Contagion

Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

Contagion models

Some large questions concerning network contagion:

1. For a given spreading mechanism on a given network, what's the probability that there will be global spreading?
2. If spreading does take off, how far will it go?
3. How do the details of the network affect the outcome?
4. How do the details of the spreading mechanism affect the outcome?
5. What if the seed is one or many nodes?

- Next up: We'll look at some fundamental kinds of spreading on generalized random networks.

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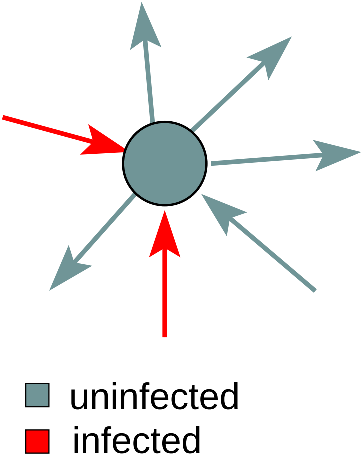

- General spreading mechanism: State of node i depends on history of i and i's neighbors' states.
- Doses of entity may be stochastic and history-dependent.
- May have multiple, interacting entities spreading at once.

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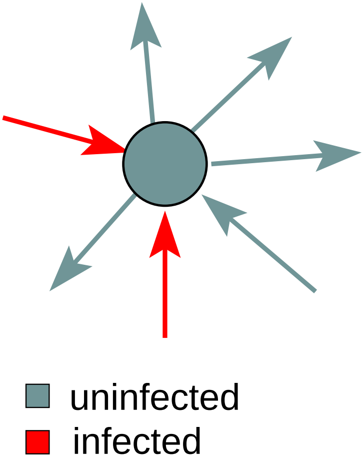

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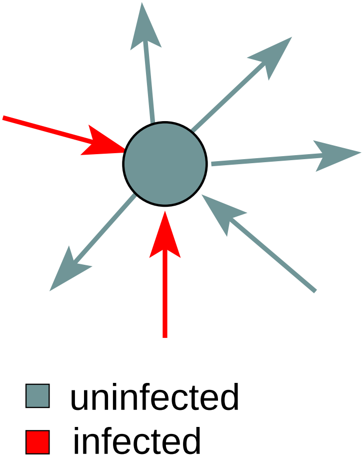

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- For random networks, we know local structure is pure branching.
- Successful spreading is $\therefore$ contingent on single edges infecting nodes.
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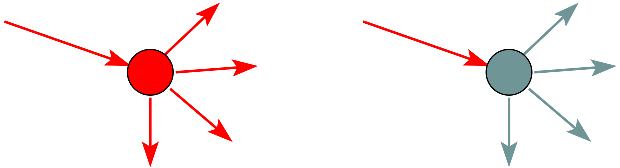

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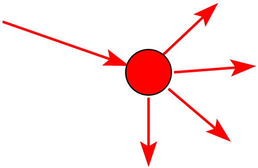

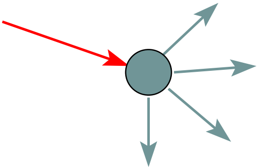

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- We need to find:
  r = the average # of infected edges that one random infected edge brings about.
- Define β_k as the probability that a node of degree k is infected by a single infected edge.
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$$r = \sum_{k=0}^{\infty} \underbrace{\frac{kP_k}{\langle k \rangle}}_{\substack{\text{prob. of} \\ \text{connecting to} \\ \text{a degree } k \text{ node}}} \cdot \underbrace{\beta_k}_{\substack{\text{Prob. of} \\ \text{infection}}} \cdot \underbrace{(k-1)}_{\substack{\#\text{ outgoing} \\ \text{infected} \\ \text{edges}}}$$

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Contagion condition

- Our contagion condition is then:

$$\boxed{r = \sum_{k=0}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k > 1.}$$

- Case 1: If $\beta_k = 1$ then

$$r = \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- Good: This is just our giant component condition again.

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- **Case 2:** If $\beta_k = \beta < 1$ then

$$r = \beta \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- A fraction (1-β) of edges do not transmit infection.
- Analogous phase transition to giant component case but critical value of $\langle k \rangle$ is increased.
- Aka bond percolation.
- Resulting degree distribution P'_k:

$$P'_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$

- We can show $F_{P'}(x) = F_P(\beta x + 1 - \beta)$.

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Contagion condition

- Cases 3, 4, 5, ...: Now allow β_k to depend on k
- Asymmetry: Transmission along an edge depends on node's degree at other end.
- Possibility: β_k increases with k... unlikely.
- Possibility: β_k is not monotonic in k... unlikely.
- Possibility: β_k decreases with k... hmmm.
- $\beta_k \searrow$ is a plausible representation of a simple kind of social contagion.
- The story:
  More well connected people are harder to influence.

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- Example: $\beta_k = 1/k$.
- $$r = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle k}$$
  $$= \sum_{k=1}^{\infty} \frac{(k-1)P_k}{\langle k \rangle} = \frac{\langle k \rangle - 1}{\langle k \rangle} = 1 - \frac{1}{\langle k \rangle}$$
- Since r is always less than 1, no spreading can occur for this mechanism.
- Decay of β_k is too fast.
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- Example: $\beta_k = H(\frac{1}{k} - \phi)$
  where $0 < \phi \leq 1$ is a threshold and H is the Heaviside function.
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- The contagion condition:

$$r = \sum_{k=1}^{\lfloor \frac{1}{\phi} \rfloor} \frac{(k-1)kP_k}{\langle k \rangle} > 1.$$

- As $\phi \to 1$, all nodes become resilient and $r \to 0$.
- As $\phi \to 0$, all nodes become vulnerable and the contagion condition matches up with the giant component condition.
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Outline

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Some important models (recap from CSYS 300)

- Tipping models—Schelling (1971) [8, 9, 10]
  - Simulation on checker boards.
  - Idea of thresholds.
- Threshold models—Granovetter (1978) [7]
- Herding models—Bikhchandani et al. (1992) [1, 2]
  - Social learning theory, Informational cascades,...

