More Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
Course 300, Fall, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

More Power-Law Mechanisms
Optimization
Minimal Cost
Mandelbrot vs. Simon
Assumptions
Model
Analysis
Extra
Robustness
HOT theory
Predicting social catastrophe
Self-Organized Criticality
COLD theory
Network robustness
References

Another approach

Benoit Mandelbrot

- Mandelbrot = father of fractals
- Mandelbrot = almond bread
- Derived Zipf's law through optimization [11]
- Idea: Language is efficient
- Communicate as much information as possible for as little cost
- Need measures of information (H) and cost (C)...
- Minimize C/H by varying word frequency
- Recurring theme: what role does optimization play in complex systems?

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Not everyone is happy...

Mandelbrot vs. Simon:

- Mandelbrot (1953): "An Informational Theory of the Statistical Structure of Languages" [11]
- Simon (1955): "On a class of skew distribution functions" [14]
- Mandelbrot (1959): "A note on a class of skew distribution function: analysis and critique of a paper by H.A. Simon"
- Simon (1960): "Some further notes on a class of skew distribution functions"

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Not everyone is happy... (cont.)

Mandelbrot vs. Simon:

- Mandelbrot (1961): “Final note on a class of skew distribution functions: analysis and critique of a model due to H.A. Simon”
- Simon (1961): “Reply to ‘final note’ by Benoit Mandelbrot”
- Mandelbrot (1961): “Post scriptum to ‘final note”’
- Simon (1961): “Reply to Dr. Mandelbrot’s post scriptum”

Not everyone is happy... (cont.)

Mandelbrot:

“We shall restate in detail our 1959 objections to Simon’s 1955 model for the Pareto-Yule-Zipf distribution. Our objections are valid quite irrespectively of the sign of p-1, so that most of Simon’s (1960) reply was irrelevant.”

Simon:

“Dr. Mandelbrot has proposed a new set of objections to my 1955 models of the Yule distribution. Like his earlier objections, these are invalid.”

Plankton:

“You can’t do this to me, I WENT TO COLLEGE!” “You weak minded fool!” “That’s it Mister! You just lost your brain privileges,” etc.

Zipfarama via Optimization

Mandelbrot’s Assumptions

- Language contains n words: $w_1, w_2, \ldots, w_n$.
- ith word appears with probability p_i
- Words appear randomly according to this distribution (obviously not true...)
- Words = composition of letters is important
- Alphabet contains m letters
- Words are ordered by length (shortest first)

Zipfarama via Optimization

Word Cost

- Length of word (plus a space)
- Word length was irrelevant for Simon’s method

Objection

- Real words don’t use all letter sequences

Objections to Objection

- Maybe real words roughly follow this pattern (?)
- Words can be encoded this way
- Na na na-na naaaaa...

Zipfarama via Optimization

Binary alphabet plus a space symbol

i	1	2	3	4	5	6	7	8
word	1	10	11	100	101	110	111	1000
length	1	2	2	3	3	3	3	4
$1 + \ln_2 i$	1	2	2.58	3	3.32	3.58	3.81	4

- Word length of 2^kth word: $= k + 1 = 1 + \log_2 2^k$
- Word length of ith word $\simeq 1 + \log_2 i$
- For an alphabet with m letters, word length of ith word $\simeq 1 + \log_m i$.

Zipfarama via Optimization

Total Cost C

- Cost of the ith word: $C_i \simeq 1 + \log_m i$
- Cost of the ith word plus space: $C_i \simeq 1 + \log_m(i+1)$
- Subtract fixed cost: $C'_i = C_i - 1 \simeq \log_m(i+1)$
- Simplify base of logarithm:

$$C'_i \simeq \log_m(i+1) = \frac{\log_e(i+1)}{\log_e m} \propto \ln(i+1)$$

- Total Cost:

$$C \sim \sum_{i=1}^{n} p_i C'_i \propto \sum_{i=1}^{n} p_i \ln(i+1)$$

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Zipfarama via Optimization

Information Measure

- Use Shannon's Entropy (or Uncertainty):

