# Lognormals and friends

## Principles of Complex Systems
## CSYS/MATH 300, Fall, 2010


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# Outline

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References



# Outline

# Alternative distributions

## There are other ‘heavy-tailed’ distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞)

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

CCDF = stretched exponential (⊞).

3. Gamma distributions (⊞), and more.





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# lognormals

## The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean $\mu$ and variance $\sigma$.
- Appears in economics and biology where growth increments are distributed normally.





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# lognormals

- Standard form reveals the mean $\mu$ and variance $\sigma^2$ of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.





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# Derivation from a normal distribution

## Take $Y$ as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{d}y \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

## Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)\mathrm{d}x$$





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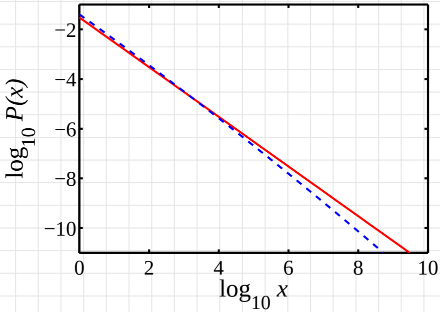
# Confusion between lognormals and pure power laws




Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.



# Confusion

## What's happening:

- $$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left( -\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$
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- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- This happens when (roughly)
- $$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$
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# Generating lognormals:

## Random multiplicative growth:

- $$x_{n+1} = r x_n$$
  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:
  $$\ln x_{n+1} = \ln r + \ln x_n$$
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# Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).
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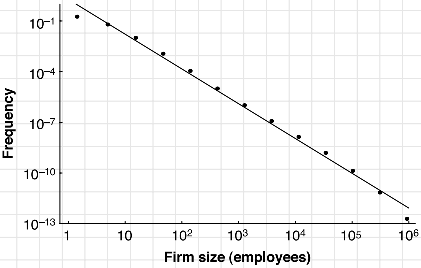


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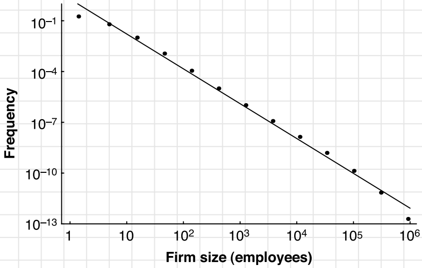


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# An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 1$
- The set up: $N$ entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where $r$ is drawn from some happy distribution
- Same as for lognormal but one extra piece.
- Each $x_i$ cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$



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- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$
- where $\gamma$ is implicitly given by

$$N = \frac{(\gamma-2)}{(\gamma-1)} \left[ \frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

$N$ = total number of firms.

- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma-2)}{(\gamma-1)} \left[ \frac{-1}{-(c/N)} \right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$
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$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

$N$ = total number of firms.

- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{-1}{-(c/N)} \right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1 - c}$$
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# An explanation

## Some math later... Insert question from assignment 6 (⊞)

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# Outline

## Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

## References





# The second tweak

## Ages of firms/people/... may not be the same

- Allow the number of updates for each size $x_i$ to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where $t$ = age.
- Back to no bottom limit: each $x_i$ follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.





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- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...





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