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Threshold model on a network

- Interactions between individuals now represented by a network
- Network is sparse
- Individual i has k_i contacts
- Influence on each link is reciprocal and of unit weight
- Each individual i has a fixed threshold ϕ_i
- Individuals repeatedly poll contacts on network
- Synchronous, discrete time updating
- Individual i becomes active when fraction of active contacts $a_i \geq \phi_i k_i$
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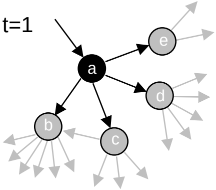

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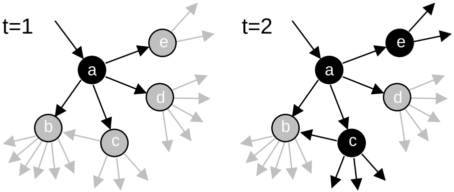

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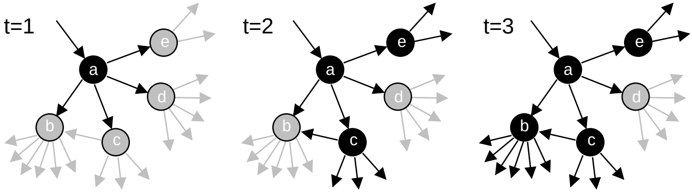

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The most gullible

Vulnerables:

- Recall definition: individuals who can be activated by just one contact being active are vulnerables.
- The vulnerability condition for node i: $1/k_i \geq \phi_i$.
- Means # contacts $k_i \leq \lfloor 1/\phi_i \rfloor$.
- Key: For global cascades on random networks, must have a *global component of vulnerables* [12]
- For a uniform threshold ϕ, our contagion condition tells us when such a component exists:

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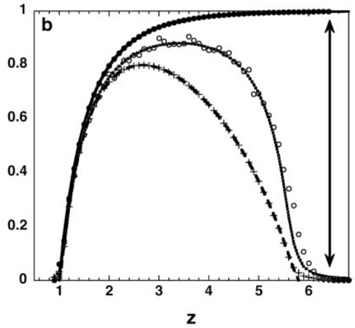

(n.b., $z = \langle k \rangle$)

- Top curve: final fraction infected if successful.
- Middle curve: chance of starting a global spreading event (cascade).
- Bottom curve: fractional size of vulnerable subcomponent.[12]

- Cascades occur only if size of vulnerable subcomponent > 0.
- System is robust-yet-fragile just below upper boundary [3, 4, 11]
- 'Ignorance' facilitates spreading.

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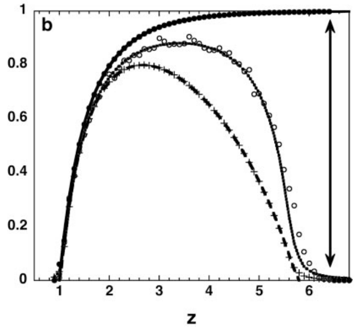

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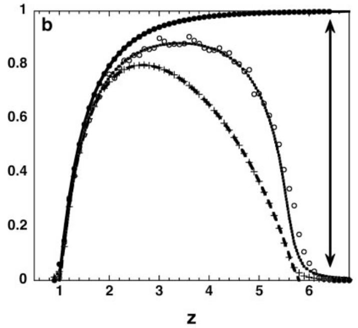

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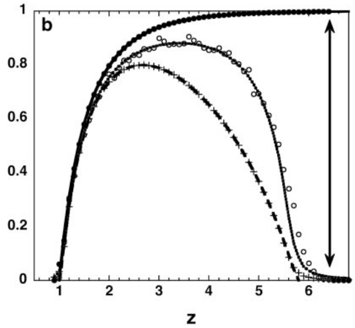

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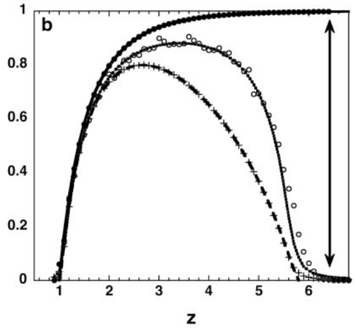

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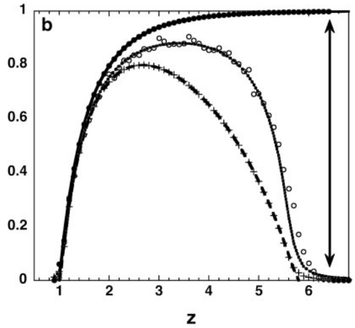

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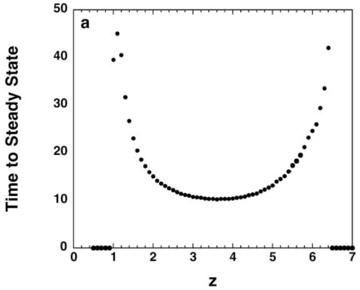

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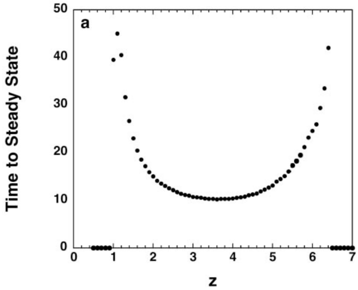

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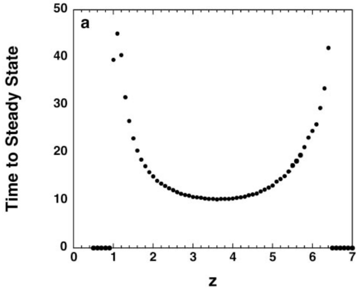

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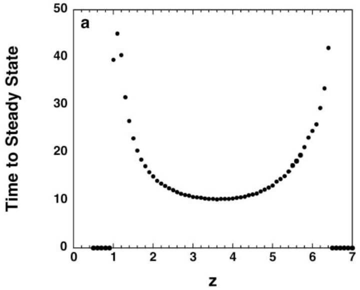

- Time taken for cascade to spread through network. [12]
- Two phase transitions.

