

# Contagion

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Basic Contagion Models

Social Contagion Models

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All-to-all networks  
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## Outline

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## Contagion models

Some large questions concerning network contagion:

1. For a given **spreading mechanism** on a given network, what's the **probability** that there will be **global spreading**?
  2. If spreading does take off, how far will it go?
  3. How do the **details** of the **network** affect the outcome?
  4. How do the **details** of the **spreading mechanism** affect the outcome?
  5. What if the **seed** is one or many nodes?
- **Next up:** We'll look at some fundamental kinds of spreading on generalized random networks.

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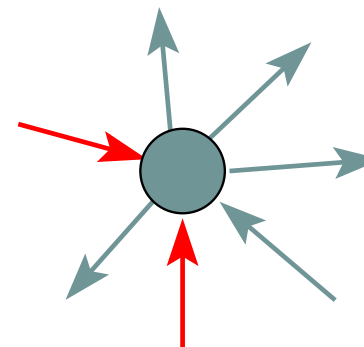
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## Spreading mechanisms



■ uninfected  
■ infected

- **General spreading mechanism:** State of node  $i$  depends on history of  $i$  and  $i$ 's neighbors' states.
- **Doses** of entity may be stochastic and history-dependent.
- May have **multiple, interacting entities** spreading at once.

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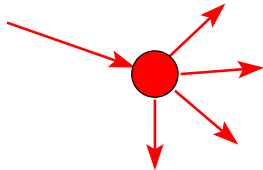
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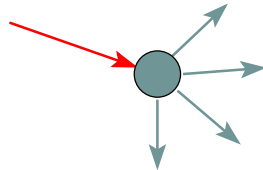
# Spreading on Random Networks

- ▶ For random networks, we know local structure is pure branching.
- ▶ Successful spreading is  $\therefore$  contingent on **single edges** infecting nodes.

Success



Failure:



- ▶ Focus on **binary** case with edges and nodes either infected or not.

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# Contagion condition

- ▶ We need to find:
  - $r$  = the average # of infected edges that one random infected edge brings about.
- ▶ Define  $\beta_k$  as the probability that a node of degree  $k$  is infected by a single infected edge.

$$r = \sum_{k=0}^{\infty} \underbrace{\frac{kP_k}{\langle k \rangle}}_{\text{prob. of connecting to a degree } k \text{ node}} \cdot \underbrace{\beta_k}_{\text{Prob. of infection}} \cdot \underbrace{(k-1)}_{\text{\# outgoing infected edges}}$$

$$+ \sum_{k=0}^{\infty} \underbrace{\frac{kP_k}{\langle k \rangle}}_{\text{Prob. of no infection}} \cdot \underbrace{0}_{\text{\# outgoing infected edges}}$$

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# Contagion condition

- ▶ Our contagion condition is then:

$$r = \sum_{k=0}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k > 1.$$

- ▶ **Case 1:** If  $\beta_k = 1$  then

$$r = \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- ▶ **Good:** This is just our giant component condition again.

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# Contagion condition

- ▶ **Case 2:** If  $\beta_k = \beta < 1$  then

$$r = \beta \frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

- ▶ A fraction  $(1-\beta)$  of edges do not transmit infection.
- ▶ Analogous phase transition to giant component case but **critical value** of  $\langle k \rangle$  is **increased**.
- ▶ Aka **bond percolation**.
- ▶ Resulting degree distribution  $P'_k$ :

$$P'_k = \beta^k \sum_{i=k}^{\infty} \binom{i}{k} (1-\beta)^{i-k} P_i.$$

- ▶ We can show  $F_{P'}(x) = F_P(\beta x + 1 - \beta)$ .

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## Contagion condition

- ▶ **Cases 3, 4, 5, ...:** Now allow  $\beta_k$  to depend on  $k$
- ▶ **Asymmetry:** Transmission along an edge depends on node's degree at other end.
- ▶ Possibility:  $\beta_k$  increases with  $k$ ... **unlikely.**
- ▶ Possibility:  $\beta_k$  is not monotonic in  $k$ ... **unlikely.**
- ▶ Possibility:  $\beta_k$  decreases with  $k$ ... **hmmm.**
- ▶  $\beta_k \searrow$  is a plausible representation of a simple kind of social contagion.
- ▶ **The story:**  
More well connected people are harder to influence.

