Optimal Supply Networks

Complex Networks, Course 303A, Spring, 2009

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Outline

Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people,
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems

River network models

Optimality:

- Optimal channel networks [5]
- Thermodynamic analogy [6]

versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

Optimization approaches

Cardiovascular networks:

- Murray's law (1926) connects branch radii at forks: [4]

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

  where r_0 = radius of main branch
  and r_1 and r_2 are radii of sub-branches
- Calculation assumes Poiseuille flow
- Holds up well for outer branchings of blood networks
- Also found to hold for trees
- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

  where Δp = pressure difference, Φ = flux

Optimization approaches

Cardiovascular networks:

- Fluid mechanics: Poiseuille impedance for smooth flow in a tube of radius r and length ℓ:

$$Z = \frac{8\eta\ell}{\pi r^4}$$

  where η = dynamic viscosity

- Power required to overcome impedance:

$$P_{\mathrm{drag}} = \Phi \Delta p = \Phi^2 Z$$

- Also have rate of energy expenditure in maintaining blood:

$$P_{\mathrm{metabolic}} = c r^2 \ell$$

  where c is a metabolic constant.

Optimization approaches

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = rate work is done = $F \cdot v$
- ΔP = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
- So $\Phi \Delta P$ = Force · velocity

Optimization approaches

Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

Optimization

Murray's law:

- Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.

Optimization

Murray's law:

- So we now have:

$$\Phi = kr^3$$

- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

Optimization

Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$r_\omega^3 = 2 r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1}$...

Optimization

Murray meets Tokunaga:

- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$\boxed{R_r^3 = R_n = R_v}$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Optimization

Murray meets Tokunaga:

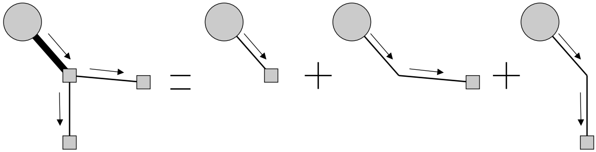
- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$\boxed{R_\ell^3 = R_v = R_n}$$

- We need one more constraint...
- West et al (1997) [12] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [10] using Tokunaga (sort of).

Geometric argument

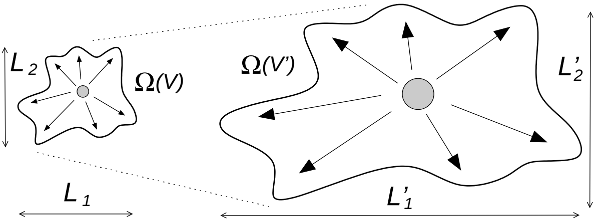
- Consider one source supplying many sinks in a d-dim. volume in a D-dim. ambient space.
- Assume sinks are invariant.
- Assume $\rho = \rho(V)$.
- See network as a bundle of virtual vessels:



- Q: how does the number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- Or: what is the highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

Geometric argument

- Allometrically growing regions:



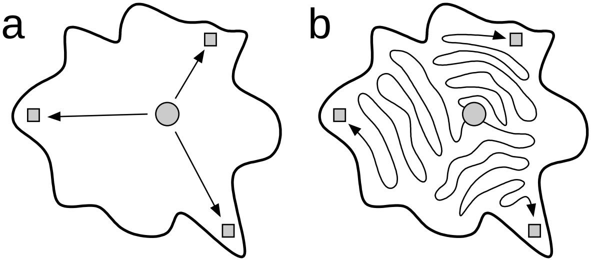
- Have d length scales which scale as

$$L_i \propto V^{\gamma_i} \text{ where } \gamma_1 + \gamma_2 + \ldots + \gamma_d = 1.$$

- For isometric growth, $\gamma_i = 1/d$.
- For allometric growth, we must have at least two of the $\{\gamma_i\}$ being different

Geometric argument

- Best and worst configurations (Banavar et al.)



- Rather obviously:
  min $V_{\mathrm{net}} \propto \sum$ distances from source to sinks.

Minimal network volume:

Real supply networks are close to optimal:

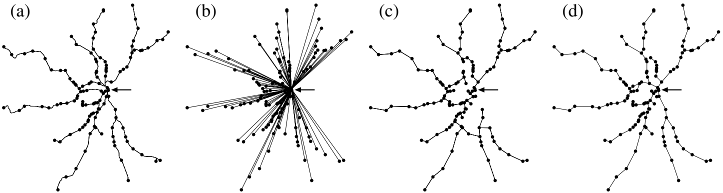


Figure 1. (a) Commuter rail network in the Boston area. The arrow marks the assumed root of the network. (b) Star graph. (c) Minimum spanning tree. (d) The model of equation (3) applied to the same set of stations.

