒ A non-trivial power-law arises out of additive aggregation or accumulation.

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$\Rightarrow$ A non-trivial power-law arises out of additive aggregation or accumulation.

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Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
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See Feller, [3] Intro to Probability Theory, Volume I

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In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

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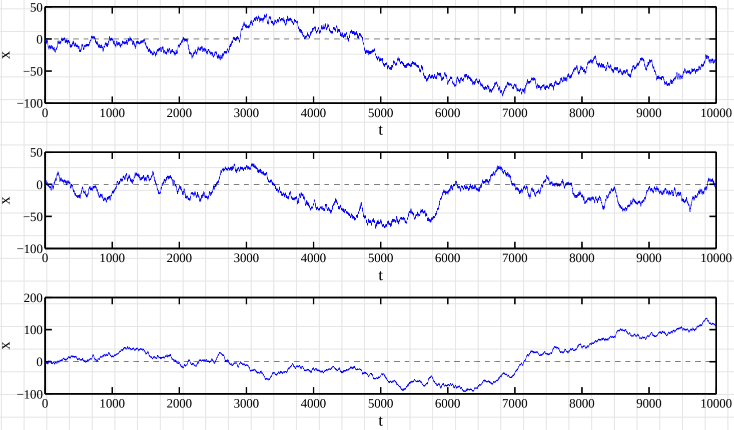
Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples



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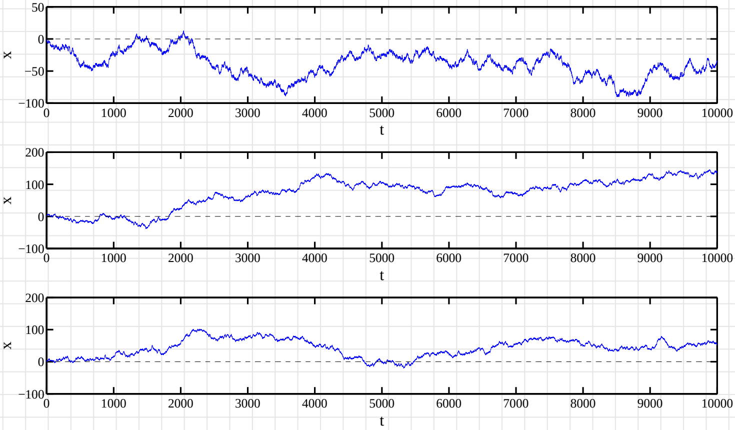

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The problem of first return:

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- Will our drunkard always return to the origin?
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Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

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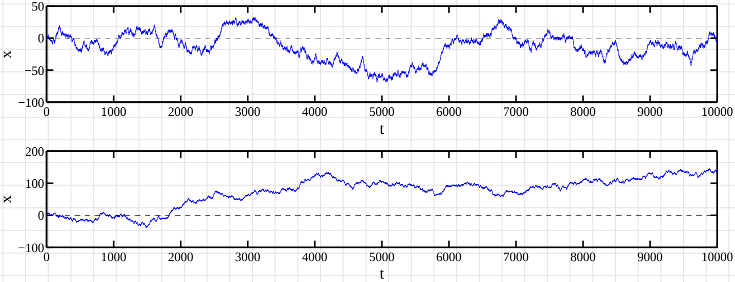

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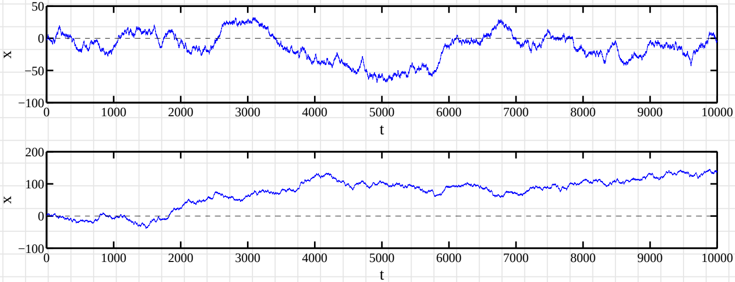
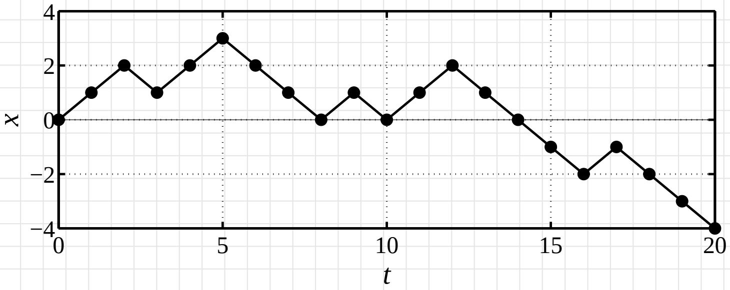