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## Contagion condition

- ▶ **Example:**  $\beta_k = 1/k$ .

$$r = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle k}$$

$$= \sum_{k=1}^{\infty} \frac{(k-1)P_k}{\langle k \rangle} = \frac{\langle k \rangle - 1}{\langle k \rangle} = 1 - \frac{1}{\langle k \rangle}$$

- ▶ Since  $r$  is always less than 1, no spreading can occur for this mechanism.
- ▶ Decay of  $\beta_k$  is too fast.
- ▶ Result is independent of degree distribution.

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## Contagion condition

- ▶ **Example:**  $\beta_k = H(\frac{1}{k} - \phi)$   
where  $0 < \phi \leq 1$  is a **threshold** and  $H$  is the Heaviside function.
- ▶ Infection only occurs for nodes with **low** degree.
- ▶ Call these nodes **vulnerables**:  
they flip when **only one** of their friends flips.

$$r = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} \beta_k = \sum_{k=1}^{\infty} \frac{(k-1)kP_k}{\langle k \rangle} H(\frac{1}{k} - \phi)$$

$$= \sum_{k=1}^{\lfloor \frac{1}{\phi} \rfloor} \frac{(k-1)kP_k}{\langle k \rangle} \quad \text{where } \lfloor \cdot \rfloor \text{ means floor.}$$

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## Contagion condition

- ▶ The contagion condition:

$$r = \sum_{k=1}^{\lfloor \frac{1}{\phi} \rfloor} \frac{(k-1)kP_k}{\langle k \rangle} > 1.$$

- ▶ As  $\phi \rightarrow 1$ , all nodes become resilient and  $r \rightarrow 0$ .
- ▶ As  $\phi \rightarrow 0$ , all nodes become vulnerable and the contagion condition matches up with the giant component condition.
- ▶ **Key:** If we fix  $\phi$  and then vary  $\langle k \rangle$ , we may see **two** phase transitions.
- ▶ Added to our standard giant component transition, we will see a cut off in spreading as nodes become more connected.

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# Social Contagion

## Some important models (recap from CSYS 300)

- ▶ Tipping models—Schelling (1971) [8, 9, 10]
  - ▶ Simulation on checker boards.
  - ▶ Idea of thresholds.
- ▶ Threshold models—Granovetter (1978) [7]
- ▶ Herding models—Bikhchandani et al. (1992) [1, 2]
  - ▶ Social learning theory, Informational cascades,...

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# Threshold model on a network

Original work:

**“A simple model of global cascades on random networks”**  
D. J. Watts. Proc. Natl. Acad. Sci., 2002 [12]

- ▶ Mean field Granovetter model → network model
- ▶ Individuals now have a limited view of the world

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# Threshold model on a network

- ▶ Interactions between individuals now represented by a network
- ▶ Network is **sparse**
- ▶ Individual  $i$  has  $k_i$  contacts
- ▶ Influence on each link is **reciprocal** and of **unit weight**
- ▶ Each individual  $i$  has a fixed threshold  $\phi_i$
- ▶ Individuals repeatedly poll contacts on network
- ▶ Synchronous, discrete time updating
- ▶ Individual  $i$  becomes active when fraction of active contacts  $a_i \geq \phi_i k_i$
- ▶ Activation is permanent (SI)

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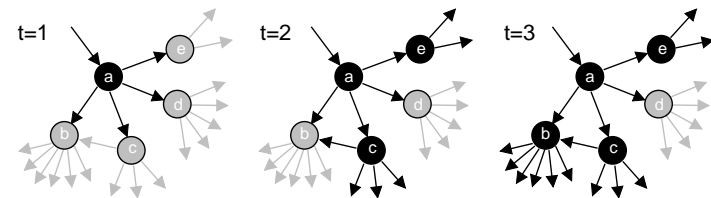
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# Threshold model on a network



- ▶ All nodes have threshold  $\phi = 0.2$ .

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# The most gullible

## Vulnerables:

- ▶ Recall definition: individuals who can be activated by just one contact being active are **vulnerables**.
- ▶ The vulnerability condition for node  $i$ :  $1/k_i \geq \phi_i$ .
- ▶ Means # contacts  $k_i \leq \lfloor 1/\phi_i \rfloor$ .
- ▶ **Key:** For global cascades on random networks, must have a *global component of vulnerables* [12]
- ▶ For a uniform threshold  $\phi$ , our contagion condition tells us when such a component exists:

