Measures of centrality

Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

How big is my node?

- Basic question: how ‘important’ are specific nodes and edges in a network?
- An important node or edge might:
  1. handle a relatively large amount of the network’s traffic (e.g., cars, information);
  2. bridge two or more distinct groups (e.g., liason, interpreter);
  3. be a source of important ideas, knowledge, or judgments (e.g., supreme court decisions, an employee who ‘knows where everything is’).
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- We generate ad hoc, reasonable measures, and examine their utility...

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Centrality

- One possible reflection of importance is centrality.
- Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network's function.
- Idea of centrality comes from social networks literature [7].
- Many flavors of centrality...
  1. Many are topological and quasi-dynamical;
  2. Some are based on dynamics (e.g., traffic).
- We will define and examine a few...
- (Later: see centrality useful in identifying communities in networks.)

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Degree centrality

- Naively estimate importance by node degree. [7]
- Doh: assumes linearity
  (If node *i* has twice as many friends as node *j*, it's twice as important.)
- Doh: doesn't take in any non-local information.

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Closeness centrality

- Idea: Nodes are more central if they can reach other nodes 'easily.'
- Measure average shortest path from a node to all other nodes.
- Define Closeness Centrality for node i as

$$\frac{N-1}{\sum_{j,j\neq i}(\text{distance from } i \text{ to } j).}$$

- Range is 0 (no friends) to 1 (single hub).
- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of 'importance.'

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Photosynth (⊞).

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Betweenness centrality

- Betweenness centrality is based on shortest paths in a network.
- Idea: If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are 'important' in terms of global cohesion.
- For each node i, count how many shortest paths pass through i.
- In the case of ties, or divide counts between paths.
- Call frequency of shortest paths passing through node i the betweenness of i, B_i.
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- Note: works for weighted and unweighted networks.

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- Computational goal: Find $\binom{N}{2}$ shortest paths (⊞) between all pairs of nodes.
- Traditionally use Floyd-Warshall (⊞) algorithm.
- Computation time grows as $O(N^3)$.
- See also:
  1. Dijkstra's algorithm (⊞) for finding shortest path between two specific nodes,
  2. and Johnson's algorithm (⊞) which outperforms Floyd-Warshall for sparse networks: $O(mN + N^2 \log N)$.
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Shortest path between node i and all others:

- Consider unweighted networks.
- Use breadth-first search:
  1. Start at node i, giving it a distance $d = 0$ from itself.
  2. Create a list of all of i's neighbors and label them being at a distance $d = 1$.
  3. Go through list of most recently visited nodes and find all of their neighbors.
  4. Exclude any nodes already assigned a distance.
  5. Increment distance d by 1.
  6. Label newly reached nodes as being at distance d.
  7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from i (former are 'predecessors' with respect to i's shortest path structure).
- Runs in $O(m)$ time and gives N shortest paths.
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  2. Create a list of all of i's neighbors and label them being at a distance $d = 1$.
  3. Go through list of most recently visited nodes and find all of their neighbors.
  4. Exclude any nodes already assigned a distance.
  5. Increment distance d by 1.
  6. Label newly reached nodes as being at distance d.
  7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from i (former are 'predecessors' with respect to i's shortest path structure).
- Runs in $O(m)$ time and gives N shortest paths.
- Find all shortest paths in $O(mN)$ time
- Much, much better than naive estimate of $O(mN^2)$.

Shortest path between node i and all others:

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Newman's Betweenness algorithm: [4]

1. Set all nodes to have a value $c_{ij} = 0$, $j = 1, ..., N$ (c for count).
2. Select one node i.
3. Find shortest paths to all other $N - 1$ nodes using breadth-first search.
4. Record # equal shortest paths reaching each node.
5. Move through nodes according to their distance from i, starting with the furthest.
6. Travel back towards i from each starting node j, along shortest path(s), adding 1 to every value of $c_{i\ell}$ at each node ℓ along the way.
7. Whenever more than one possibility exists, apportion according to total number of short paths coming through predecessors.
8. Exclude starting node j and i from increment.
9. Repeat steps 2–8 for every node i and obtain betweenness as $B_j = \sum_{i=1}^{N} c_{ij}$.