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i$$

- (allegedly) von Neumann suggested 'entropy'...
- Proportional to average number of bits needed to encode each 'word' based on frequency of occurrence
- $-\log_2 p_i = \log_2 1/p_i$ = minimum number of bits needed to distinguish event i from all others
- If $p_i = 1/2$, need only 1 bit ($log_2 1/p_i = 1$)
- If $p_i = 1/64$, need 6 bits ($log_2 1/p_i = 6$)

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Zipfarama via Optimization

Information Measure

- Use a slightly simpler form:

$$H = -\sum_{i=1}^{n} p_i \log_e p_i / \log_e 2 = -g \sum_{i=1}^{n} p_i \ln p_i$$

where $g = 1/\ln 2$

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Zipfarama via Optimization

- Minimize
$$F(p_1, p_2, \ldots, p_n) = C/H$$
subject to constraint
$$\sum_{i=1}^{n} p_i = 1$$
- Tension:
(1) Shorter words are cheaper
(2) Longer words are more informative (rarer)
- (Good) question: how much does choice of C/H as function to minimize affect things?

Zipfarama via Optimization

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Time for Lagrange Multipliers:

- Minimize
$$\Psi(p_1, p_2, \ldots, p_n) =$$
$$F(p_1, p_2, \ldots, p_n) + \lambda G(p_1, p_2, \ldots, p_n)$$
where
$$F(p_1, p_2, \ldots, p_n) = \frac{C}{H} = \frac{\sum_{i=1}^{n} p_i \ln(i+1)}{-g\sum_{i=1}^{n} p_i \ln p_i}$$
and the constraint function is
$$G(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i - 1 = 0$$
- [Insert assignment problem...]

Zipfarama via Optimization

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Some mild suffering leads to:

- $$p_j = e^{-1-\lambda H^2/gC}(j+1)^{-H/gC} \propto (j+1)^{-H/gC}$$
- A power law appears [applause]: $\boxed{\alpha = H/gC}$
- Next: sneakily deduce λ in terms of g, C, and H.
- Find
$$p_j = (j+1)^{-H/gC}$$

Zipfarama via Optimization

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Differentiate with respect to p_j:

- $$\frac{\partial \Psi}{\partial p_j} = \frac{\partial (C/H)}{\partial p_j} + \frac{\partial}{\partial p_j}\lambda\left(\sum_{i=1}^{n} p_i - 1\right) \quad (= 0)$$
- $$= \frac{\frac{\partial C}{\partial p_j}H - C\frac{\partial H}{\partial p_j}}{H^2} + \lambda$$
- $$= \frac{\frac{\partial}{\partial p_j}\left(\sum_{i=1}^{n} p_i \ln(i+1)\right)H - C\frac{\partial}{\partial p_j}\left(-g\sum_{i=1}^{n} p_i \ln p_i\right)}{H^2} + \lambda$$
- $$\frac{H\ln(j+1) + gC(\ln p_j + p_j/p_j\cancel{p_j/p_j}\ 1 1)}{H^2} + \lambda \ = 0$$

Zipfarama via Optimization

Keep going...

- $$\frac{H \ln (j+1) + gC(\ln p_j + 1)}{H^2} + \lambda = 0.$$
- $$\ln p_j = -1 - \frac{H \ln (j+1) + \lambda H^2}{gC}$$
- $$p_j = \exp\left\{-1 - \frac{H \ln (j+1) + \lambda H^2}{gC}\right\}$$
- $$p_j = e^{-1-\lambda H^2/gC}(j+1)^{-H/gC} \propto (j+1)^{-H/gC}$$
- A power law appears [applause]: $\boxed{\alpha = H/gC}$

Zipfarama via Optimization

Finding the exponent

- Expressions for H and C are implicit functions.
- Not terribly obvious what the exponent will be.
- Let's find out...
- First: Determine λ
- Sneakiness: Subsitute form for p_j into H
- Find $\lambda = -gC/H^2$
  Our form for p_j reduces:

$$p_j = e^{-1-\lambda H^2/gC}(j+1)^{-H/gC}$$

$$= e^{-1+1}(j+1)^{-H/gC} = (j+1)^{-H/gC}$$

Zipfarama via Optimization

Finding the exponent

- Now use the normalization constraint:

$$1 = \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} (j+1)^{-H/gC} = \sum_{j=1}^{n} (j+1)^{-\alpha}$$