(n.b., $z = \langle k \rangle$)

- Largest vulnerable component = critical mass.
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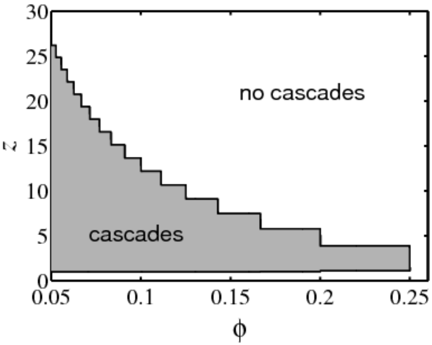
Cascade window for random networks



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- Outline of cascade window for random networks.

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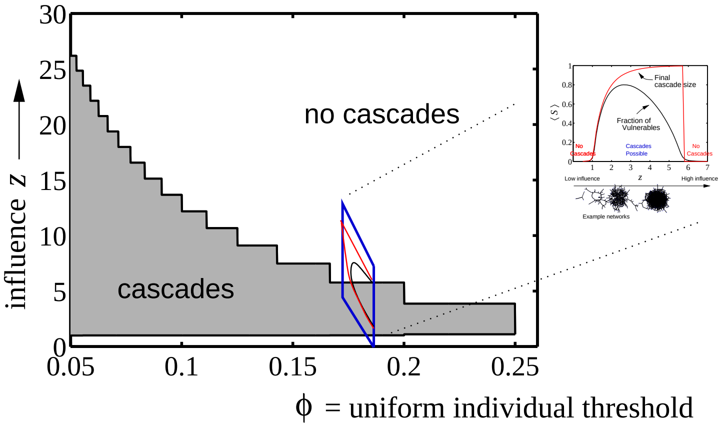
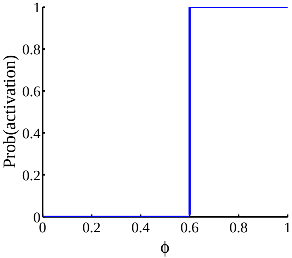


Outline

Social Contagion

Granovetter's Threshold model—recap



- Assumes deterministic response functions
- ϕ_* = threshold of an individual.
- $f(\phi_*)$ = distribution of thresholds in a population.
- $F(\phi_*)$ = cumulative distribution = $\int_{\phi'_*=0}^{\phi_*} f(\phi'_*)\mathrm{d}\phi'_*$
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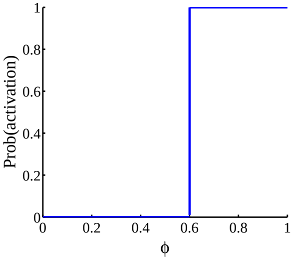

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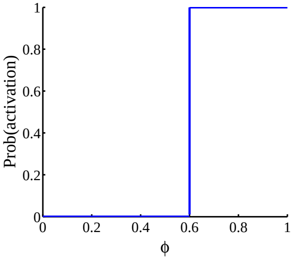

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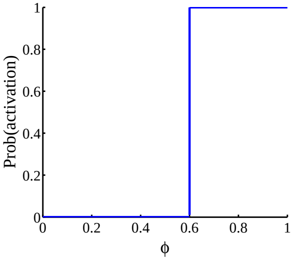

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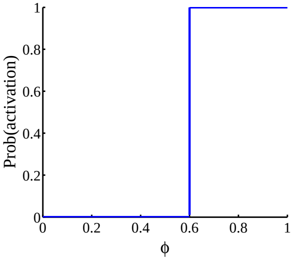

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Social Sciences—Threshold models

- At time $t+1$, fraction rioting = fraction with $\phi_* \leq \phi_t$.
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$$\phi_{t+1} = \int_0^{\phi_t} f(\phi_*)\mathrm{d}\phi_* = F(\phi_*)|_0^{\phi_t} = F(\phi_t)$$
- $\Rightarrow$ Iterative maps of the unit interval $[0, 1]$.

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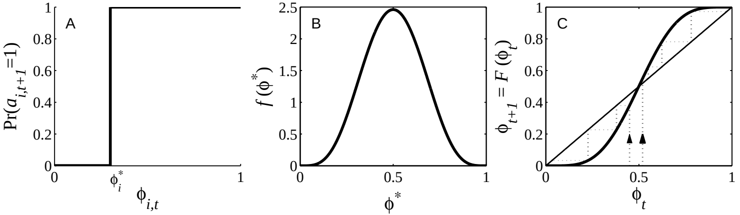
Social Sciences—Threshold models

Action based on perceived behavior of others.



