Branching Networks I

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Outline

Introduction

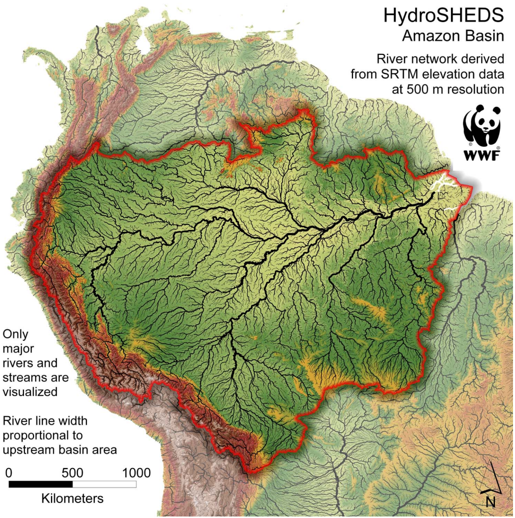
Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

Branching networks are everywhere...



http://hydrosheds.cr.usgs.gov/ (⊞)

Branching networks are everywhere...

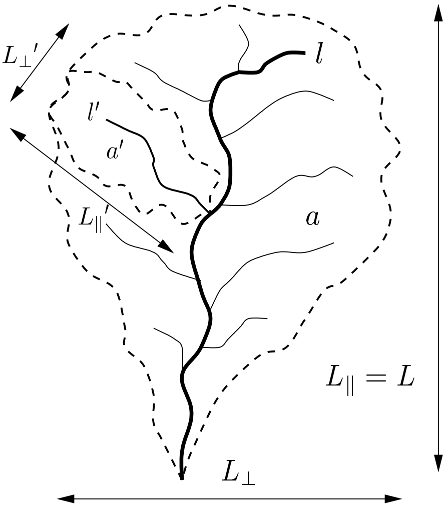
http://en.wikipedia.org/wiki/Image:Applebox.JPG (⊞)

Geomorphological networks

Definitions

- Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...

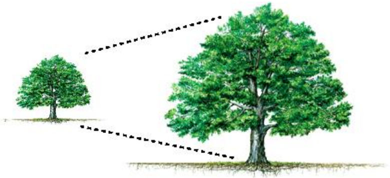
Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:

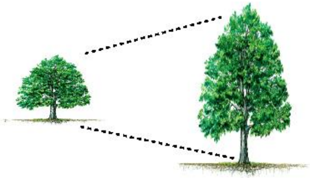


- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)
- $L = L_{\parallel}$ = longitudinal length of basin
- $L = L_{\perp}$ = width of basin

Allometry

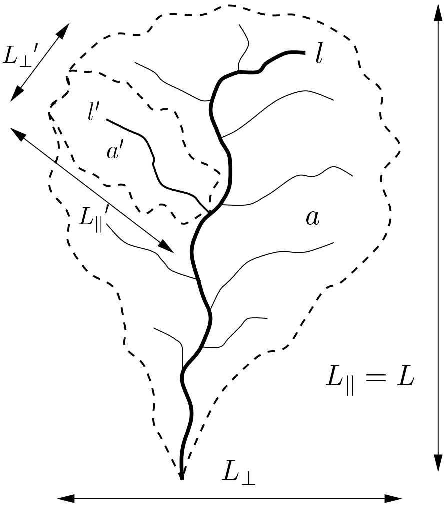
Isometry: dimensions scale linearly with each other.





Allometry: dimensions scale nonlinearly.

Basin allometry



Allometric relationships:

- $$\ell \propto a^h$$
- $$\ell \propto L^d$$
- Combine above:

$$a \propto L^{d/h} \equiv L^D$$

'Laws'

- Hack's law (1957) [2]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$

- Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^{d}$$

reportedly $1.0 < d < 1.1$

- Basin allometry:

$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

$D < 2 \rightarrow$ basins elongate.

There are a few more 'laws': [1]

Relation:	Name or description:
$T_k = T_1(R_T)^k$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law

Reported parameter values:[1]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05

Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...

Stream Ordering:

Method for describing network architecture:

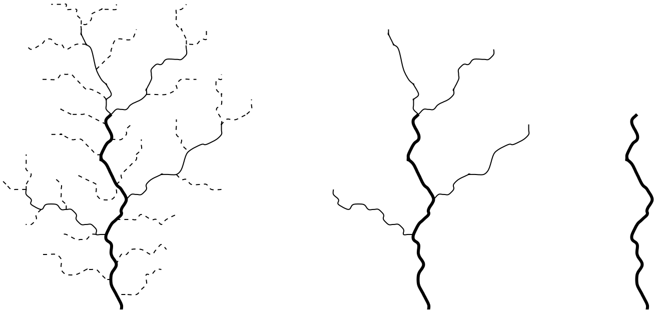
- Introduced by Horton (1945) [3]
- Modified by Strahler (1957) [6]
- Term: Horton-Strahler Stream Ordering [4]
- Can be seen as iterative trimming of a network.