- As $n \to \infty$, we end up with $\zeta(H/gC) = 2$ where ζ is the Riemann Zeta Function
- Gives $\alpha \simeq 1.73$ (> 1, too high)
- If cost function changes ($j + 1 \to j + a$) then exponent is tunable
- Increase a, decrease α

Zipfarama via Optimization

All told:

- Reasonable approach: Optimization is at work in evolutionary processes
- But optimization can involve many incommensurate elements: monetary cost, robustness, happiness,...
- Mandelbrot's argument is not super convincing
- Exponent depends too much on a loose definition of cost

More

Reconciling Mandelbrot and Simon

- Mixture of local optimization and randomness
- Numerous efforts...

1. Carlson and Doyle, 1999:
   Highly Optimized Tolerance
   (HOT)—Evolved/Engineered Robustness [5]
2. Ferrer i Cancho and Solé, 2002:
   Zipf's Principle of Least Effort [8]
3. D'Souza et al., 2007:
   Scale-free networks [7]

More

Other mechanisms:

Much argument about whether or not monkeys typing could produce Zipf's law... (Miller, 1957) [12]

Others are also not happy

Krugman and Simon

- "The Self-Organizing Economy" (Paul Krugman, 1995) [10]
- Krugman touts Zipf's law for cities, Simon's model
- "Déjà vu, Mr. Krugman" (Berry, 1999)
- Substantial work done by Urban Geographers

Who needs a hug?

From Berry [4]

- Déjà vu, Mr. Krugman. Been there, done that. The Simon-Ijiri model was introduced to geographers in 1958 as an explanation of city size distributions, the first of many such contributions dealing with the steady states of random growth processes, ...
- But then, I suppose, even if Krugman had known about these studies, they would have been discounted because they were not written by professional economists or published in one of the top five journals in economics!

Who needs a hug?

From Berry [4]

- ... [Krugman] needs to exercise some humility, for his world view is circumscribed by folkways that militate against recognition and acknowledgment of scholarship beyond his disciplinary frontier.
- Urban geographers, thank heavens, are not so afflicted.

Robustness

- Many complex systems are prone to cascading catastrophic failure: exciting!!!
  - Blackouts
  - Disease outbreaks
  - Wildfires
  - Earthquakes
- But complex systems also show persistent robustness (not as exciting but important...)
- Robustness and Failure may be a power-law story...

More Power-Law Mechanisms

Robustness

- System robustness may result from
  1. Evolutionary processes
  2. Engineering/Design
- Idea: Explore systems optimized to perform under uncertain conditions.
- The handle: ‘Highly Optimized Tolerance’ (HOT) [5, 6, 15]
- The catchphrase: Robust yet Fragile
- The people: Jean Carlson and John Doyle

More Power-Law Mechanisms

Robustness

Features of HOT systems: [6]

- High performance and robustness
- Designed/evolved to handle known environmental variability
- Fragile in the face of unpredicted environmental signals
- Highly specialized, low entropy configurations
- Power-law distributions

More Power-Law Mechanisms

Robustness

HOT combines things we've seen:

- Variable transformation
- Constrained optimization

- Need power law transformation between variables: $(Y = X^{-\alpha})$
- MIWO is good: Mild In, Wild Out
- X has a characteristic size but Y does not

Robustness

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Forest fire example [6]

- Square $N \times N$ grid
- Sites contain a tree with probability ρ = density
- Sites are empty with probability $1 - \rho$
- Fires start at location according to some distribution P_{ij}
- Fires spread from tree to tree
- Connected clusters of trees burn completely
- Empty sites block fire
- Best case scenario:
  Maximize average # trees left intact

Robustness

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Forest fire example [6]

- Build a forest by adding one tree at a time
- Test D ways of adding one tree
- D = design parameter
- Average over P_{ij} = spark probability
- $D = 1$: random addition
- $D = N^2$: test all possibilities