(2006) Gastner and Newman [3]: "Shape and efficiency in spatial distribution networks"

Minimal network volume:

Add one more element:

- Vessel cross-sectional area may vary with distance from the source.
- Flow rate increases as cross-sectional area decreases.
- e.g., a collection network may have vessels tapering as they approach the central sink.
- Find that vessel volume v must scale with vessel length ℓ to affect overall system scalings.
- Consider vessel radius $r \propto (\ell + 1)^{-\epsilon}$, tapering from $r = r_{\max}$ where $\epsilon \geq 0$.
- Gives $v \propto \ell^{1-2\epsilon}$ if $\epsilon < 1/2$
- Gives $v \propto 1 - \ell^{-(2\epsilon-1)} \to 1$ for large ℓ if $\epsilon > 1/2$
- Previously, we looked at $\epsilon = 0$ only.

Minimal network volume:

For $0 \leq \epsilon < 1/2$, approximate network volume by integral over region:

$$\min V_{\text{net}} \propto \int_{\Omega_{d,D}(V)} \rho \, ||\vec{x}||^{1-2\epsilon} \, \mathrm{d}\vec{x}$$

Insert question from assignment 2 (⊞)

$$\propto \rho V^{1+\gamma_{\max}(1-2\epsilon)}$$

For $\epsilon > 1/2$, find simply that

$$\min V_{\text{net}} \propto \rho V$$

- So if supply lines can taper fast enough and without limit, minimum network volume can be made negligible.

Geometric argument

For $0 \leq \epsilon < 1/2$:

- $\boxed{\min V_{\text{net}} \propto \rho V^{1+\gamma_{\max}(1-2\epsilon)}}$
- If scaling is isometric, we have $\gamma_{\max} = 1/d$:
$$\min V_{\text{net/iso}} \propto \rho V^{1+(1-2\epsilon)/d}$$
- If scaling is allometric, we have $\gamma_{\max} = \gamma_{\text{allo}} > 1/d$: and
$$\min V_{\text{net/allo}} \propto \rho V^{1+(1-2\epsilon)\gamma_{\text{allo}}}$$
- Isometrically growing volumes require less network volume than allometrically growing volumes:
$$\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \to 0 \text{ as } V \to \infty$$

Geometric argument

For $0 \leq \epsilon < 1/2$:

- $\boxed{\min V_{\text{net}} \propto \rho V}$
- Network volume scaling is now independent of overall shape scaling.

Limits to scaling

- Can argue that ϵ must effectively be 0 for real networks over large enough scales.
- Limit to how fast material can move, and how small material packages can be.
- e.g., blood velocity and blood cell size.

Blood networks

- Velocity at capillaries and aorta approximately constant across body size [11]: $\epsilon = 0$.
- Material costly $\Rightarrow$ expect lower optimal bound of $V_{\mathrm{net}} \propto \rho V^{(d+1)/d}$ to be followed closely.
- For cardiovascular networks, $d = D = 3$.
- Blood volume scales linearly with blood volume [7], $V_{\mathrm{net}} \propto V$.
- Sink density must $\therefore$ decrease as volume increases:

$$\rho \propto V^{-1/d}.$$

- Density of suppliable sinks decreases with organism size.

Blood networks

- Then P, the rate of overall energy use in Ω, can at most scale with volume as

$$P \propto \rho V \propto \rho M \propto M^{(d-1)/d}$$

- For $d = 3$ dimensional organisms, we have

$$\boxed{P \propto M^{2/3}}$$

- Including other constraints may raise scaling exponent to a higher, less efficient value.

Recap:

- The exponent $\alpha = 2/3$ works for all birds and mammals up to 10–30 kg
- For mammals $>$ 10–30 kg, maybe we have a new scaling regime
- Economos: limb length break in scaling around 20 kg
- White and Seymour, 2005: unhappy with large herbivore measurements. Find $\alpha \simeq 0.686 \pm 0.014$

River networks

- View river networks as collection networks.
- Many sources and one sink.
- ϵ?
- Assume ρ is constant over time and $\epsilon = 0$:

$$V_{\mathrm{net}} \propto \rho V^{(d+1)/d} = \mathrm{constant} \times V^{3/2}$$

- Network volume grows faster than basin 'volume' (really area).
- It's all okay:
  Landscapes are d=2 surfaces living in D=3 dimension.
- Streams can grow not just in width but in depth...
- If $\epsilon > 0$, V_{net} will grow more slowly but 3/2 appears to be confirmed from real data.

