Lecture 1/25—Chapter 2

Linear Algebra
MATH 124, Fall, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Outline

Importance

Usages

Key problems

Three ways of looking...

Colbert on Equations

References

Basics:

- Instructor: Prof. Peter Dodds
- Lecture room and meeting times:
  209 Votey Hall, Tuesday and Thursday, 10:00 am to 11:15 am
- Office: Farrell Hall, second floor, Trinity Campus
- E-mail: peter.dodds@uvm.edu
- Course website: http://www.uvm.edu/~pdodds/teaching/courses/2010-08UVM-124 (⊞)
- Textbook: "Introduction to Linear Algebra" (4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press). The 3rd edition is okay too.

Admin:

Paper products:

1. Outline

Papers to read:

1. “The Fundamental Theorem of Linear Algebra” [1]
2. “Too Much Calculus” [2]

Office hours:

- 1:00 pm to 4:00 pm, Wednesday,
  Farrell Hall, second floor, Trinity Campus

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Grading breakdown:

1. Assignments (40%)
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.
2. Midterm exams (35%)
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.
3. Final exam (24%)
   - $\leq$ Three hours of joyful celebration.
   - Saturday, December 11, 7:30 am to 10:15 am, 209 Votey

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Grading breakdown:

1. Homework (0%)—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.
2. General attendance (1%)—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline.

How grading works:

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.

Schedule:

The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

Week # (dates)	**Tuesday**	**Thursday**
1 (8/31, 9/2)	Lecture	Lecture + A1
2 (9/7, 9/9)	Lecture	Lecture + A2
3 (9/14, 9/16)	Lecture	Lecture + A3
4 (9/21, 9/23)	Lecture	*Test 1*
5 (9/28, 9/30)	Lecture	Lecture + A4
6 (10/5, 10/7)	Lecture	Lecture + A5
7 (10/12, 10/14)	Lecture	Lecture + A6
8 (10/19, 10/21)	Lecture	*Test 2*
9 (10/26, 10/29)	Lecture	Lecture + A7
10 (11/2, 11/4)	Lecture	Lecture + A8
11 (11/9, 11/11)	Lecture	Lecture + A9
12 (11/16, 11/18)	Lecture	*Test 3*
13 (11/23, 11/25)	Thanksgiving	Thanksgiving
14 (11/30, 12/2)	Lecture	Lecture + A10
15 (12/7, 12/9)	Lecture	Lecture

Important dates:

1. Classes run from Monday, August 31 to Wednesday, December 9.
2. Add/Drop, Audit, Pass/No Pass deadline—Monday, September 14.
3. Last day to withdraw—Friday, November 6.
4. Reading and exam period—Thursday, December 10 to Friday, December 18.

More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).
2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See `http://www.uvm.edu/cses/` for guidelines.

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Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Students are encouraged to use Matlab or something similar to check their work.

Note: for assignment problems, written details of calculations will be required.

Grading:

A+	97–100	B+	87–89	C+	77–79	D+	67–69
A	93–96	B	83–86	C	73–76	D	63–66
A-	90–92	B-	80–82	C-	70–72	D-	60–62

Why are we doing this?

Many things are discrete:

- Information (0's & 1's, letters, words)
- People (sociology)
- Networks (the Web, people again, food webs, ...)
- Sounds (musical notes)

Even more:

If real data is continuous, we almost always discretize it (0's and 1's)

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Big deal: Linear Algebra is a body of mathematics that deals with discrete problems.

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Linear Algebra is used in many fields to solve problems:

- Engineering
- Computer Science (Google's Pagerank)
- Physics
- Economics
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Linear Algebra is as important as Calculus...

Calculus ≡ the blue pill...

You are now choosing the red pill:

Matrices as gadgets:

A matrix A transforms a vector $\vec{x}$ into a new vector $\vec{x}'$ through matrix multiplication (whatever that is):

$$\vec{x}' = A\vec{x}$$

We can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things in different directions
- Reveal the true ur-dystopian reality.

