# Chapter 6: Lecture 25

## Linear Algebra, Course 124C, Spring, 2009

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont





## All the way with $A\vec{x} = \vec{b}$:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^{\mathrm{T}}$.

### Where $\vec{x}$ lives:

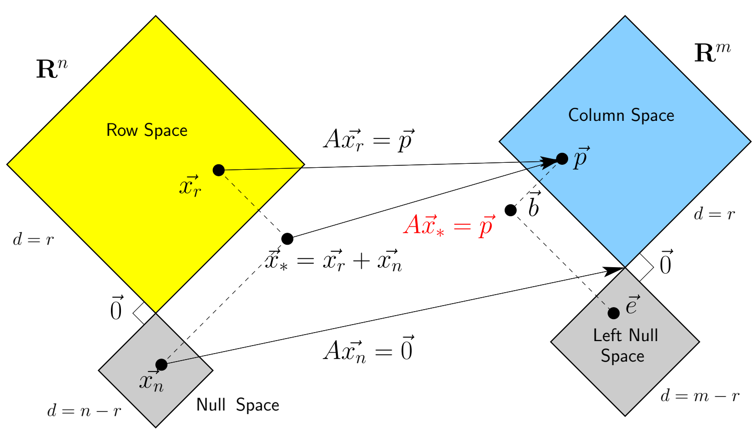
- Row space $C(A^{\mathrm{T}}) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- $\dim C(A^{\mathrm{T}}) + \dim N(A) = r + (n - r) = n$
- Orthogonality: $C(A^{\mathrm{T}}) \bigotimes N(A) = R^n$

### Where $\vec{b}$ lives:

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^{\mathrm{T}}) \subset R^m$.
- $\dim C(A) + \dim N(A^{\mathrm{T}}) = r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^{\mathrm{T}}) = R^m$



## Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:






## Fundamental Theorem of Linear Algebra

### Now we see:

- Each of the four fundamental subspaces has a 'best' orthonormal basis
- The $\hat{v}_i$ span $R^n$
- We find the $\hat{v}_i$ as eigenvectors of $A^{\mathrm{T}}A$.
- The $\hat{u}_i$ span $R^m$
- We find the $\hat{u}_i$ as eigenvectors of $AA^{\mathrm{T}}$.

### Happy bases

- $\{\hat{v}_1, \ldots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \ldots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space

# Fundamental Theorem of Linear Algebra



## How $A\vec{x}$ works:

- $A = U\Sigma V^{\mathrm{T}}$
- $A$ sends each $\vec{v}_i \in C(A^{\mathrm{T}})$ to its partner $\vec{u}_i \in C(A)$ with a stretch/shrink factor $\sigma_i > 0$.
- $A$ is diagonal with respect to these bases and has positive entries (all $\sigma_i > 0$).
- When viewed the right way, any $A$ is a diagonal matrix $\Sigma$.

# Image approximation (80x60)



## Idea: use SVD to approximate images

- Interpret elements of matrix $A$ as color values of an image.
- Truncate series SVD representation of $A$:

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

- Use fact that $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.
- Rank $r = \min(m, n)$.
- Rank $r =$ # of pixels on shortest side.
- For color: approximate 3 matrices (RGB).