Many sources, many sinks

How do we distribute sources?

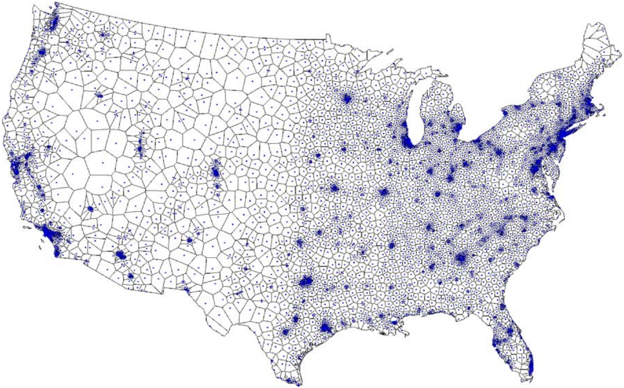
- Focus on 2-d (results generalize to higher dimensions)
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly
- Which lattice is optimal? The hexagonal lattice
  Q1: How big should the hexagons be?
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan [8, 9] and by Gastner and Newman (2006) [2] and work cited by them.

Optimal source allocation

Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

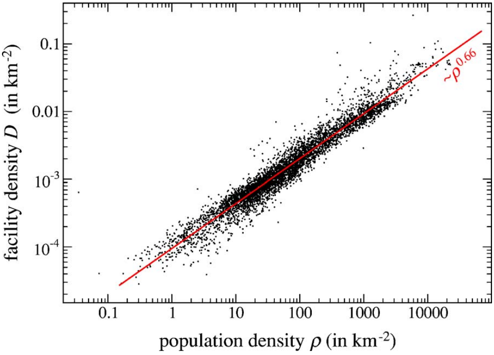
Optimal source allocation



From Gastner and Newman (2006) [2]

- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



From Gastner and Newman (2006) [2]

- Optimal facility density D vs. population density ρ.
- Fit is $D \propto \rho^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power...

Optimal source allocation

Size-density law:

- $$\boxed{D \propto \rho^{2/3}}$$
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region...

Optimal source allocation

- We first examine Stephan's treatment (1977) [8, 9]
- "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- Zipf-like approach: invokes principle of minimal effort.
- Also known as the Homer principle.

Optimal source allocation

- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center is $\bar{d}$ and assume average speed of travel is $\bar{v}$.
- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

Optimal source allocation

- Next assume facility requires regular maintenance (person-hours per day)
- Call this quantity τ
- If burden of mainenance is shared then average cost per person is τ/P.
- Replace P by ρA where ρ is density.
- Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho A) = gA^{1/2}/\bar{v} + \tau/(\rho A).$$

- Now Minimize with respect to A...

Optimal source allocation

- Differentiating...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho A)\right)$$

$$= \frac{c}{2\bar{v}A^{1/2}} - \frac{\tau}{\rho A^2} = 0$$

- Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho}\right)^{2/3} \propto \rho^{-2/3}$$

- # facilities per unit area $\propto$

$$A^{-1} \propto \rho^{2/3}$$

- Groovy...

Optimal source allocation

An issue:

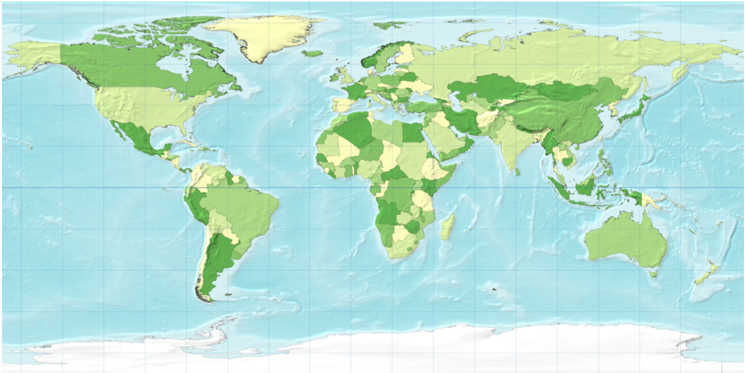
- Maintenance (τ) is assumed to be independent of population and area (P and A)

Optimal source allocation

Stephan's online book
"The Division of Territory in Society" is here (⊞).

