Scale-Free Networks

Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

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Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

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- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, . . . (non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not. . .

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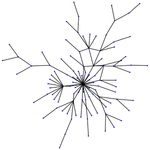
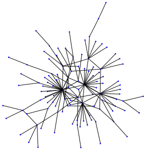
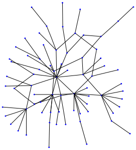
Random networks: largest components



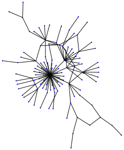
γ = 2.5
$\langle k \rangle$ = 1.8



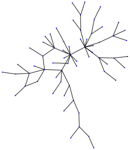
γ = 2.5
$\langle k \rangle$ = 2.05333



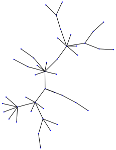
γ = 2.5
$\langle k \rangle$ = 1.66667



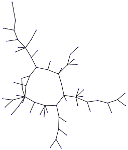
γ = 2.5
$\langle k \rangle$ = 1.92



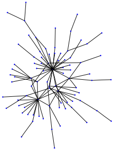
γ = 2.5
$\langle k \rangle$ = 1.6



γ = 2.5
$\langle k \rangle$ = 1.50667



γ = 2.5
$\langle k \rangle$ = 1.62667



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Scale-free networks

The big deal:

- We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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Heritage

Work that presaged scale-free networks

- 1924: G. Udny Yule [9]:
  # Species per Genus
- 1926: Lotka [4]:
  # Scientific papers per author
- 1953: Mandelbrot [5]):
  Zipf's law for word frequency through optimization
- 1955: Herbert Simon [8, 10]:
  Zipf's law, city size, income, publications, and species per genus
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BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
  3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.

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- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\max}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

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Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t}$:

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- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

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- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\, t^{1/2}.}$$

- Next find c_i ...

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Approximate analysis

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- Early nodes do best (First-mover advantage).

Approximate analysis

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

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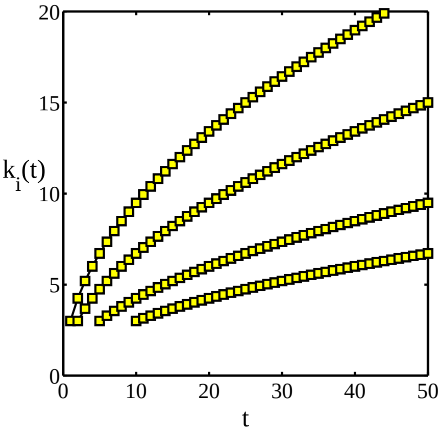

- $m = 3$
- $t_{i,\mathrm{start}} =$ $1, 2, 5,$ and 10.

Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\mathrm{start}})\mathrm{d}t_{i,\mathrm{start}} \simeq \frac{\mathrm{d}t_{i,\mathrm{start}}}{t+m_0}$$

- Using

$$k_i(t) = m\left(\frac{t}{t_{i,\mathrm{start}}}\right)^{1/2} \Rightarrow t_{i,\mathrm{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

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- Using the uniformity of start times:

$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

- Differentiate to find $\mathbf{Pr}(k)$:

$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k}\mathbf{Pr}(k_i < k) = \frac{2m^2 t}{(t + m_0)k^3}$$

$$\sim 2m^2 k^{-3} \text{ as } m \to \infty.$$

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- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
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Real data

From Barabási and Albert's original paper [2]:

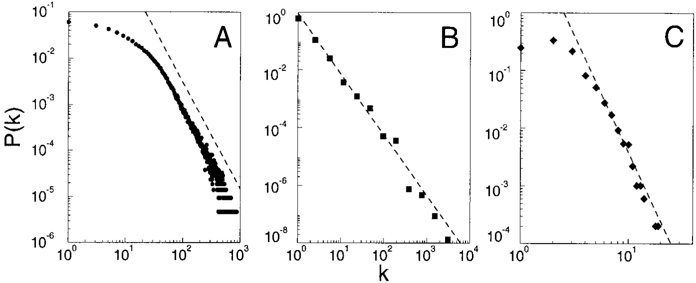

Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
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Outline

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- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
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Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

- So rich-gets-richer scheme can now be seen to work in a natural way.