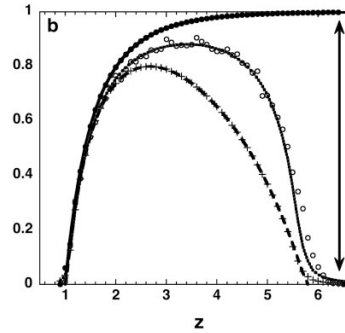
$$r = \sum_{k=1}^{\lfloor \frac{1}{\phi} \rfloor} \frac{(k-1)kP_k}{\langle k \rangle} > 1.$$

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# Cascades on random networks



- ▶ **Top curve:** final fraction infected if successful.
- ▶ **Middle curve:** chance of starting a global spreading event (cascade).
- ▶ **Bottom curve:** fractional size of vulnerable subcomponent. [12]

( n.b.,  $z = \langle k \rangle$ )

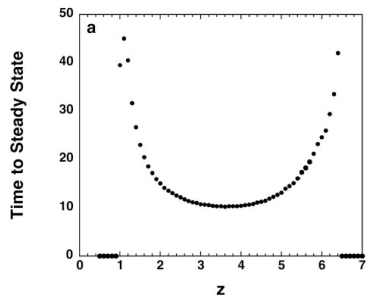
- ▶ Cascades occur only if size of vulnerable subcomponent  $> 0$ .
- ▶ System is robust-yet-fragile just below upper boundary [3, 4, 11]
- ▶ 'Ignorance' facilitates spreading.

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# Cascades on random networks



- ▶ Time taken for cascade to spread through network. [12]
- ▶ Two phase transitions.

( n.b.,  $z = \langle k \rangle$ )

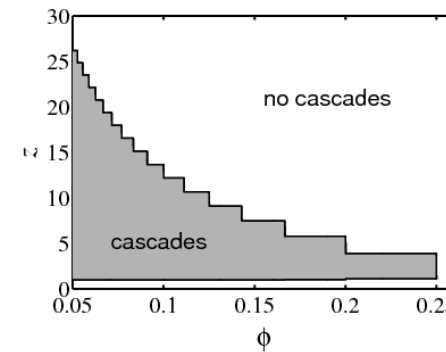
- ▶ Largest vulnerable component = **critical mass**.
- ▶ Now have endogenous mechanism for spreading from an individual to the critical mass and then beyond.

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# Cascade window for random networks



( n.b.,  $z = \langle k \rangle$ )

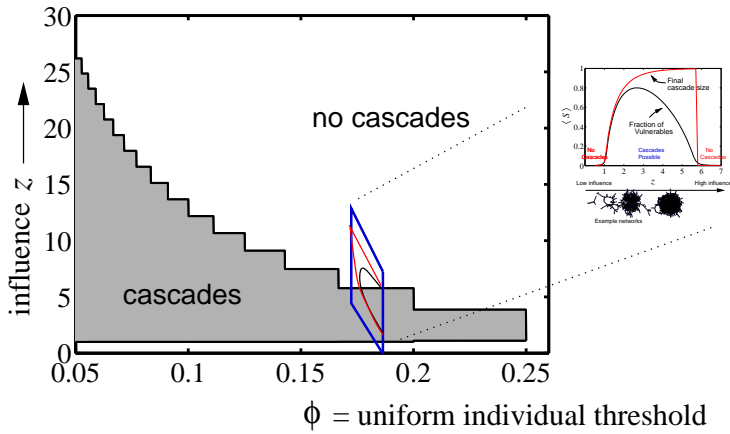
- ▶ Outline of cascade window for random networks.

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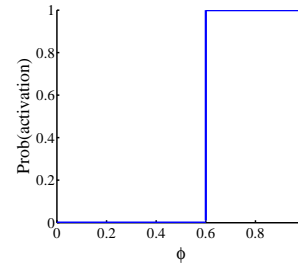
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# Cascade window for random networks



# Social Contagion

## Granovetter's Threshold model—recap



- ▶ Assumes deterministic response functions
- ▶  $\phi_*$  = threshold of an individual.
- ▶  $f(\phi_*)$  = distribution of thresholds in a population.
- ▶  $F(\phi_*)$  = cumulative distribution =  $\int_{\phi'_*=0}^{\phi_*} f(\phi'_*)d\phi'_*$
- ▶  $\phi_t$  = fraction of people 'rioting' at time step  $t$ .