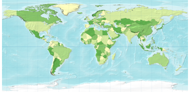
Cartograms

Standard world map:



Cartograms

Cartogram of countries ‘rescaled’ by population:



Cartograms

Diffusion-based cartograms:

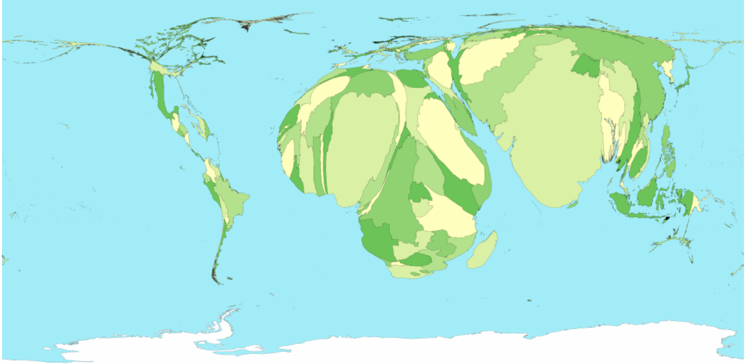
- Idea of cartograms is to distort areas to more accurately represent some local density ρ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [1] is based on standard diffusion:

$$\nabla^2 \rho - \frac{\partial \rho}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
- Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}$.

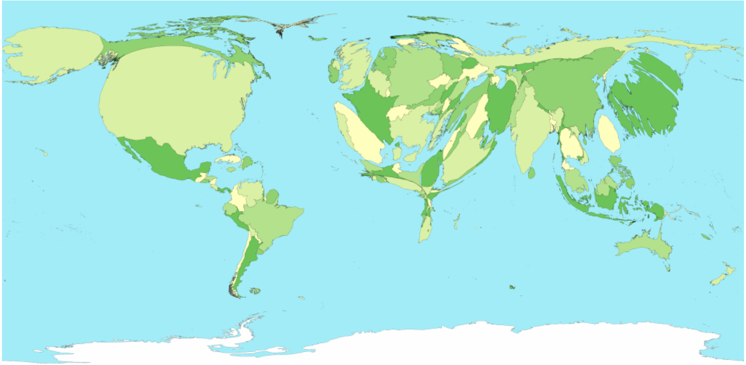
Cartograms

Child mortality:



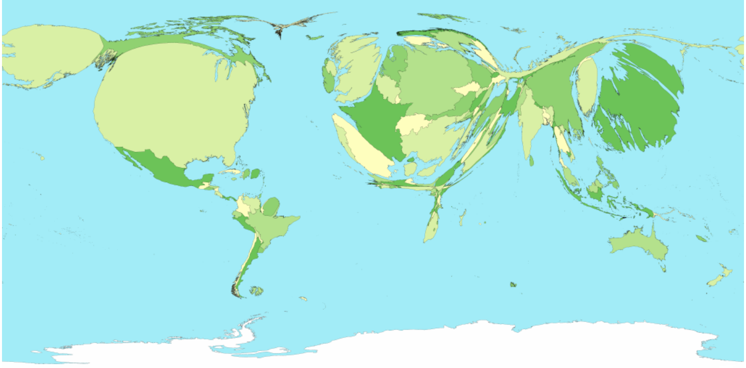
Cartograms

Energy consumption:



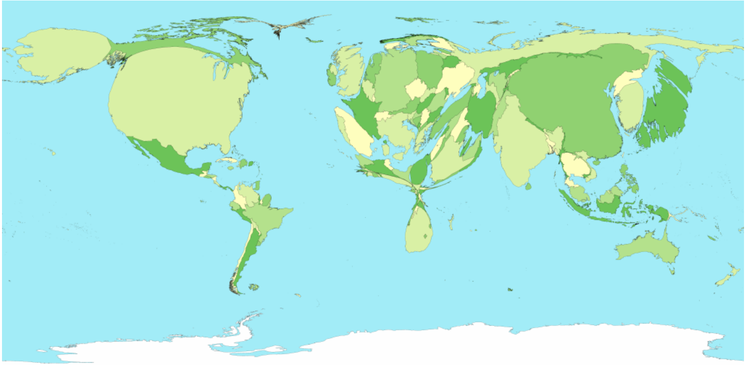
Cartograms

Gross domestic product:



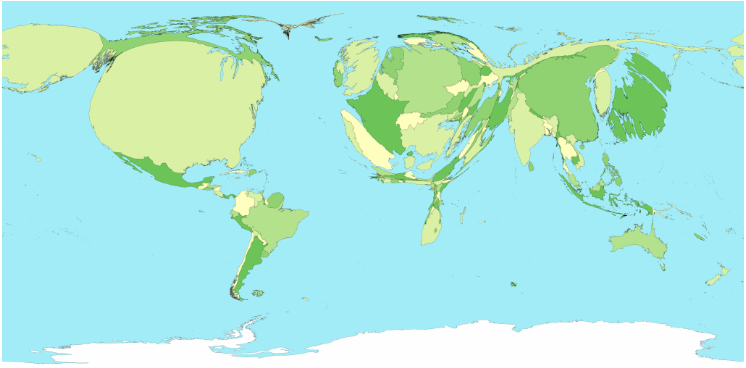
Cartograms

Greenhouse gas emissions:



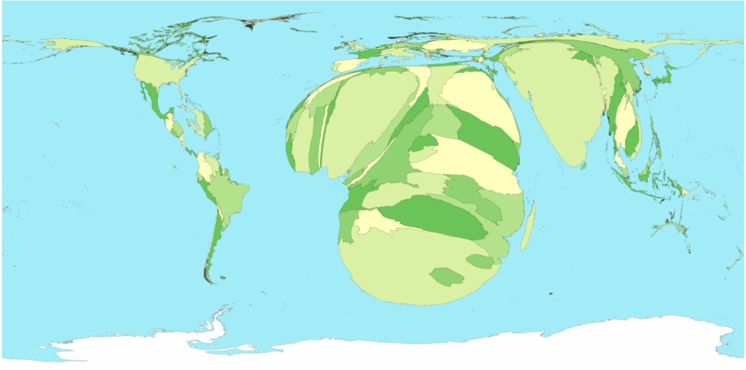
Cartograms

Spending on healthcare:



Cartograms

People living with HIV:

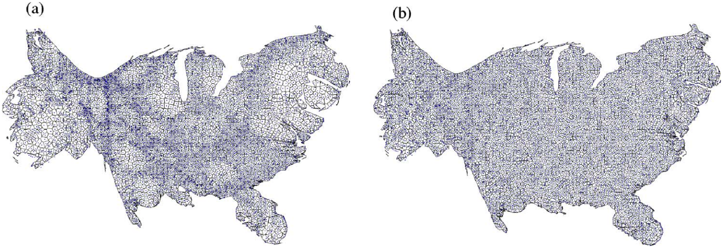


Cartograms

- The preceding sampling of Gastner & Newman's cartograms lives here (⊞).
- A larger collection can be found at worldmapper.org (⊞).

WORLDMAPPER The world as you've never seen it before

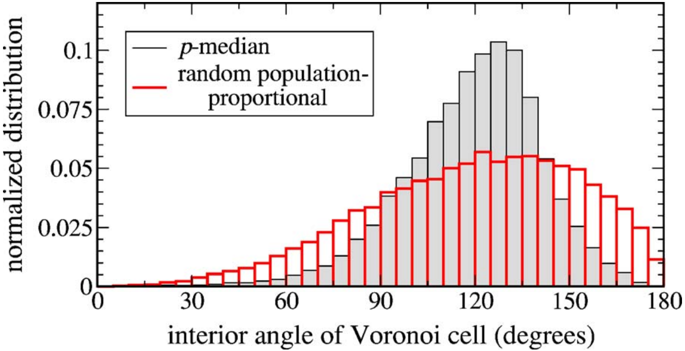
Size-density law



- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [2]

Size-density law



From Gastner and Newman (2006) [2]

- Cartogram's Voronoi cells are somewhat hexagonal.

Size-density law

Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility. [1]
- Assume given a fixed population density ρ defined on a spatial region Ω.
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \ldots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \ldots, \vec{x}_n\}) = \int_\Omega \rho(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x}\,.$$

- Also known as the p-median problem.
- Not easy... in fact this one is an NP-hard problem. [1]

Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (⊞), one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

  where c_i is a shape factor for the ith Voronoi cell.
- Approximate c_i as a constant c.

Size-density law

Carrying on:

- The cost function is now
$$F = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x}\,.$$
- We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:
$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$
- Within each cell, $A(\vec{x})$ is constant.
- So... integral over each of the n cells equals 1.

