Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
Course 300, Fall, 2008

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University of Vermont

Outline

Mechanisms

A powerful theme in complex systems:

- structure arises out of randomness.

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- Exhibit A: Random walks... (⊞)

Random walks

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

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Variances sum: (⊞)*

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$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

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$$\sigma = t^{1/2}$$

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Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
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- If you are behind early on, what are the chances you will make a comeback?
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See Feller, [3] Intro to Probability Theory, Volume I

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In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

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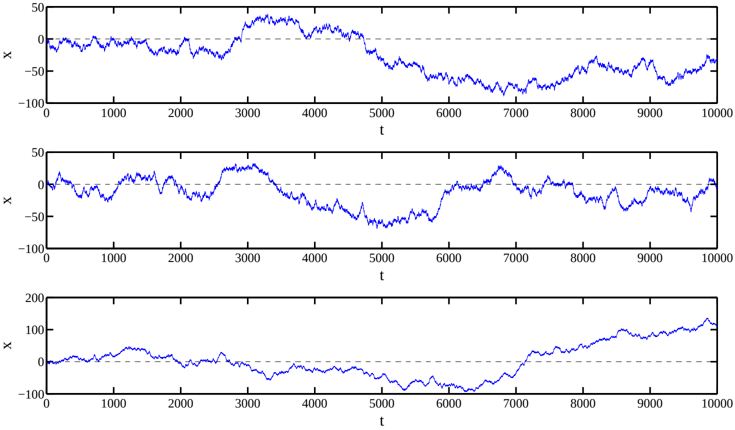
Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples



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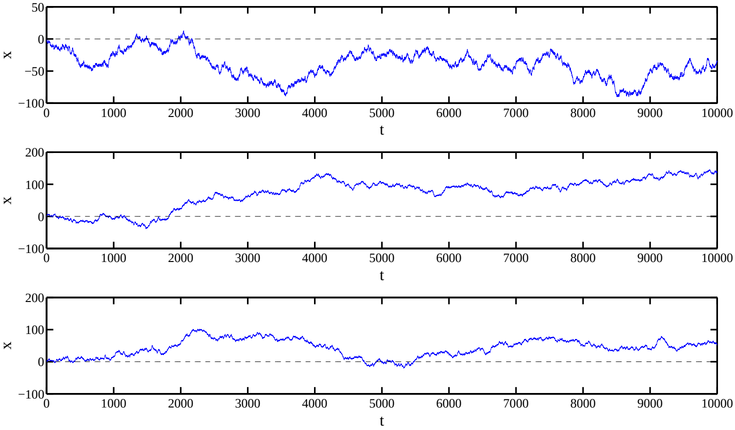

Random walks

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

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Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

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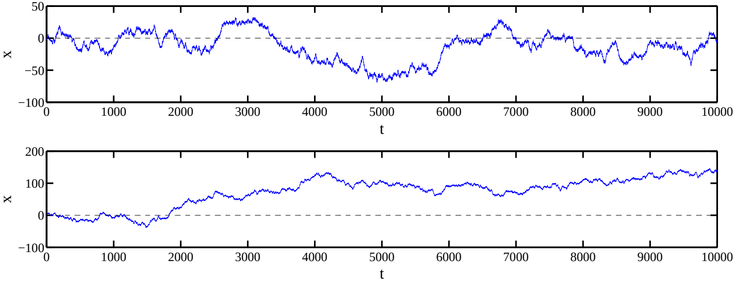