Stream Ordering:

Some definitions:

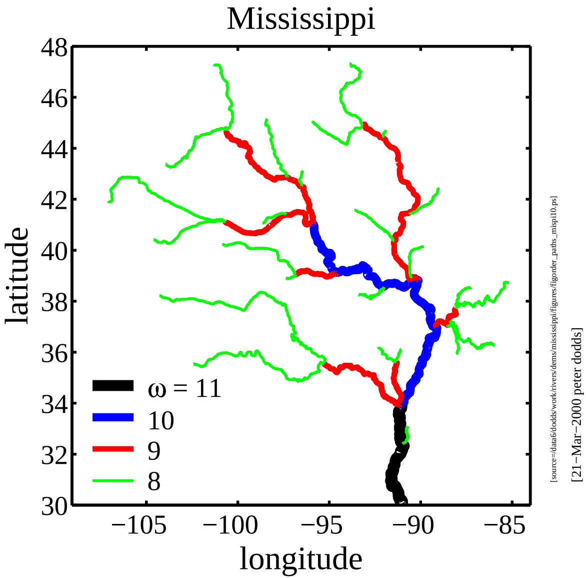
- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.

Stream Ordering:



1. Label all source streams as order $\omega = 1$ and remove.
2. Label all new source streams as order $\omega = 2$ and remove.
3. Repeat until one stream is left (order = Ω)
4. Basin is said to be of the order of the last stream removed.
5. Example above is a basin of order $\Omega = 3$.

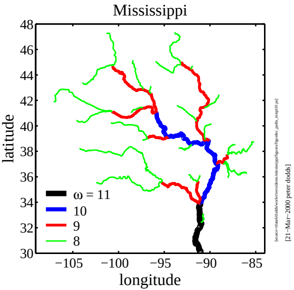
Stream Ordering—A large example:



Stream Ordering:

Another way to define ordering:

- As before, label all source streams as order $\omega = 1$.
- Follow all labelled streams downstream
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).



- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.

Stream Ordering:

One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.

Stream Ordering:

Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture

Stream Ordering:

Resultant definitions:

- A basin of order Ω has n_ω streams (or sub-basins) of order ω.
  - $n_\omega > n_{\omega+1}$
- An order ω basin has area a_ω.
- An order ω basin has a main stream length ℓ_ω.
- An order ω basin has a stream segment length s_ω
  1. an order ω stream segment is only that part of the stream which is actually of order ω
  2. an order ω stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams

Horton's laws

Self-similarity of river networks

- First quantified by Horton (1945) [3], expanded by Schumm (1956) [5]

Three laws:

- Horton's law of stream numbers:

$$\boxed{n_\omega / n_{\omega+1} = R_n > 1}$$

- Horton's law of stream lengths:

$$\boxed{\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell > 1}$$

- Horton's law of basin areas:

$$\boxed{\bar{a}_{\omega+1} / \bar{a}_\omega = R_a > 1}$$

Horton's laws

Horton's Ratios:

- So... Horton's laws are defined by three ratios:

$$R_n, \ R_\ell, \ \text{and} \ R_a.$$

- Horton's laws describe exponential decay or growth:

$$\begin{aligned} n_\omega &= n_{\omega-1}/R_n \\ &= n_{\omega-2}/R_n^2 \\ &\vdots \\ &= n_1/R_n^{\omega-1} \\ &= n_1 e^{-(\omega-1)\ln R_n} \end{aligned}$$

Horton's laws

Similar story for area and length:

- $$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$
- $$\bar{\ell}_\omega = \bar{\ell}_1 e^{(\omega-1)\ln R_\ell}$$
- As stream order increases, number drops and area and length increase.

Horton's laws

A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.
- Horton's ratios go a long way to defining a branching network...
- But we need one other piece of information...