# Social Sciences—Threshold models

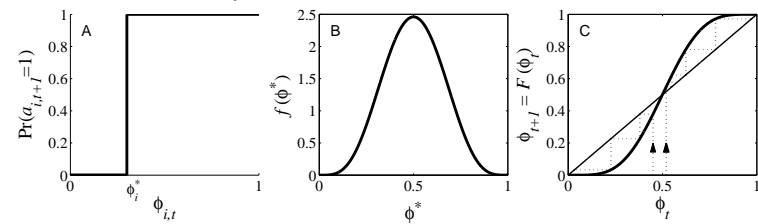
- ▶ At time  $t + 1$ , fraction rioting = fraction with  $\phi_* \leq \phi_t$ .

$$\phi_{t+1} = \int_0^{\phi_t} f(\phi_*)d\phi_* = F(\phi_*)|_0^{\phi_t} = F(\phi_t)$$

- ▶  $\Rightarrow$  Iterative maps of the unit interval  $[0, 1]$ .

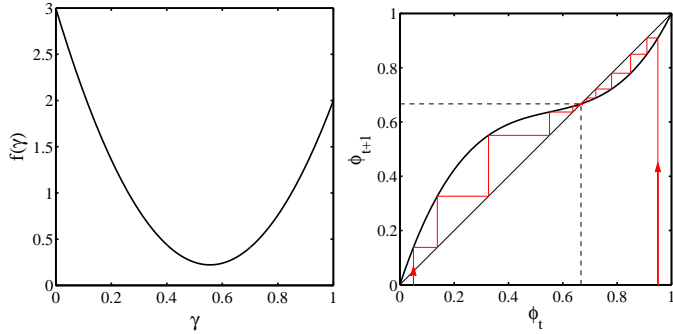
# Social Sciences—Threshold models

## Action based on perceived behavior of others.



- ▶ Two states: S and I
- ▶ Recover now possible (SIS)
- ▶  $\phi$  = fraction of contacts 'on' (e.g., rioting)
- ▶ Discrete time, synchronous update (strong assumption!)
- ▶ This is a **Critical mass model**

# Social Sciences—Threshold models



- ▶ Example of single stable state model

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# Social Sciences—Threshold models

## Implications for collective action theory:

1. Collective uniformity  $\not\Rightarrow$  individual uniformity
2. Small individual changes  $\Rightarrow$  large global changes

## Next:

- ▶ Connect mean-field model to network model.
- ▶ Single seed for network model:  $1/N \rightarrow 0$ .
- ▶ Comparison between network and mean-field model sensible for vanishing seed size for the latter.

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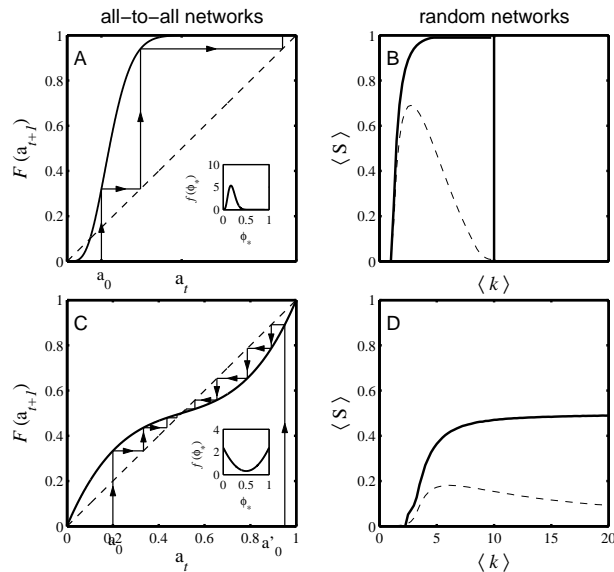
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# All-to-all versus random networks



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# Threshold contagion on random networks

## Three key pieces to describe analytically:

1. The fractional size of the largest subcomponent of vulnerable nodes,  $S_{\text{vuln}}$ .
2. The chance of starting a global spreading event,  $P_{\text{trig}} = S_{\text{trig}}$ .
3. The expected final size of any successful spread,  $S$ .
  - ▶ n.b., the distribution of  $S$  is almost always bimodal.

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## Threshold contagion on random networks

- ▶ **First goal:** Find the largest component of vulnerable nodes.
- ▶ Recall that for finding the giant component's size, we had to solve:

$$F_\pi(x) = xF_P(F_\rho(x)) \text{ and } F_\rho(x) = xF_R(F_\rho(x))$$

- ▶ We'll find a similar result for the subset of nodes that are vulnerable.
- ▶ This is a node-based percolation problem.
- ▶ For a general monotonic threshold distribution  $f(\phi)$ , a degree  $k$  node is vulnerable with probability

$$\beta_k = \int_0^{1/k} f(\phi) d\phi.$$

## Threshold contagion on random networks

- ▶ Everything now revolves around the **modified** generating function:

$$F_P^{(\text{vuln})}(x) = \sum_{k=0}^{\infty} \beta_k P_k x^k.$$

- ▶ Generating function for friends-of-friends distribution is related in same way as before:

$$F_R^{(\text{vuln})}(x) = \frac{\frac{d}{dx} F_P^{(\text{vuln})}(x)}{\frac{d}{dx} F_P^{(\text{vuln})}(x)|_{x=1}}.$$