- Two states: S and I
- Recover now possible (SIS)
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- Discrete time, synchronous update (strong assumption!)
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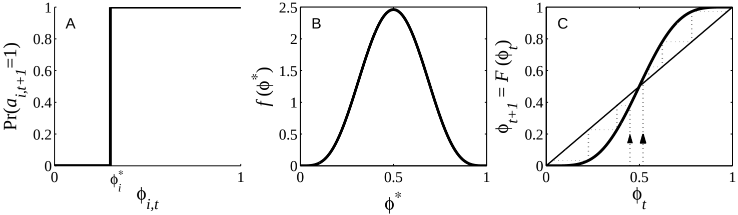

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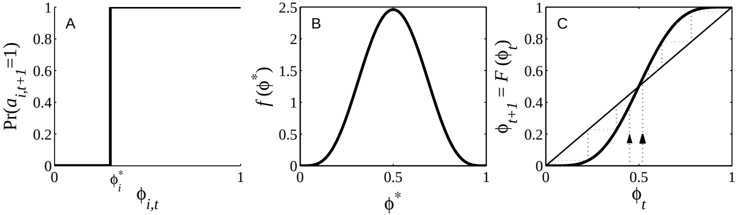

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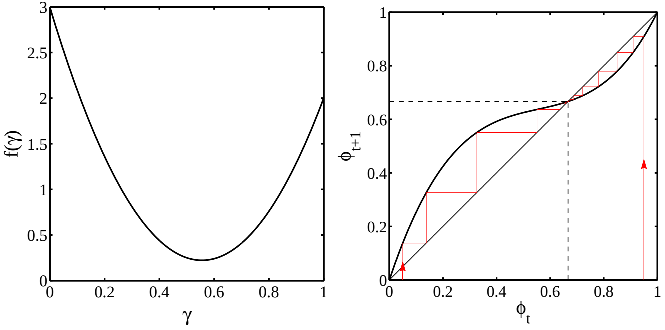

- Example of single stable state model

Social Sciences—Threshold models

Implications for collective action theory:

1. Collective uniformity $\not\Rightarrow$ individual uniformity
2. Small individual changes $\Rightarrow$ large global changes

Next:

- Connect mean-field model to network model.
- Single seed for network model: $1/N \to 0$.
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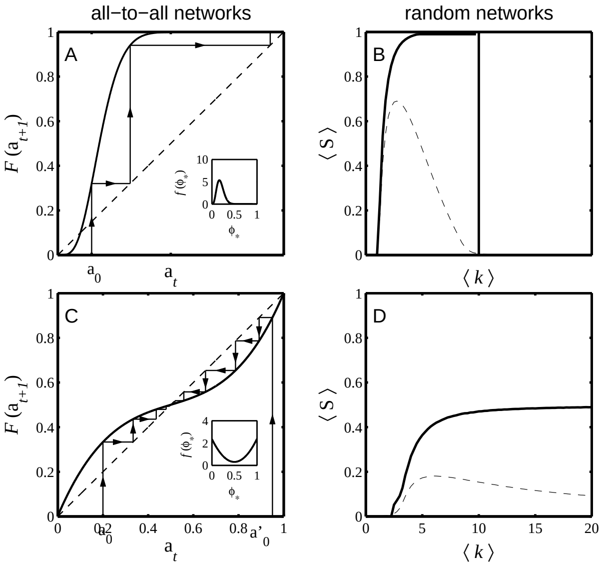
All-to-all versus random networks



Outline

Threshold contagion on random networks

Three key pieces to describe analytically:

1. The fractional size of the largest subcomponent of vulnerable nodes, S_{vuln}.
2. The chance of starting a global spreading event, $P_{\mathrm{trig}} = S_{\mathrm{trig}}$.
3. The expected final size of any successful spread, S.
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- First goal: Find the largest component of vulnerable nodes.
- Recall that for finding the giant component's size, we had to solve:

$$F_\pi(x) = xF_P\left(F_\rho(x)\right) \text{ and } F_\rho(x) = xF_R\left(F_\rho(x)\right)$$

- We'll find a similar result for the subset of nodes that are vulnerable.
- This is a node-based percolation problem.
- For a general monotonic threshold distribution $f(\phi)$, a degree k node is vulnerable with probability

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- Everything now revolves around the modified generating function:

$$F_P^{(\mathrm{vuln})}(x) = \sum_{k=0}^{\infty} \beta_k P_k x^k.$$

- Generating function for friends-of-friends distribution is related in same way as before:

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- Functional relations for component size g.f.'s are almost the same...

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- Second goal: Find probability of triggering largest vulnerable component.
- Assumption is first node is randomly chosen.
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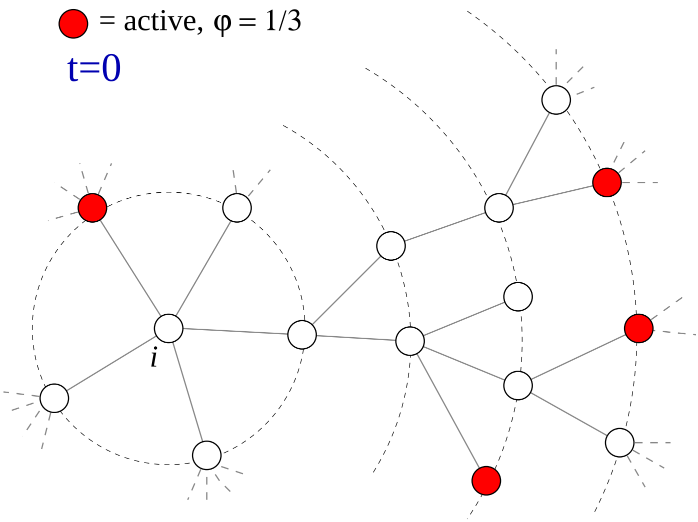