Horton's laws

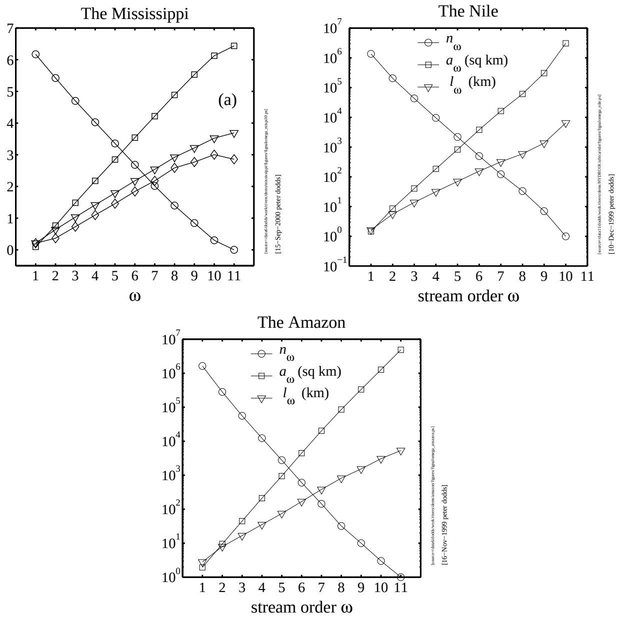
A bonus law:

- Horton's law of stream segment lengths:

$$\boxed{\bar{s}_{\omega+1}/\bar{s}_{\omega} = R_s > 1}$$

- Can show that $R_s = R_\ell$.

Horton's laws in the real world:



Horton's laws-at-large

Blood networks:

- Horton's laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton's law.

Horton's laws

Observations:

- Horton's ratios vary:

R_n	3.0–5.0
R_a	3.0–6.0
R_ℓ	1.5–3.0

- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...
- ... but they don't explain how networks are structured.

Tokunaga's law

Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure [7, 8, 9]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
- Tokunaga's law is also a law of averages.

Network Architecture

Definition:

- $T_{\mu,\nu}$ = the average number of side streams of order ν that enter as tributaries to streams of order μ
- μ, ν = 1, 2, 3, ...
- $\mu \geq \nu + 1$
- Recall each stream segment of order μ is 'generated' by two streams of order $\mu - 1$
- These generating streams are not considered side streams.

Network Architecture

Tokunaga's law

- Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

- Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1 (R_T)^{\mu-\nu-1}$$

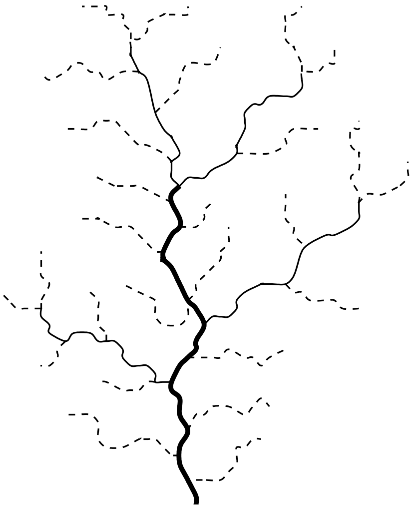
- We usually write Tokunaga's law as:

$$\boxed{T_k = T_1 (R_T)^{k-1}} \quad \text{where } R_T \simeq 2$$

Tokunaga's law—an example:

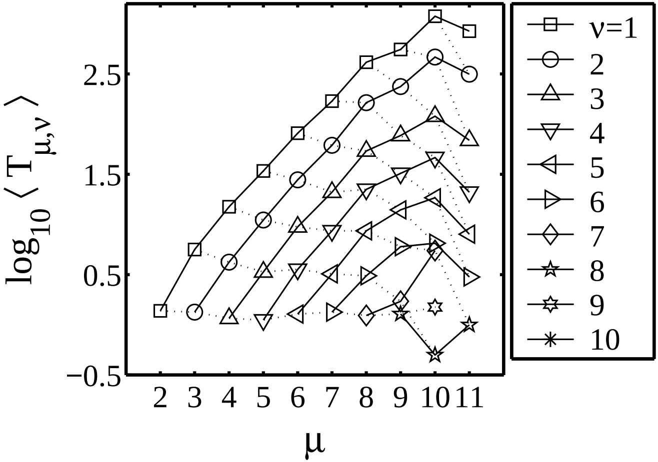
$T_1 \simeq 2$

$R_T \simeq 4$



The Mississippi

A Tokunaga graph:



Nutshell:

Branching networks I:

- Show remarkable self-similarity over many scales.
- There are many interrelated scaling laws.
- Horton-Strahler Stream ordering gives one useful way of getting at the architecture of branching networks.
- Horton's laws reveal self-similarity.
- Horton's laws can be misinterpreted as suggesting a pure hierarchy.
- Tokunaga's laws neatly describe network architecture.
- Branching networks exhibit a mixed hierarchical structure.
- Horton and Tokunaga can be connected analytically (next up).

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