HOT Forests

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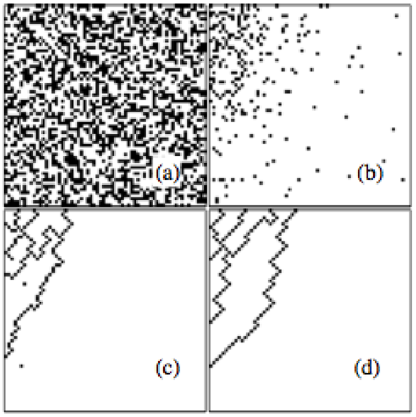

[6]

$N = 64$

(a) $D = 1$
(b) $D = 2$
(c) $D = N$
(d) $D = N^2$

P_{ij} has a Gaussian decay

Optimized forests do well on average (robustness)
but rare extreme events occur (fragility)

Random Forests

$D = 1$: Random forests = Percolation

- Randomly add trees
- Below critical density ρ_c, no fires take off
- Above critical density ρ_c, percolating cluster of trees burns
- Only at ρ_c, the critical density, is there a power-law distribution of tree cluster sizes
- Forest is random and featureless

HOT forests

HOT forests

- Highly structured
- Power law distribution of tree cluster sizes for $\rho > \rho_c$
- No specialness of ρ_c
- Forest states are tolerant
- Uncertainty is okay if well characterized
- If P_{ij} is characterized poorly, failure becomes highly likely

More Power-Law Mechanisms

HOT theory

The abstract story:

- Given $y_i = x_i^{-\alpha}$, $i = 1, \ldots, N$
- Design system to minimize $\langle y \rangle$ subject to a constraint on the x_i
- Minimize cost:

$$C = \sum_{i=1}^{N} Pr(y_i) y_i$$

Subject to $\sum_{i=1}^{N} x_i = \text{constant}$

- Drag out the Lagrange Multipliers, battle away and find:

$$p_i \propto y_i^{-\gamma}$$

More Power-Law Mechanisms

HOT theory

Optimal fire walls in d dimensions:

- Two costs:
  1. Expected size of fire

$$C_{\text{fire}} \propto \sum_{i=1}^{N} (p_i a_i) a_i = \sum_{i=1}^{N} p_i a_i^2$$

a_i = area of ith region
p_i = average probability of fire at site in ith region

  2. Cost of building and maintaining firewalls

$$C_{\text{firewalls}} \propto \sum_{i=1}^{N} a_i^{1/2}$$

In d dimensions, 1/2 is replaced by $(d-1)/d$

More Power-Law Mechanisms

HOT theory

- Minimize C_{fire} given $C_{\text{firewalls}} = \text{constant}$.
- $$\frac{\partial}{\partial a_j}\left(C_{\text{fire}} - \lambda C_{\text{firewalls}}\right) = 0$$
  $$\propto \frac{\partial}{\partial a_j}\left(\sum_{i=1}^{N} p_i a_i^2 - \lambda' a_i^{(d-1)/d}\right)$$
- $$p_i \propto a_i^{-\gamma} = a_i^{-(1+1/d)}$$
- For $d = 2, \gamma = 3/2$

HOT theory

Basic idea of designed tolerance

- Build more firewalls in areas where sparks are likely
- Small connected regions in high-danger areas
- Large connected regions in low-danger areas
- Routinely see many small outbreaks (robust)
- Rarely see large outbreaks (fragile)
- Sensitive to changes in the environment (P_{ij})

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Predicting social catastrophe isn't easy...

"Greenspan Concedes Error on Regulation"

- . . . humbled Mr. Greenspan admitted that he had put too much faith in the self-correcting power of free markets . . .
- "Those of us who have looked to the self-interest of lending institutions to protect shareholders' equity, myself included, are in a state of shocked disbelief"
- Rep. Henry A. Waxman: "Do you feel that your ideology pushed you to make decisions that you wish you had not made?"
- Mr. Greenspan conceded: "Yes, I've found a flaw. I don't know how significant or permanent it is. But I've been very distressed by that fact."

New York Times, October 23, 2008 (⊞)

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Economics, Schmeconomics

Alan Greenspan (September 18, 2007):

"I've been dealing with these big mathematical models of forecasting the economy ...

If I could figure out a way to determine whether or not people are more fearful or changing to more euphoric,

I don't need any of this other stuff.