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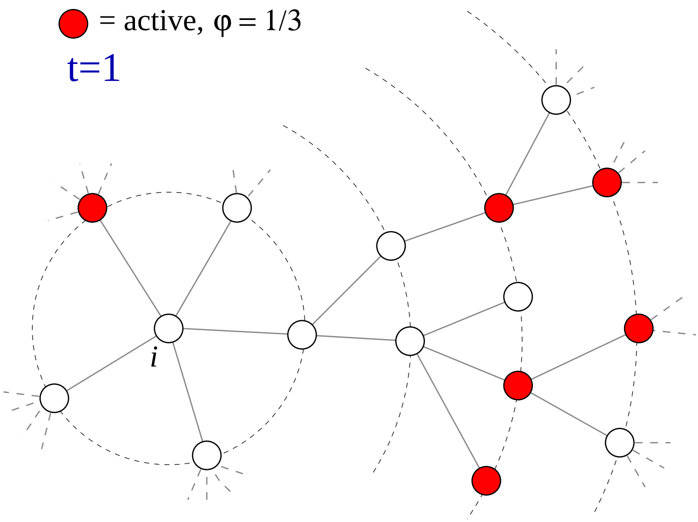

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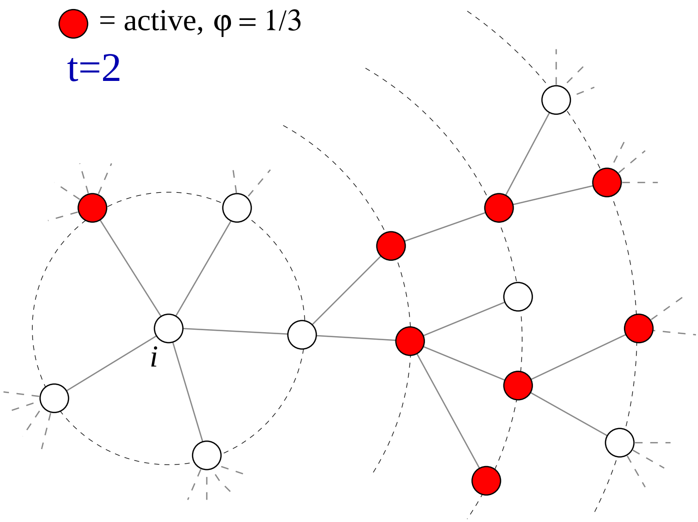

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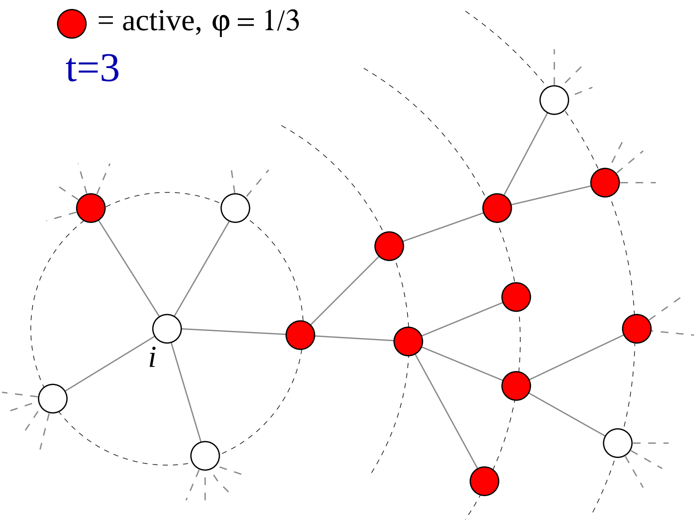

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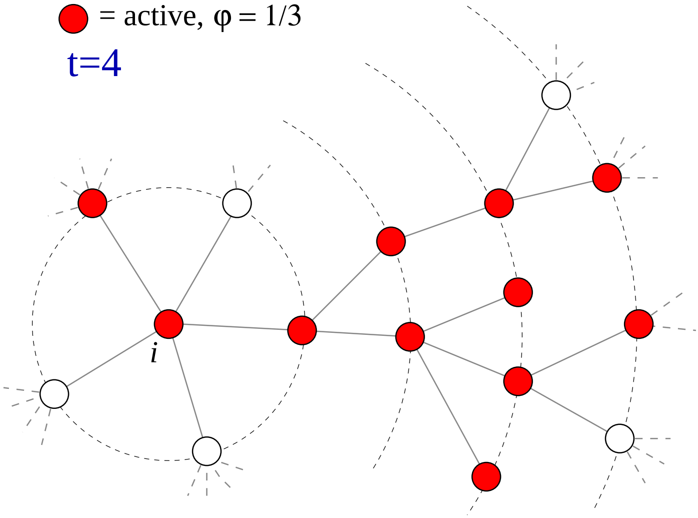

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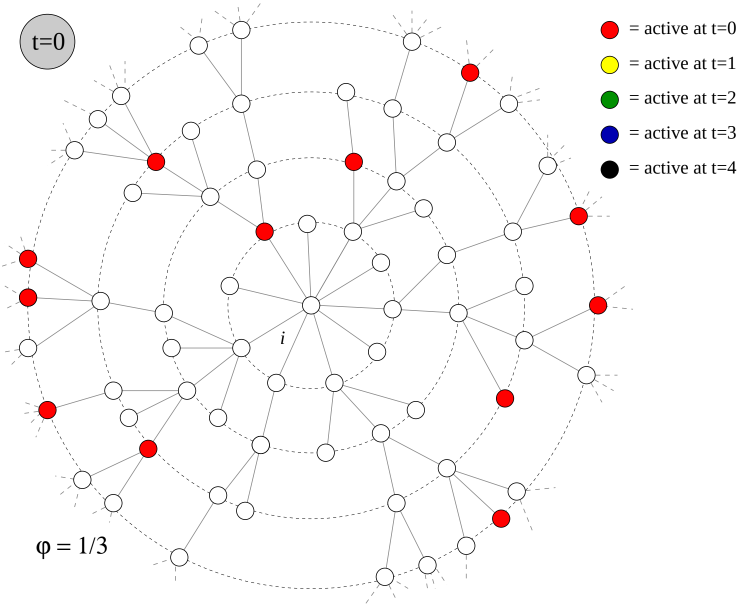