## Threshold contagion on random networks

- ▶ Functional relations for component size g.f.'s are almost the same...

$$F_\pi^{(\text{vuln})}(x) = \underbrace{1 - F_P^{(\text{vuln})}(1)}_{\substack{\text{central node} \\ \text{is not} \\ \text{vulnerable}}} + xF_P^{(\text{vuln})}(F_\rho^{(\text{vuln})}(x))$$

$$F_\rho^{(\text{vuln})}(x) = \underbrace{1 - F_R^{(\text{vuln})}(1)}_{\substack{\text{first node} \\ \text{is not} \\ \text{vulnerable}}} + xF_R^{(\text{vuln})}(F_\rho^{(\text{vuln})}(x))$$

- ▶ Can now solve as before to find  $S_{\text{vuln}} = 1 - F_\pi^{(\text{vuln})}(1)$ .

## Threshold contagion on random networks

- ▶ **Second goal:** Find probability of triggering largest vulnerable component.
- ▶ Assumption is **first node** is **randomly chosen**.
- ▶ **Same set up** as for vulnerable component except now we don't care if the initial node is vulnerable or not:

$$F_\pi^{(\text{trig})}(x) = xF_P(F_\rho^{(\text{vuln})}(x))$$

$$F_\rho^{(\text{vuln})}(x) = 1 - F_R^{(\text{vuln})}(1) + xF_R^{(\text{vuln})}(F_\rho^{(\text{vuln})}(x))$$

- ▶ Solve as before to find  $P_{\text{trig}} = S_{\text{trig}} = 1 - F_\pi^{(\text{trig})}(1)$ .



# Threshold contagion on random networks

- ▶ **Third goal:** Find expected fractional size of spread.
- ▶ Not obvious even for uniform threshold problem.
- ▶ Difficulty is in figuring out if and when nodes that need  $\geq 2$  hits switch on.
- ▶ Problem **solved** for infinite seed case by Gleeson and Cahalane:  
 "Seed size strongly affects cascades on random networks," Phys. Rev. E, 2007. [6]
- ▶ Developed further by Gleeson in "Cascades on correlated and modular random networks," Phys. Rev. E, 2008. [5]

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# Expected size of spread

## Idea:

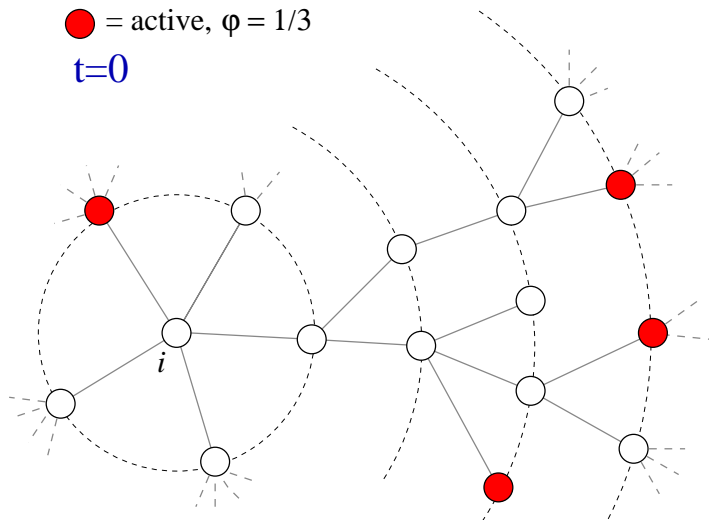
- ▶ Randomly turn on a fraction  $\phi_0$  of nodes at time  $t = 0$
- ▶ Capitalize on local branching network structure of random networks (again)
- ▶ Now think about what must happen for a specific node  $i$  to become active at time  $t$ :
  - $t = 0$ :  $i$  is one of the seeds (prob =  $\phi_0$ )
  - $t = 1$ :  $i$  was not a seed but enough of  $i$ 's friends switched on at time  $t = 0$  so that  $i$ 's threshold is now exceeded.
  - $t = 2$ : enough of  $i$ 's friends and friends-of-friends switched on at time  $t = 0$  so that  $i$ 's threshold is now exceeded.
  - $t = n$ : enough nodes within  $n$  hops of  $i$  switched on at  $t = 0$  and their effects have propagated to reach  $i$ .

