Principles of Complex Systems, CSYS/MATH 300—Assignment 4
University of Vermont, Fall 2010

Dispersed: Thursday, October 14, 2010.
Due: By start of lecture, 1:00 pm, Thursday, October 21, 2010.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 1:00 pm to 4:00 pm, Wednesday
Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2010-08UVM-300`

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. *Discrete random walks:*

   In class, we argued that the number of random walks returning to the origin for the first time after $2n$ time steps is given by

   $$N_{\textrm{first return}}(2n) = N_{\textrm{fr}}(2n) = N(1,1,2n-2) - N(-1,1,2n-2)$$

   where

   $$N(i,j,t) = \binom{t}{(t+j-i)/2}.$$

   Find the leading order term for $N_{\textrm{fr}}(2n)$ as $n \to \infty$.

   Hint: combine the terms and use Stirling's sterling approximation [1, 2].

   (If you enjoy this sort of thing, you may like to explore the same problem for random walks in higher dimensions. George Pólya is your man.)

   (And we are connecting to some good stuff in combinatorics; more to come in the solutions.)

2. *Simon's model I:*

   For Herbert Simon's model of what we've called Random Competitive Replication, we found in class that the normalized number of groups in the long time limit, n_k, satisfies the following difference equation:

   $$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \qquad (1)$$

where $k \geq 2$. The model parameter ρ is the probability that a newly arriving node forms a group of its own (or is a novel word, starts a new city, has a unique flavor, etc.). For $k = 1$, we have instead

$$n_1 = \rho - (1 - \rho)n_1 \quad (2)$$

which directly gives us n_1 in terms of ρ.

Derive the exact solution for n_k in terms of gamma functions and ultimately the beta function. From this exact form, determine the large k behavior for n_k ($\sim k^{-\gamma}$) and identify the exponent γ in terms of ρ.

3. *Simon's model II:*

   (a) A missing piece from the lectures: Obtain γ in terms of ρ by expanding Eq. 1 in terms of $1/k$. In the end, you will need to express n_k/n_{k-1} as $(1 - 1/k)^{\theta}$; from here, you will be able to identify γ. Taylor expansions and Procrustean truncations will be in order.

   This (dirty) method avoids finding the exact form for n_k.

   (b) What happens to γ in the limits $\rho \to 0$ and $\rho \to 1$? Explain in a sentence or two what's going on in these cases and how the specific limiting value of γ makes sense.

References

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions*. Dover Publications, New York, 1974.

[2] I. Gradshteyn and I. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, San Diego, fifth edition, 1994.