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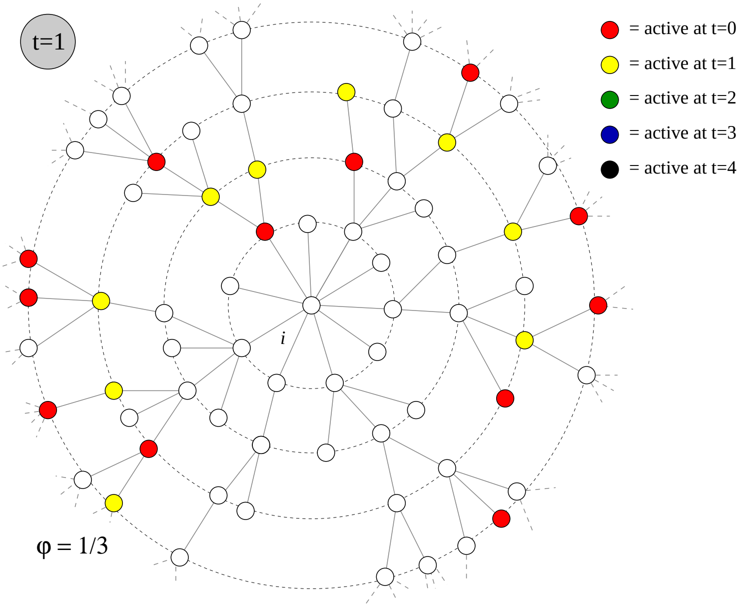

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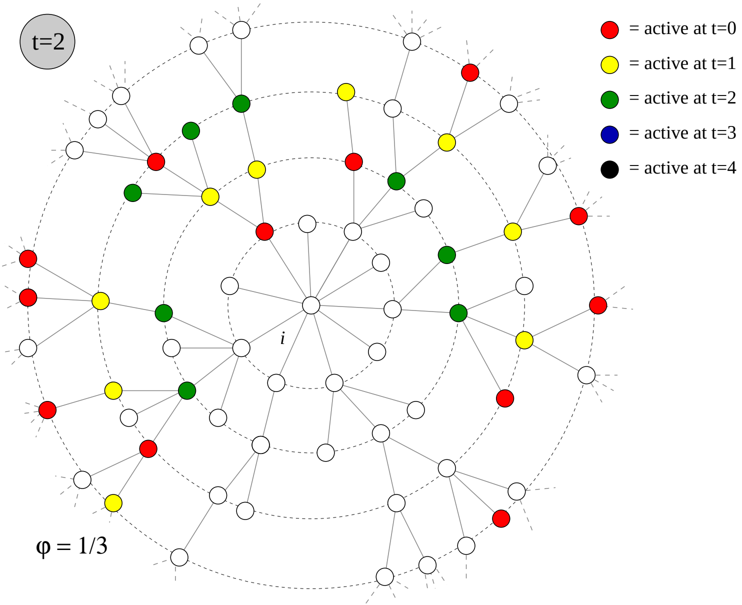

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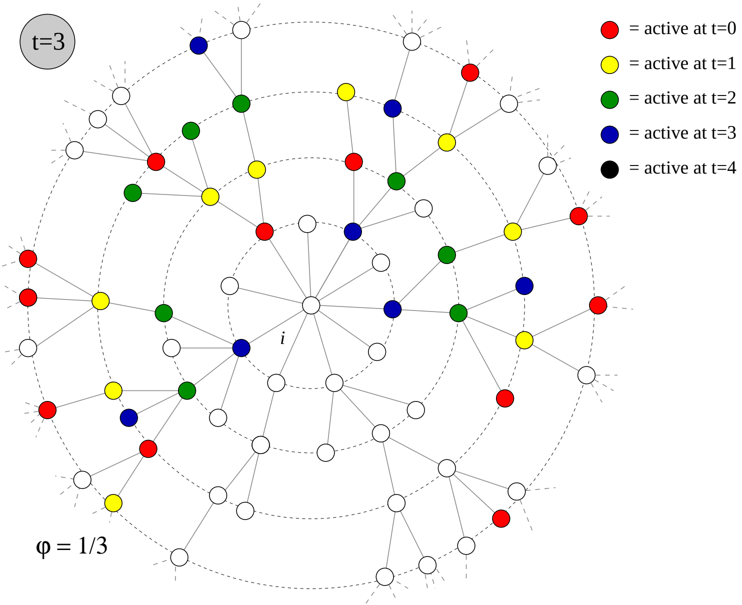

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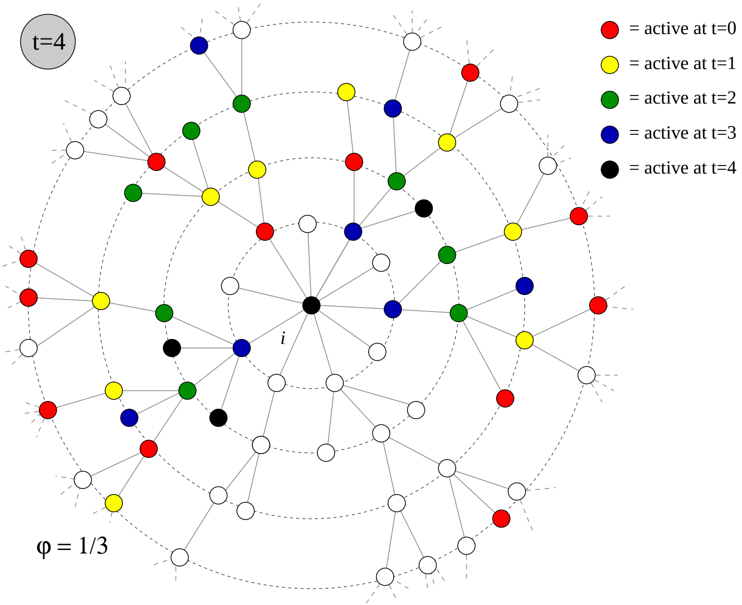