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# Expected size of spread

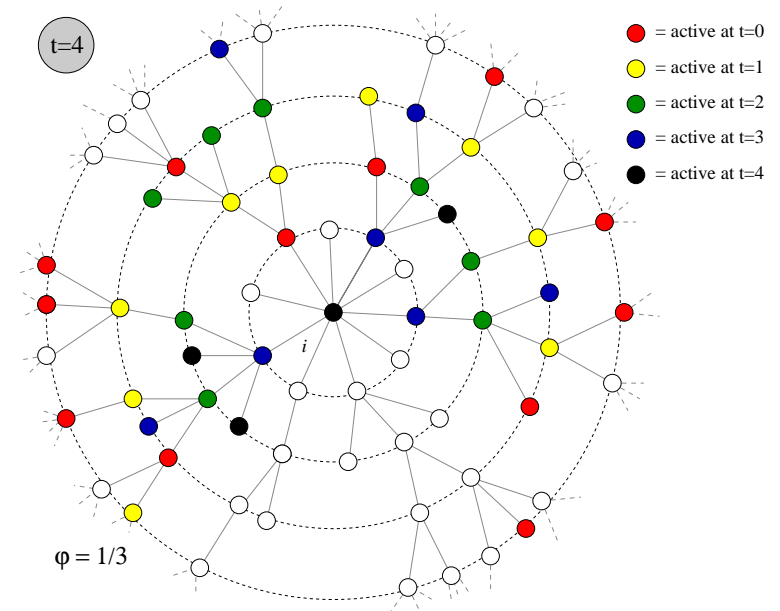


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# Expected size of spread



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## Expected size of spread

### Notes:

- ▶ Calculations are possible nodes do not become inactive.
- ▶ Not just for threshold model—works for a wide range of contagion processes.
- ▶ We can analytically determine the entire time evolution, not just the final size.
- ▶ We can in fact determine  $\Pr(\text{node of degree } k \text{ switches on at time } t)$ .
- ▶ Asynchronous updating can be handled too.

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## Expected size of spread

### Pleasantness:

- ▶ Taking off from a single seed story is about **expansion** away from a node.
- ▶ Extent of spreading story is about **contraction** at a node.

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## Expected size of spread

- ▶ **Notation:**  $\Pr(\text{node } i \text{ becomes active at time } t) = \phi_{i,t}$ .
- ▶ **Notation:**  $\beta_{kj} = \Pr(\text{a degree } k \text{ node becomes active if } j \text{ neighbors are active})$ .
- ▶ Our starting point:  $\phi_{i,0} = \phi_0$ .
- ▶  $\binom{k_i}{j} \phi_0^j (1 - \phi_0)^{k_i - j} = \Pr(j \text{ of node } i\text{'s } k_i \text{ neighbors were seeded at time } t = 0)$ .
- ▶ Probability node  $i$  was a seed at  $t = 0$  is  $\phi_0$  (as above).
- ▶ Probability node  $i$  was not a seed at  $t = 0$  is  $(1 - \phi_0)$ .
- ▶ Combining everything, we have:

$$\phi_{i,1} = \phi_0 + (1 - \phi_0) \sum_{j=0}^{k_i} \binom{k_i}{j} \phi_0^j (1 - \phi_0)^{k_i - j} \beta_{k_i j}$$

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## Expected size of spread

- ▶ For general  $t$ , we need to know the probability an edge coming into node  $i$  at time  $t$  is active.
- ▶ **Notation:** call this probability  $\theta_t$ .
- ▶ We already know  $\theta_0 = \phi_0$ .
- ▶ Story analogous to  $t = 1$  case:

$$\phi_{i,t+1} = \phi_0 + (1 - \phi_0) \sum_{j=0}^{k_i} \binom{k_i}{j} \theta_t^j (1 - \theta_t)^{k_i - j} \beta_{k_i j}$$

- ▶ Average over all nodes to obtain expression for  $\phi_{t+1}$ :

$$\phi_{t+1} = \phi_0 + (1 - \phi_0) \sum_{k=0}^{\infty} P_k \sum_{j=0}^k \binom{k}{j} \theta_t^j (1 - \theta_t)^{k - j} \beta_{kj}$$

- ▶ So we need to compute  $\theta_t \dots$  massive excitement...