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- For a pure tree network, c_{ij} is the number of nodes beyond j from i's vantage point.
- Same algorithm for computing drainage area in river networks (with 1 added across the board).
- For edge betweenness, use exact same algorithm but now
  1. j indexes edges,
  2. and we add one to each edge as we traverse it.
- For both algorithms, computation time grows as

$$O(mN).$$

- For sparse networks with relatively small average degree, we have a fairly digestible time growth of

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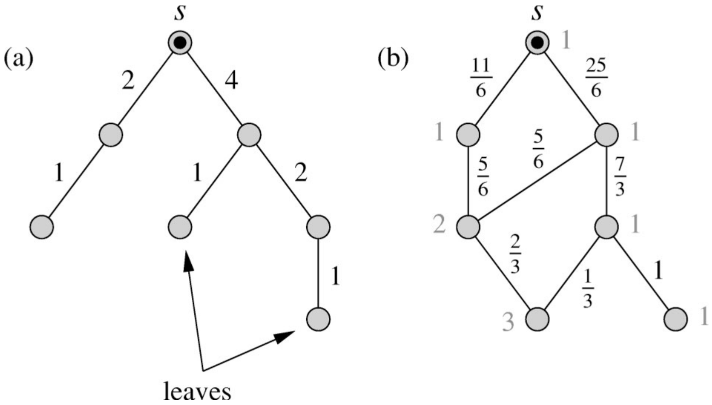

Outline

Important nodes have important friends:

- Define x_i as the 'importance' of node i.
- Idea: x_i depends (somehow) on x_j
  if j is a neighbor of i.
- Recursive: importance is transmitted through a network.
- Simplest possibility is a linear combination:

$$x_i \propto \sum_j a_{ji} x_j$$

- Assume further that constant of proportionality, c, is independent of i.
- Above gives $\vec{x} = c\mathbf{A}^{\mathrm{T}}\vec{x}$ or $\boxed{\mathbf{A}^{\mathrm{T}}\vec{x} = c^{-1}\vec{x} = \lambda\vec{x}}$.
- Eigenvalue equation based on adjacency matrix...
- Note: Lots of despair over size of the largest eigenvalue. [7] Lose sight of original assumption's non-physicality.

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- But which eigenvalue and eigenvector?
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  2. λ to be real.
  3. Entries of $\vec{x}$ to be real.
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  5. λ to actually mean something...
  6. Values of x_i to mean something
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Perron-Frobenius theorem: (⊞)

If an $N \times N$ matrix A has non-negative entries then:

1. A has a real eigenvalue $\lambda_1 \geq |\lambda_i|$ for $i = 2, \ldots, N$.
2. λ_1 corresponds to left and right 1-d eigenspaces for which we can choose a basis vector that has non-negative entries.
3. The dominant real eigenvalue λ_1 is bounded by the minimum and maximum row sums of A:

$$\min_i \sum_{j=1}^{N} a_{ij} \leq \lambda_1 \leq \max_i \sum_{j=1}^{N} a_{ij}$$

4. All other eigenvectors have one or more negative entries.

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- Assuming our network is irreducible (⊞), meaning there is only one component, is reasonable: just consider one component at a time if more than one exists.
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Big pumpkins (⊞)

Outline

Background

Centrality measures

- Degree centrality
- Closeness centrality
- Betweenness
- Eigenvalue centrality
- Hubs and Authorities

References

Hubs and Authorities

- Generalize eigenvalue centrality to allow nodes to have two attributes:
  1. Authority: how much knowledge, information, etc., held by a node on a topic.
  2. Hubness (or Hubosity or Hubbishness): how well a node 'knows' where to find information on a given topic.
- Original work due to the legendary Jon Kleinberg. [2]
- Best hubs point to best authorities.
- Recursive: nodes can be both hubs and authorities.
- More: look for dense links between sets of good hubs pointing to sets of good authorities.
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where c_1 and c_2 must be positive.

- Above equations combine to give

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We can do this:

- $A^T A$ is symmetric.
- $A^T A$ is semi-positive definite so its eigenvalues are all ≥ 0.
- $A^T A$'s eigenvalues are the square of A's singular values.
- $A^T A$'s eigenvectors form a joyful orthogonal basis.
- Perron-Frobenius tells us that only the dominant eigenvalue's eigenvector can be chosen to have non-negative entries.
- So: linear assumption leads to a solvable system.
- What would be very good: find networks where we have independent measures of node 'importance' and see how importance is actually distributed.

We can do this:

- $A^T A$ is symmetric.
- $A^T A$ is semi-positive definite so its eigenvalues are all ≥ 0.
- $A^T A$'s eigenvalues are the square of A's singular values.
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