Again: expected time between ties = ∞...
Let's find out why... [3]

First Returns



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- The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

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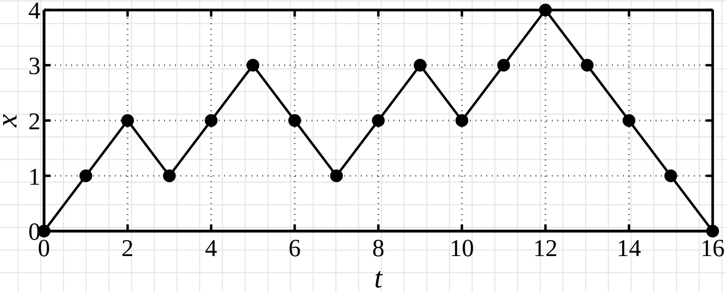

- A useful restatement: $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

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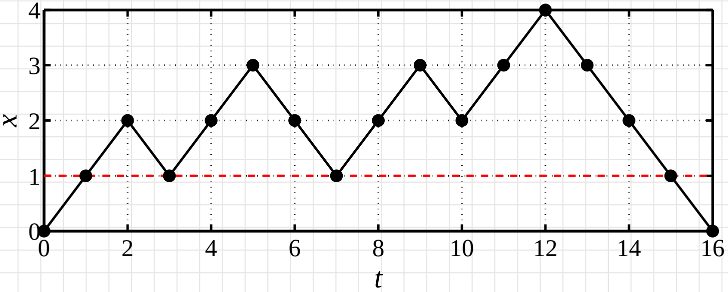

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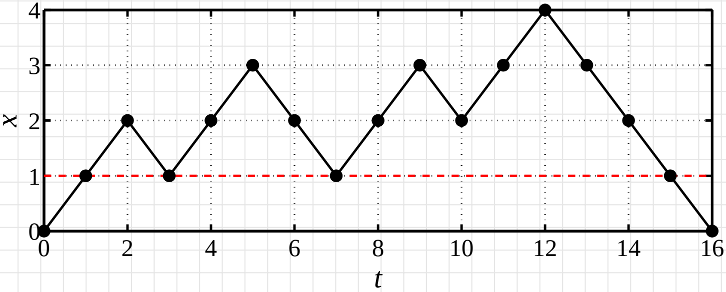

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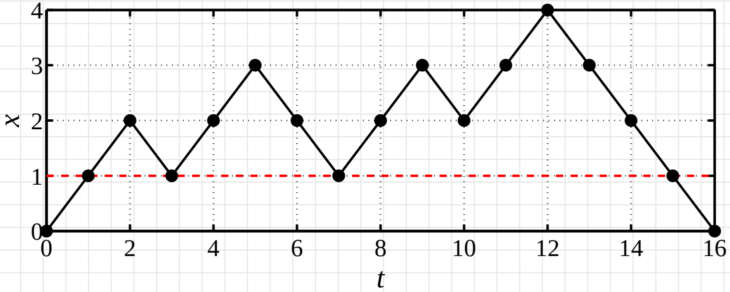

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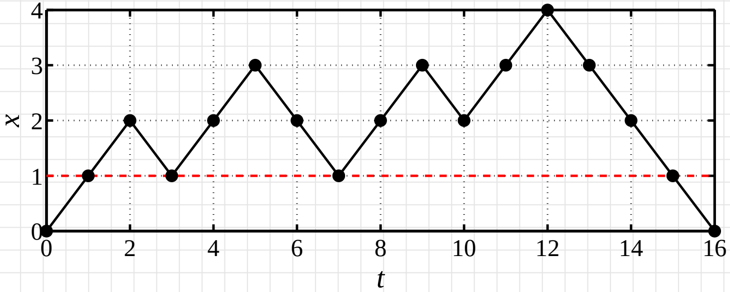

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- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
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See this 1-1 correspondence visually...