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## Expected size of spread

First connect  $\theta_0$  to  $\theta_1$ :

▶  $\theta_1 = \phi_0 +$

$$(1 - \phi_0) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_0^j (1 - \theta_0)^{k-1-j} \beta_{kj}$$

- ▶  $\frac{kP_k}{\langle k \rangle} = R_k = \mathbf{Pr}$  (edge connects to a degree  $k$  node).
- ▶  $\sum_{j=0}^{k-1}$  piece gives  $\mathbf{Pr}$ (degree node  $k$  activates) of its neighbors  $k - 1$  incoming neighbors are active.
- ▶  $\phi_0$  and  $(1 - \phi_0)$  terms account for state of node at time  $t = 0$ .
- ▶ See this all generalizes to give  $\theta_{t+1}$  in terms of  $\theta_t$ ...

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## Expected size of spread

Two pieces:

1.  $\theta_{t+1} = \phi_0 +$

$$(1 - \phi_0) \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_t^j (1 - \theta_t)^{k-1-j} \beta_{kj}$$

with  $\theta_0 = \phi_0$ .

2.  $\phi_{t+1} = \phi_0 +$

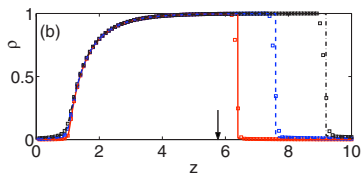
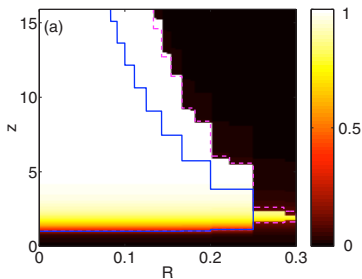
$$(1 - \phi_0) \sum_{k=0}^{\infty} P_k \sum_{j=0}^k \binom{k}{j} \theta_t^j (1 - \theta_t)^{k-j} \beta_{kj}$$

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## Comparison between theory and simulations



From Gleeson and Cahalane [6]

- ▶ Pure random networks with simple threshold responses
- ▶  $R =$  uniform threshold (our  $\phi_*$ );  $z =$  average degree;  $\rho = \phi$ ;  $q = \theta$ ;  $N = 10^5$ .
- ▶  $\phi_0 = 10^{-3}$ ,  $0.5 \times 10^{-2}$ , and  $10^{-2}$ .
- ▶ Cascade window is for  $\phi = 10^{-2}$  case.
- ▶ Sensible expansion of cascade window as  $\phi_0$  increases.

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## Notes:

- ▶ Retrieve cascade condition for spreading from a single seed in limit  $\phi_0 \rightarrow 0$ .
- ▶ Depends on map  $\theta_{t+1} = G(\theta_t; \phi_0)$ .
- ▶ First: if self-starters are present, some activation is assured:

$$G(0; \phi_0) = \sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \beta_{k0} > 0.$$

meaning  $\beta_{k0} > 0$  for at least one value of  $k \geq 1$ .

- ▶ If  $\theta = 0$  is a fixed point of  $G$  (i.e.,  $G(0; \phi_0) = 0$ ) then spreading occurs if

$$G'(0; \phi_0) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k - 1) k P_k \beta_{k1} > 1.$$

Insert question from assignment 6 (田)

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## Notes:

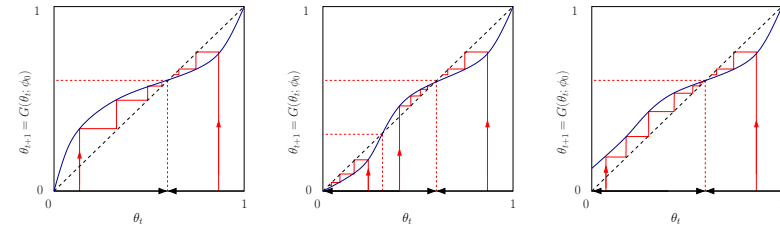
### In words:

- ▶ If  $G(0; \phi_0) > 0$ , spreading must occur because some nodes turn on for free.
- ▶ If  $G$  has an **unstable fixed point** at  $\theta = 0$ , then cascades are also always possible.