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- We've looked at some aspects of contagion on scale-free networks:
  1. Facilitate disease-like spreading.
  2. Inhibit threshold-like spreading.
- Another simple story concerns system robustness.
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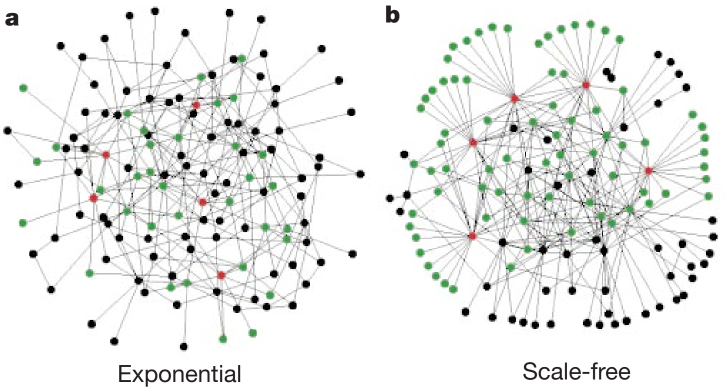
Robustness

- Standard random networks (Erdős-Rényi) versus Scale-free networks



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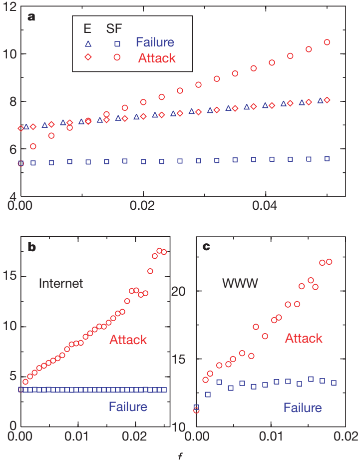

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- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
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Outline

Generalized model

Fooling with the mechanism:

- 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

$$\mathbf{Pr}(\text{attach to node } i) \propto A_k = k_i^{\nu}$$

  where A_k is the attachment kernel and $\nu > 0$.
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- Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A}\left[A_{k-1}N_{k-1} - A_k N_k\right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

1. The first term corresponds to degree $k-1$ nodes becoming degree k nodes.
2. The second term corresponds to degree k nodes becoming degree $k-1$ nodes.
3. Detail: $A_0 = 0$
4. One node is added per unit time.
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$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t}\left[(k-1)N_{k-1} - kN_k\right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
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- Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

- Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

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Generalized model

- Now find n_k:

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Outline

Universality?

- As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3} \text{ for large } k.$$

- Now: what happens if we start playing around with the attachment kernel A_k?
- Again, is the result $\gamma = 3$ universal (⊞)?
- Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- Keep A_k linear in k but tweak details.
- Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

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$$n_k = \frac{1}{\mu}\left[A_{k-1}n_{k-1} - A_k n_k\right] + \delta_{k1}$$

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- Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1};$$

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$$n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1+\frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{A_j}{A_j+\mu}$$

$$= \frac{\mu}{\cancel{A_k}} \frac{A_1}{(A_1+\mu)} \frac{A_2}{(A_2+\mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{\cancel{k}}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi} k^{k+1/2} e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2} e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

- Since μ depends on A_k, details matter...

Universality?

- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
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- Closed form expression for μ.
- We can solve for μ in some cases.
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- Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
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$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- Now use result that [3]

$$\sum_{k=2}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with $a = 1$ and $b = \mu + 1$.

- $$\mu = \alpha\frac{\Gamma(3)}{(\mu+1-1-1)\Gamma(2+\mu)}\Gamma(2+\mu)$$

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Outline

Sublinear attachment kernels

- Rich-get-somewhat-richer:

$$A_k \sim k^{\nu} \text{ with } 0 < \nu < 1.$$

- General finding by Krapivsky and Redner:[3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}.$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
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Sublinear attachment kernels

Details:

- For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu\left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu}\right)}$$

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Superlinear attachment kernels

- Rich-get-much-richer:

$$A_k \sim k^{\nu} \text{ with } \nu > 1.$$

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
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