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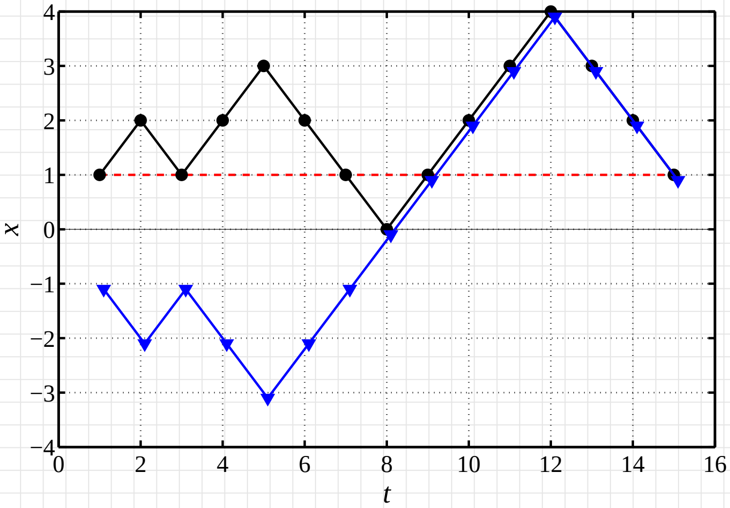

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- For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
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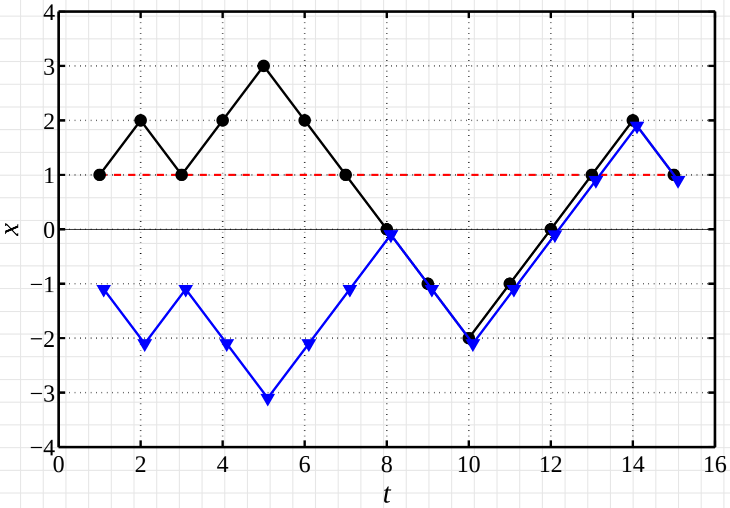

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- Next problem: what is $N(i, j, t)$?
- # positive steps + # negative steps = t.
- Random walk must displace by $j - i$ after t steps.
- # positive steps - # negative steps = $j - i$.
- # positive steps = $(t + j - i)/2$.
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We now have

$$N_{\text{first return}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$$

where

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First Returns

Insert question from assignment 4 (⊞)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths $= 2^{2n}$
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$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

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- Same scaling holds for continuous space/time walks.
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$$P(t) \propto t^{-3/2}, \ \gamma = 3/2$$
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Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

First Returns

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On finite spaces:

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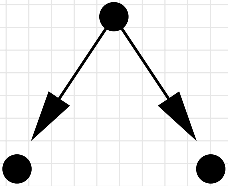
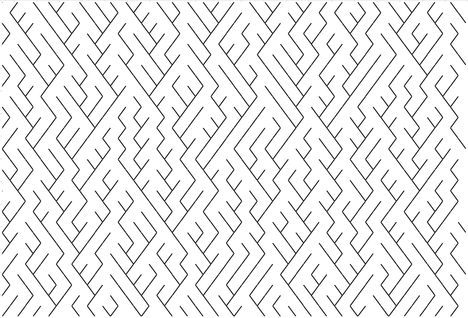
Outline

Scheidegger Networks [11, 2]





- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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- x_0 could be > 0.
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Understand power laws as arising from

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- Random variable X with known distribution P_x
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- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$
  $= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$

Figure...

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Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

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$$P_y(y)\mathrm{d}y = P_x \overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)} \overbrace{\frac{c^{1/\alpha}}{\alpha} y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- If $P_x(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If $P_x(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

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- Exponentials arise from randomness...
- More later when we cover robustness.

Outline

Gravity

- Select a random point in the universe $\vec{x}$
- (possible all of space-time)
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

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Ingredients [13]

Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

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- $$dF \propto d(r^{-2})$$
- $$\propto r^{-3} dr$$
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Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)...
- Don't do this!!! (slap, slap)
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Outline

Aggregation

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
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Work of Yore

- 1924: G. Udny Yule [14]:
  # Species per Genus
- 1926: Lotka [6]:
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- 1953: Mandelbrot [8]:
  Optimality argument for Zipf's law; focus on language.
- 1955: Herbert Simon [12, 15]:
  Zipf's law for word frequency, city size, income, publications, and species per genus.
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Examples

Evidence for Zipf's law...