### Non-vanishing seed case:

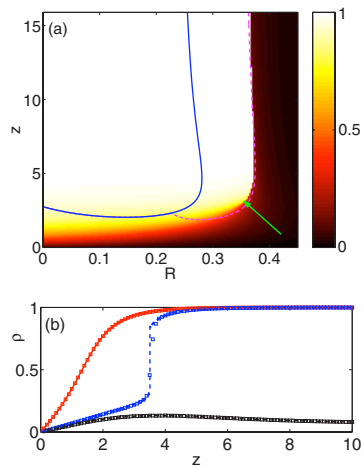
- ▶ Cascade condition is more complicated for  $\phi_0 > 0$ .
- ▶ If  $G$  has a **stable fixed point** at  $\theta = 0$ , and an **unstable fixed point** for some  $0 < \theta_* < 1$ , then for  $\theta_0 > \theta_*$ , spreading takes off.
- ▶ Tricky point:  $G$  depends on  $\phi_0$ , so as we change  $\phi_0$ , we also change  $G$ .

## General fixed point story:



- ▶ Given  $\theta_0 (= \phi_0)$ ,  $\theta_\infty$  will be the nearest stable fixed point, either above or below.
- ▶ n.b., adjacent fixed points must have opposite stability types.
- ▶ **Important:** Actual form of  $G$  depends on  $\phi_0$ .
- ▶ So choice of  $\phi_0$  dictates both  $G$  and starting point—can't start anywhere for a given  $G$ .

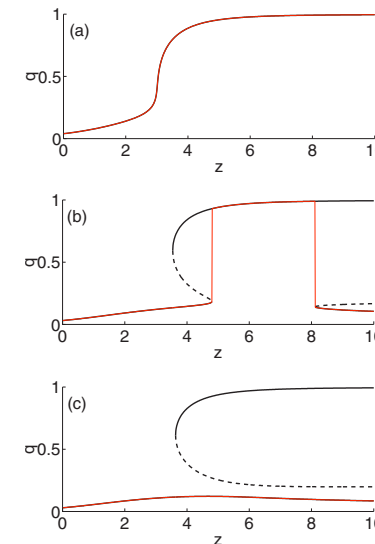
## Comparison between theory and simulations



- ▶ Now allow thresholds to be distributed according to a Gaussian with mean  $R$ .
- ▶  $R = 0.2, 0.362$ , and  $0.38$ ;  $\sigma = 0.2$ .
- ▶  $\phi_0 = 0$  but some nodes have thresholds  $\leq 0$  so effectively  $\phi_0 > 0$ .
- ▶ Now see a (nasty) discontinuous phase transition for low  $\langle k \rangle$ .

From Gleeson and Cahalane<sup>[6]</sup>

## Comparison between theory and simulations



- ▶ Plots of stability points for  $\theta_{t+1} = G(\theta_t; \phi_0)$ .
- ▶ n.b.: 0 is not a fixed point here:  $\theta_0 = 0$  always takes off.
- ▶ Top to bottom:  $R = 0.35, 0.371$ , and  $0.375$ .
- ▶ n.b.: higher values of  $\theta_0$  for (b) and (c) lead to higher fixed points of  $G$ .
- ▶ Saddle node bifurcations appear and merge (b and c).

From Gleeson and Cahalane<sup>[6]</sup>

# Spreadarama

## Bridging to single seed case:

- ▶ Consider largest vulnerable component as initial set of seeds.
- ▶ Not quite right as spreading must move through vulnerables.
- ▶ But we can usefully think of the vulnerable component as activating at time  $t = 0$  because order doesn't matter.
- ▶ Rebuild  $\phi_t$  and  $\theta_t$  expressions...

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# Spreadarama

## Two pieces modified for single seed:

1.  $\theta_{t+1} = \theta_{\text{vuln}} +$

$$(1 - \theta_{\text{vuln}}) \sum_{k=1}^{\infty} \frac{k P_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} \theta_t^j (1 - \theta_t)^{k-1-j} \beta_{kj}$$

with  $\theta_0 = \theta_{\text{vuln}} = \mathbf{Pr}$  an edge leads to the giant vulnerable component (if it exists).

2.  $\phi_{t+1} = S_{\text{vuln}} +$

$$(1 - S_{\text{vuln}}) \sum_{k=0}^{\infty} P_k \sum_{j=0}^k \binom{k}{j} \theta_t^j (1 - \theta_t)^{k-j} \beta_{kj}$$

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# Time-dependent solutions

## Synchronous update

- ▶ Done: Evolution of  $\phi_t$  and  $\theta_t$  given exactly by the maps we have derived.

## Asynchronous updates

- ▶ Update nodes with probability  $\alpha$ .
- ▶ As  $\alpha \rightarrow 0$ , updates become effectively independent.
- ▶ Now can talk about  $\phi(t)$  and  $\theta(t)$ .
- ▶ More on this later...

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