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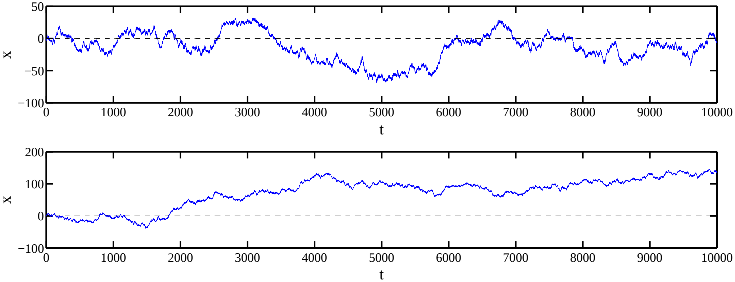
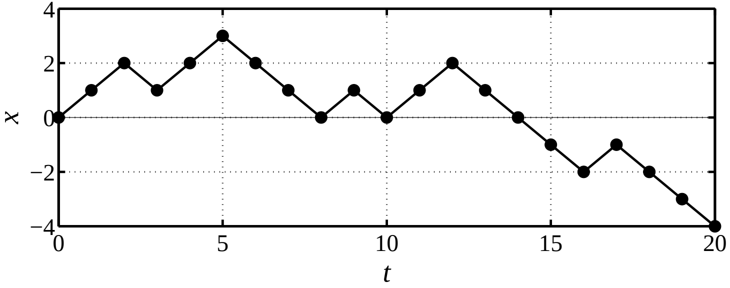


Again: expected time between ties = ∞...
Let's find out why... [3]

First Returns



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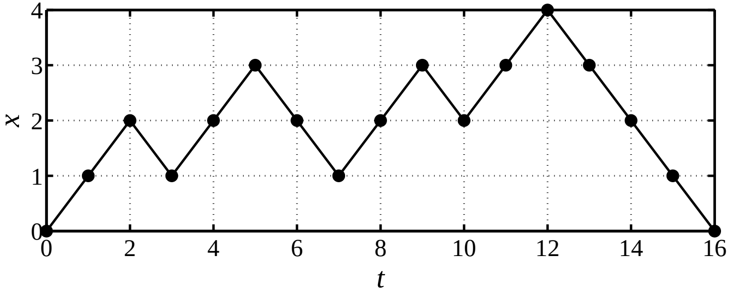
- The problem

$$P_{\text{fr}}(2n) = Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

First Returns



- A useful restatement: $P_{\rm fr}(2n) = \frac{1}{2}Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 or -2 instead of 0 at $t = 2n$.)

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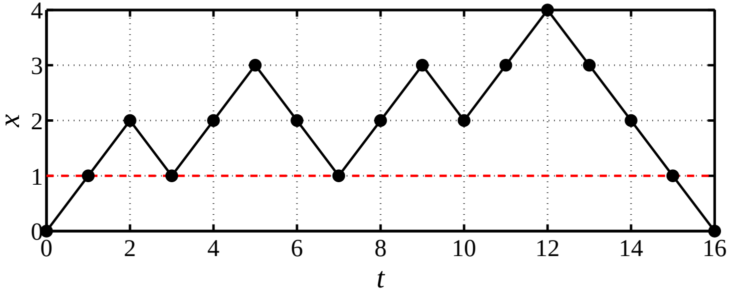

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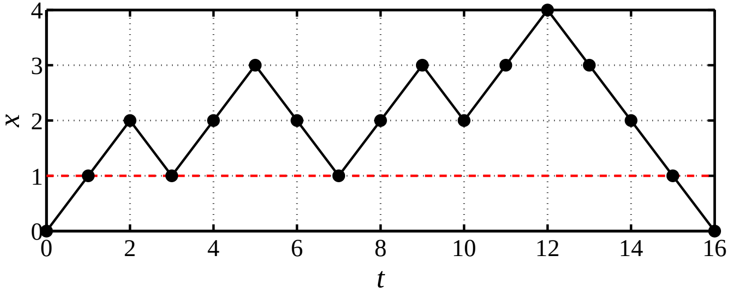

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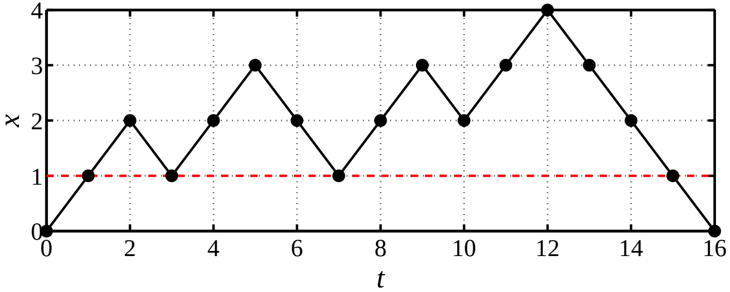