I could forecast the economy better than any way I know."

http://wikipedia.org

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Economics, Schmeconomics

Greenspan continues:

"The trouble is that we can't figure that out. I've been in the forecasting business for 50 years. I'm no better than I ever was, and nobody else is. Forecasting 50 years ago was as good or as bad as it is today. And the reason is that human nature hasn't changed. We can't improve ourselves."

Jon Stewart:

"You just bummed the @*!# out of me."

wildbluffmedia.com

- From the Daily Show (⊞) (September 18, 2007)
- The full inteview is here (⊞).

Economics, Schmeconomics

James K. Galbraith:

- But there are at least 15,000 professional economists in this country, and you're saying only two or three of them foresaw the mortgage crisis? Ten or 12 would be closer than two or three.
- What does that say about the field of economics, which claims to be a science? It's an enormous blot on the reputation of the profession. There are thousands of economists. Most of them teach. And most of them teach a theoretical framework that has been shown to be fundamentally useless.

From the New York Times, 11/02/2008 (⊞)

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Avalanches on Sand and Rice

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SOC theory

SOC = Self-Organized Criticality

- Idea: natural dissipative systems exist at 'critical states'
- Analogy: Ising model with temperature somehow self-tuning
- Power-law distributions of sizes and frequencies arise 'for free'
- Introduced in 1987 by Bak, Tang, and Weisenfeld [3, 2, 9]:
  "Self-organized criticality - an explanation of 1/f noise"
- Problem: Critical state is a very specific point
- Self-tuning not always possible
- Much criticism and arguing...

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Robustness

HOT versus SOC

- Both produce power laws
- Optimization versus self-tuning
- HOT systems viable over a wide range of high densities
- SOC systems have one special density
- HOT systems produce specialized structures
- SOC systems produce generic structures

COLD forests

Avoidance of large-scale failures

- Constrained Optimization with Limited Deviations [13]
- Weight cost of larges losses more strongly
- Increases average cluster of trees...
- ... but reduces chances of catastrophe
- Power law distribution of fire sizes is truncated

Cutoffs

Aside:

- Power law distributions often have an exponential cutoff
- $$P(x) \sim x^{-\gamma} \exp^{-x/x_c}$$
- where x_c is the approximate cutoff scale.

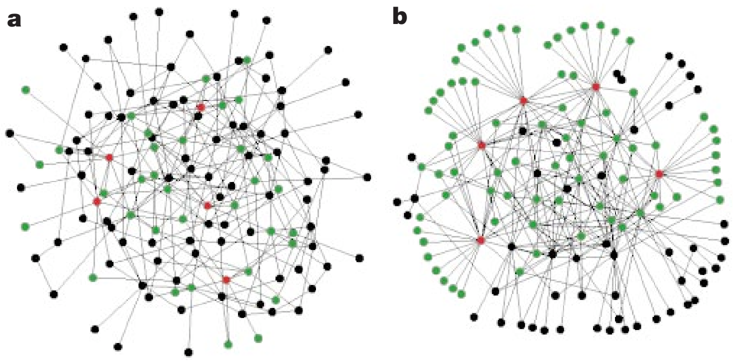
Robustness

And we've already seen this...

- network robustness.
- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [1]

Robustness

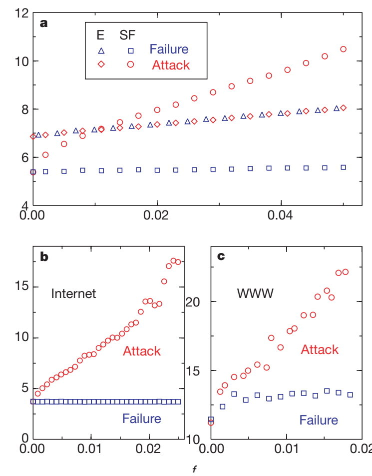
- Standard random networks (Erdös-Rényi) versus Scale-free networks



Exponential Scale-free

from Albert et al., 2000 [1]

Robustness



from Albert et al., 2000 [1]

- Plots of network diameter as a function of fraction of nodes removed
- Erdös-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

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