Size-density law

Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_{\Omega} [A(\vec{x})]^{-1} \mathrm{d}\vec{x} \right)$$

- Next compute $\delta G / \delta A$, the functional derivative (⊞) of the functional $G(A)$.
- This gives

$$\int_{\Omega} \left[\frac{c}{2} \rho(\vec{x}) A(\vec{x})^{-1/2} - \lambda \left[A(\vec{x}) \right]^{-2} \right] \mathrm{d}\vec{x} = 0.$$

- Setting the integrand to be zilch, we have:

$$\rho(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

Size-density law

Now a Lagrange multiplier story:

- Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3}\rho^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $D(\vec{x})$, an approximation of the local source density.
- Substituting $D = 1/A$, we have

$$D(\vec{x}) = \left(\frac{c}{2\lambda}\rho\right)^{2/3}.$$

- Normalizing (or solving for λ):

$$D(\vec{x}) = n\frac{[\rho(\vec{x})]^{2/3}}{\int_{\Omega}[\rho(\vec{x})]^{2/3}\mathrm{d}\vec{x}} \propto [\rho(\vec{x})]^{2/3}.$$

Global redistribution networks

One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gaster and Newman model: cost is a function of basic maintenance and travel time:

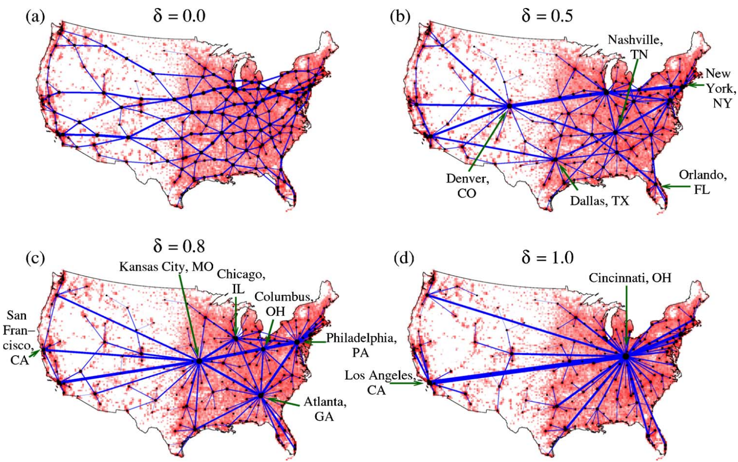
$$C_{\mathrm{maint}} + \gamma C_{\mathrm{travel}}.$$

- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1 - \delta)\ell_{ij} + \delta(\#\mathrm{hops}).$$

- When $\delta = 1$, only number of hops matters.

Global redistribution networks



From Gastner and Newman (2006) [2]

References I

[1] M. T. Gastner and M. E. J. Newman.
Diffusion-based method for producing density-equalizing maps.
Proc. Natl. Acad. Sci., 101:7499–7504, 2004. pdf (⊞)

[2] M. T. Gastner and M. E. J. Newman.
Optimal design of spatial distribution networks.
Phys. Rev. E, 74:Article # 016117, 2006. pdf (⊞)

[3] M. T. Gastner and M. E. J. Newman.
Shape and efficiency in spatial distribution networks.
J. Stat. Mech.: Theor. & Exp., 1:01015–, 2006. pdf (⊞)

[4] C. D. Murray.
The physiological principle of minimum work. I. The vascular system and the cost of blood volume.
Proc. Natl. Acad. Sci. U.S.A, 12:207–214, 1926.

References II

[5] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.

[6] A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.

[7] W. R. Stahl.
Scaling of respiratory variables in mammals.
Journal of Applied Physiology, 22:453–460, 1967.

[8] G. E. Stephan.
Territorial division: The least-time constraint behind the formation of subnational boundaries.
Science, 196:523–524, 1977. pdf (⊞)

References III

[9] G. E. Stephan.
Territorial subdivision.
Social Forces, 63:145–159, 1984. pdf (⊞)

[10] D. L. Turcotte, J. D. Pelletier, and W. I. Newman.
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998.

[11] P. D. Weinberg and C. R. Ethier.
Twenty-fold difference in hemodynamic wall shear stress between murine and human aortas.
Journal of Biomechanics, 40(7):1594–1598, 2007. pdf (⊞)

[12] G. B. West, J. H. Brown, and B. J. Enquist.
A general model for the origin of allometric scaling laws in biology.
Science, 276:122–126, 1997. pdf (⊞)