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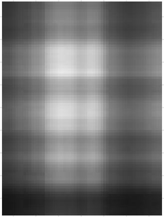
Image approximation (80x60)

$$A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



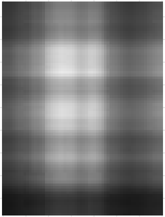
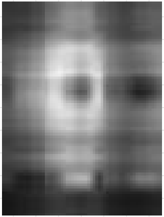
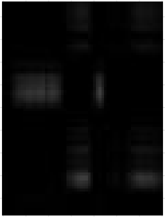


Image approximation (80x60)

$$A = \sum_{i=1}^{2} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$





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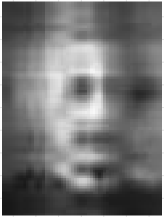

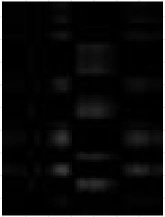

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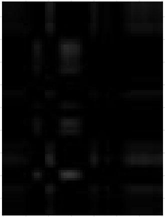
$$A = \sum_{i=1}^{4} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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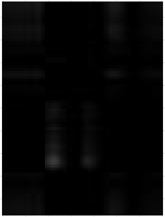
$$A = \sum_{i=1}^{5} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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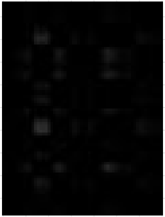
$$A = \sum_{i=1}^{6} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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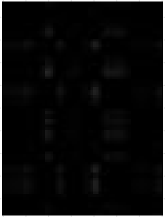
$$A = \sum_{i=1}^{7} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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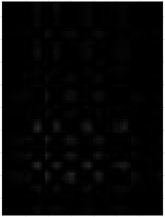

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$$A = \sum_{i=1}^{9} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

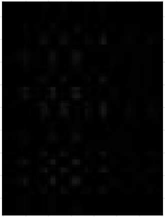
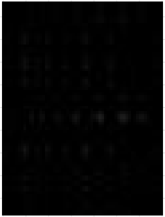


Image approximation (80x60)

$$A = \sum_{i=1}^{10} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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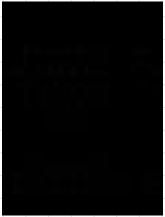
$$A = \sum_{i=1}^{\color{red}{20}} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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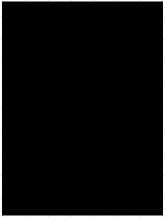

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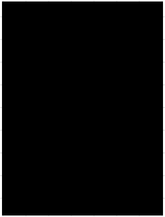

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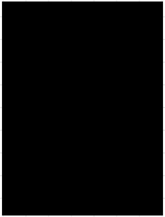

Image approximation (80x60)

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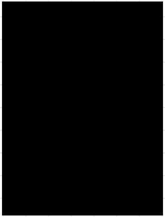

Three key problems of Linear Algebra

1. Given a matrix A and a vector $\vec{b}$, find $\vec{x}$ such that

$$A\vec{x} = \vec{b}.$$

2. Eigenvalue problem: Given A, find λ and $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

3. Coupled linear differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = A\, y(t)$$

- Our focus will be largely on #1, partly on #2.

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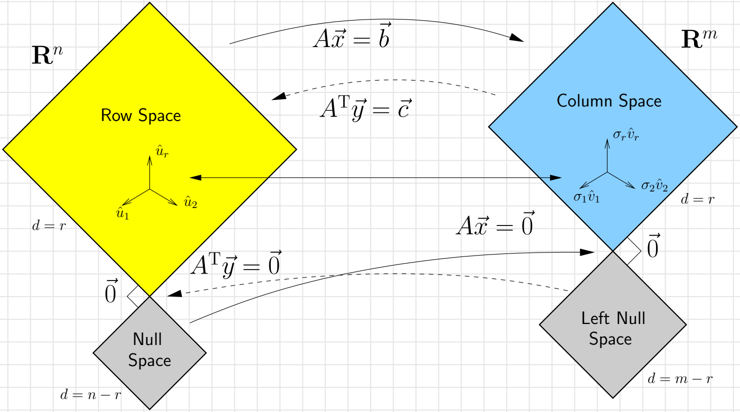
Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:



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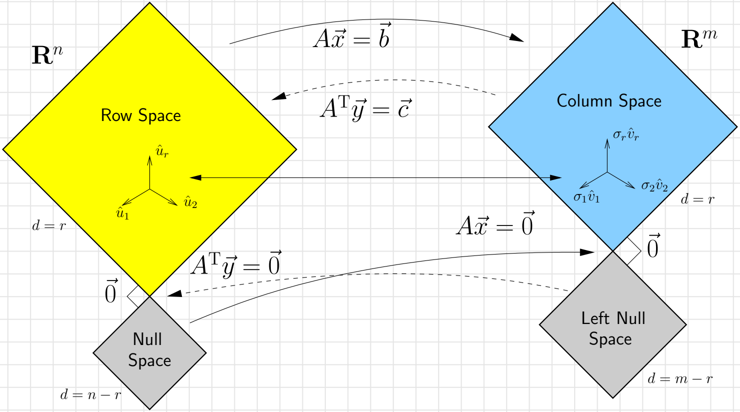

What is going on here? We have 25 lectures to find out...

Is this your left nullspace?:

Our new BFF: $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:

- $\vec{b}$ represents reality (e.g., music, structure)
- A contains building blocks (e.g., notes, shapes)
- $\vec{x}$ specifies how we combine our building blocks to make $\vec{b}$ (as best we can).

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How can we disentangle an orchestra's sound?

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- $\vec{b}$ represents reality (e.g., music, structure)
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- $\vec{x}$ specifies how we combine our building blocks to make $\vec{b}$ (as best we can).

How can we disentangle an orchestra's sound?

What about pictures, waves, signals, ...?

Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?

- Compress information
- See how we can alter information (filtering)
- Find a system's simplest representation
- Find a system's most important elements
- See how to adjust a system in a principled way

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Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The Row Picture
- Way 2: The Column Picture
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Example:

$$\begin{aligned} -x_1 &+ x_2 &= 1 \\ 2x_1 &+ x_2 &= 4 \end{aligned}$$

- Call this a 2 by 2 system of equations.
- 2 equations with 2 unknowns.
- Standard method of simultaneous equations: solve above by adding and subtracting multiples of equations to each other

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Row Picture—what we are doing:

- (a) Finding intersection of two lines
- (b) Finding the values of x_1 and x_2 for which both equations are satisfied (true/happy)
- A splendid and deep connection:
  (a) Geometry $\rightleftharpoons$ (b) Algebra

Three possible kinds of solution:

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The University of Vermont

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The column picture:

- Column vectors are our 'building blocks'
- Key idea: try to 'reach' $\vec{b}$ by combining (summing) multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$.

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Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \not\parallel \vec{a}_2$: 1 solution
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(assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$)

Three ways to understand $A\vec{x} = \vec{b}$:

Difficulties:

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- No! We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
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The Matrix Picture:

A is now an operator:

- A transforms $\vec{x}$ into $\vec{b}$.
- Roughly speaking, A does two things to $\vec{x}$:
  1. Rotation/Flipping
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Three ways to understand $A\vec{x} = \vec{b}$:

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Now see

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Key idea in linear algebra:

- Decomposition or factorization of matrices.
- Matrices can often be written as products or sums of simpler matrices
- $A = LU$, $A = QR$, $A = U\Sigma V^{\mathrm{T}}$, $A = \sum_i \lambda_i \vec{v}\vec{v}^{\mathrm{T}}$, ...

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The truth about mathematics

The Colbert Report on Math (⊞) (February 7, 2006)

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