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- Calculations are possible nodes do not become inactive.
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First connect θ_0 to θ_1:

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Two pieces:

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   with $\theta_0 = \phi_0$.

2. $\phi_{t+1} = \phi_0 +$

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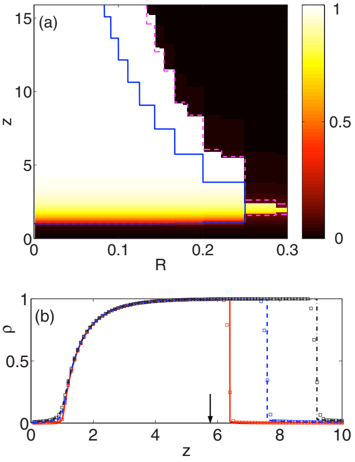
Comparison between theory and simulations



From Gleeson and Cahalane [6]

- Pure random networks with simple threshold responses
- R = uniform threshold (our ϕ_*); z = average degree; $\rho = \phi$; $q = \theta$; $N = 10^5$.
- $\phi_0 = 10^{-3}$, 0.5×10^{-2}, and 10^{-2}.
- Cascade window is for $\phi = 10^{-2}$ case.
- Sensible expansion of cascade window as ϕ_0 increases.

Notes:

- Retrieve cascade condition for spreading from a single seed in limit $\phi_0 \rightarrow 0$.
- Depends on map $\theta_{t+1} = G(\theta_t; \phi_0)$.
- First: if self-starters are present, some activation is assured:

$$G(0; \phi_0) = \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \beta_{k0} > 0.$$

meaning $\beta_{k0} > 0$ for at least one value of $k \geq 1$.

- If $\theta = 0$ is a fixed point of G (i.e., $G(0; \phi_0) = 0$) then spreading occurs if

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Notes:

In words:

- If $G(0; \phi_0) > 0$, spreading must occur because some nodes turn on for free.
- If G has an unstable fixed point at $\theta = 0$, then cascades are also always possible.

Non-vanishing seed case:

- Cascade condition is more complicated for $\phi_0 > 0$.
- If G has a stable fixed point at $\theta = 0$, and an unstable fixed point for some $0 < \theta_* < 1$, then for $\theta_0 > \theta_*$, spreading takes off.
- Tricky point: G depends on ϕ_0, so as we change ϕ_0, we also change G.

Notes:

In words:

- If $G(0; \phi_0) > 0$, spreading must occur because some nodes turn on for free.
- If G has an unstable fixed point at $\theta = 0$, then cascades are also always possible.

Non-vanishing seed case:

- Cascade condition is more complicated for $\phi_0 > 0$.
- If G has a stable fixed point at $\theta = 0$, and an unstable fixed point for some $0 < \theta_* < 1$, then for $\theta_0 > \theta_*$, spreading takes off.
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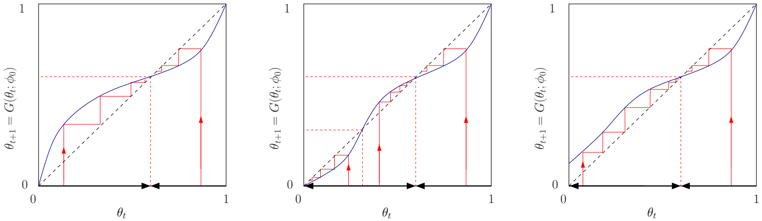
General fixed point story:



- Given $\theta_0 (= \phi_0)$, θ_∞ will be the nearest stable fixed point, either above or below.
- n.b., adjacent fixed points must have opposite stability types.
- Important: Actual form of G depends on ϕ_0.
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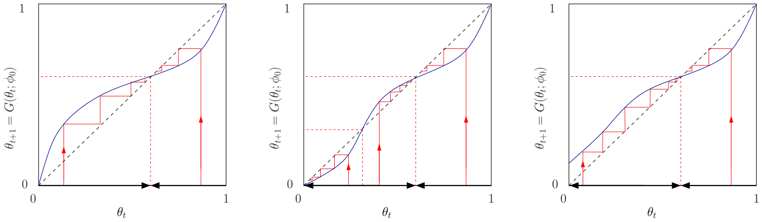

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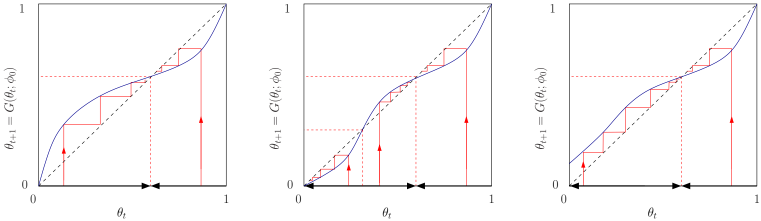