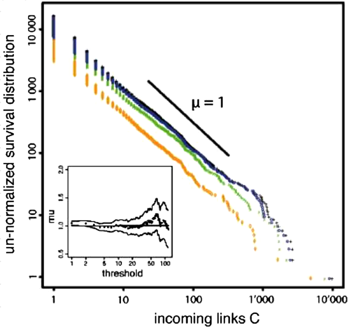

FIG. 1 (color online). (Color Online) Log-log plot of the number of packages in four Debian Linux Distributions with more than C in-directed links. The four Debian Linux Distributions are Woody (19.07.2002) (orange diamonds), Sarge (06.06.2005) (green crosses), Etch (15.08.2007) (blue circles), Lenny (15.12.2007) (black+'s). The inset shows the maximum likelihood estimate (MLE) of the exponent μ together with two boundaries defining its 95% confidence interval (approximately given by $1 \pm 2/\sqrt{n}$, where n is the number of data points using in the MLE), as a function of the lower threshold. The MLE has been modified from the standard Hill estimator to take into account the discreteness of C.

Maillart et al., PRL, 2008:
"Empirical Tests of Zipf's Law Mechanism in Open Source Linux Distribution" [7]

Essential Extract of a Growth Model

Random Competitive Replication (RCR):

1. Start with 1 element of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
    - With probability ρ, create a new element with a new flavor
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    - Elements of the same flavor form a group

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Random Competitive Replication

Example: Words in a text

- Consider words as they appear sequentially.
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- Competition for growth between groups is not random
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     (But mechanism for selection is not as simple...)

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Definitions:

- k_i = size of a group i
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Basic question: How does $N_k(t)$ evolve with time?

First: $\sum_k kN_k(t) = t =$ number of elements at time t

Random Competitive Replication

$P_k(t)$ = Probability of choosing an element that belongs to a group of size k:

- $N_k(t)$ size k groups
- $\Rightarrow kN_k(t)$ elements in size k groups
- t elements overall

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$$P_k(t) = \frac{kN_k(t)}{t}$$

Random Competitive Replication

$N_k(t)$, the number of groups with k elements, changes at time t if

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2. An element belonging to a group with $k-1$ elements is replicated

Random Competitive Replication

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Special case for $N_1(t)$:

1. The new element is a new flavor:

2. A unique element is replicated.

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Random Competitive Replication

Put everything together:

For $k > 1$:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho)\left((k-1)\frac{N_{k-1}(t)}{t} - k\frac{N_k(t)}{t} \right)$$

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For $k = 1$:

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$

Random Competitive Replication

Assume distribution stabilizes: $N_k(t) = n_k t$

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional
- Okay over large time scales
- n_k/ρ = the fraction of groups that have size k.

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Stochastic difference equation:

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$$\Rightarrow n_k\left(1+(1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$

Random Competitive Replication

We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in k large (the tail of the distribution)
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  Insert question from assignment 4 (⊞)
- To get at tail: Expand as a series of powers of $1/k$

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$$\frac{n_k}{n_{k-1}} \simeq \left(1 - \frac{1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

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Random Competitive Replication

$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- Observe $2 < \gamma < \infty$ as ρ varies.
- For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- Recalls Zipf's law: $s_r \sim r^{-\alpha}$
  (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma - 1)$
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We had one other equation:

- $$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho) 1 \cdot \frac{N_1(t)}{t}$$
- As before, set $N_1(t) = n_1 t$ and drop expectations
- $$n_1(t+1) - n_1 t = \rho - (1-\rho) 1 \cdot \frac{n_1 t}{t}$$
- $$n_1 = \rho - (1-\rho) n_1$$
- Rearrange:
$$n_1 + (1-\rho) n_1 = \rho$$
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So... $$N_1(t) = n_1 t = \frac{\rho t}{2-\rho}$$

- Recall number of distinct elements $= \rho t$.
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2-\rho}$$

(also = fraction of groups of size 1)

- For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ρ increases, fraction increases
- Can show fraction of groups with two elements $\sim 1/6$
- Model does well at both ends of the distribution

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Outline

Words

From Simon [12]:

Estimate $\rho_{\text{est}} = \#\text{ unique words}/\#\text{ all words}$

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For Joyce's Ulysses: $\rho_{\text{est}} \simeq 0.115$

N_1 (real)	N_1 (est)	N_2 (real)	N_2 (est)
16,432	15,850	4,776	4,870

Evolution of catch phrases

- Yule's paper (1924) [14]:
  "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) [12]:
  "On a class of skew distribution functions" (snore)

From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only be a similarity in the structure of the underlying probability mechanisms.

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And just to be clear...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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