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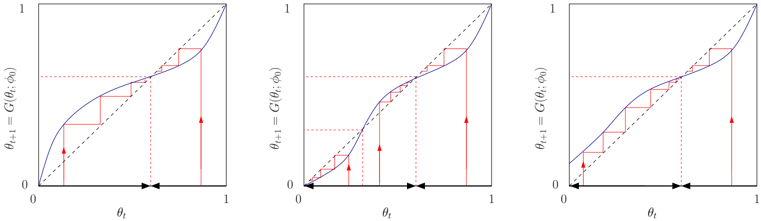

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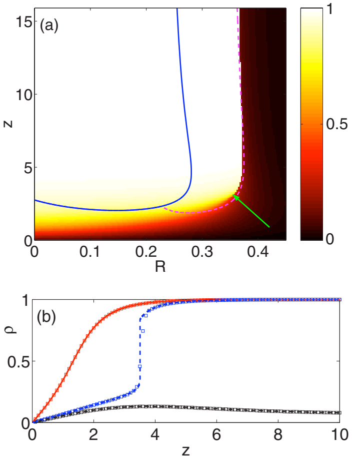

From Gleeson and Cahalane [6]

- Now allow thresholds to be distributed according to a Gaussian with mean R.
- R = 0.2, 0.362, and 0.38; $\sigma = 0.2$.
- $\phi_0 = 0$ but some nodes have thresholds ≤ 0 so effectively $\phi_0 > 0$.
- Now see a (nasty) discontinuous phase transition for low $\langle k \rangle$.

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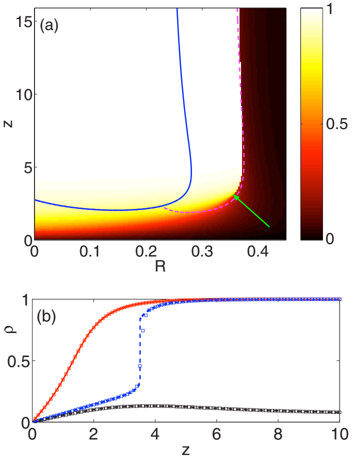

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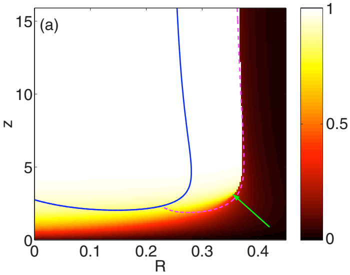

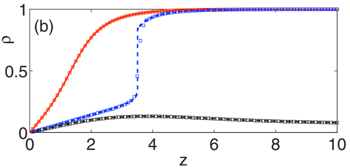

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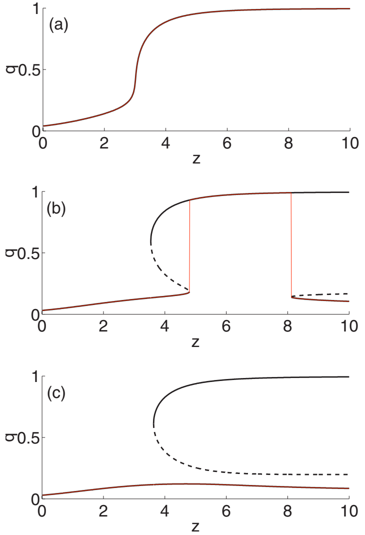

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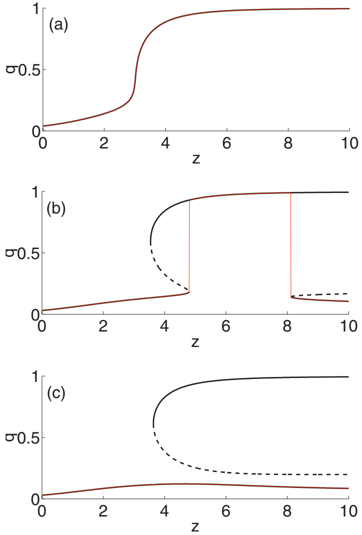

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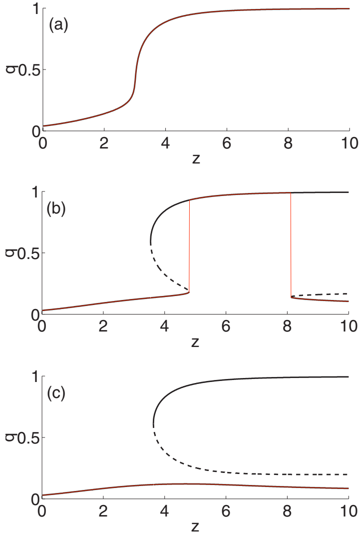

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Spreadarama

Bridging to single seed case:

- Consider largest vulnerable component as initial set of seeds.
- Not quite right as spreading must move through vulnerables.
- But we can usefully think of the vulnerable component as activating at time $t = 0$ because order doesn't matter.
- Rebuild ϕ_t and θ_t expressions...

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Two pieces modified for single seed:

1. $\theta_{t+1} = \theta_{\mathrm{vuln}} +$

$$(1-\theta_{\mathrm{vuln}})\sum_{k=1}^{\infty}\frac{kP_k}{\langle k\rangle}\sum_{j=0}^{k-1}\binom{k-1}{j}\theta_t^{\,j}(1-\theta_t)^{k-1-j}\beta_{kj}$$

   with $\theta_0 = \theta_{\mathrm{vuln}}$ = **Pr** an edge leads to the giant vulnerable component (if it exists).

2. $\phi_{t+1} = S_{\mathrm{vuln}} +$

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Time-dependent solutions

Synchronous update

- Done: Evolution of ϕ_t and θ_t given exactly by the maps we have derived.

Asynchronous updates

- Update nodes with probability α.
- As $\alpha \to 0$, updates become effectively independent.
- Now can talk about $\phi(t)$ and $\theta(t)$.
- More on this later...

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