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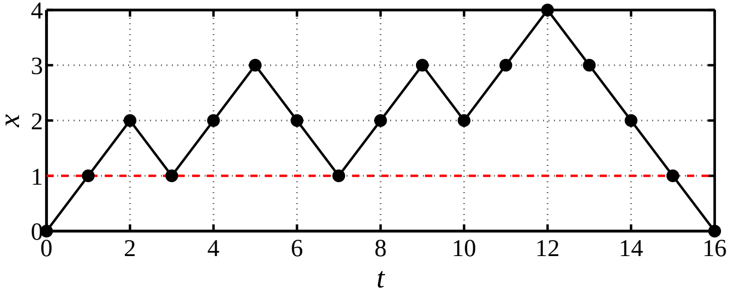

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- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Subtract how many hit $x = 0$.

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See this 1-1 correspondence visually...

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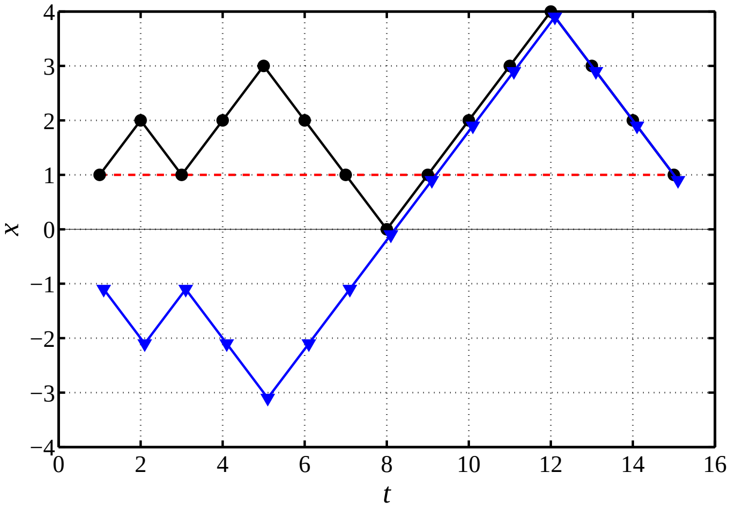

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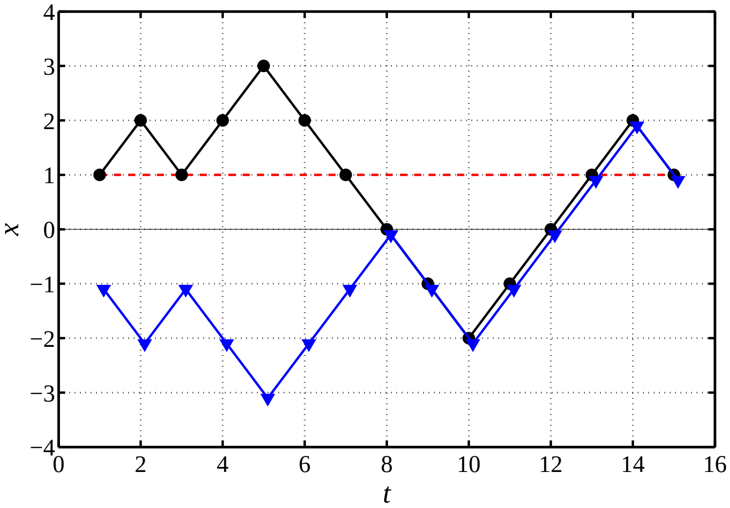

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- Next problem: what is $N(i, j, t)$?
- # positive steps + # negative steps = t.
- Random walk must displace by $j - i$ after t steps.
- # positive steps - # negative steps = $j - i$.
- # positive steps = $(t + j - i)/2$.
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We now have

$$N_{\text{first return}}(2n) = \frac{1}{2}\left[N(1,1,2n-2) - N(-1,1,2n-2)\right]$$

where

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[Assignment question occurs]

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths = 2^{2n}
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$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

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- Same scaling holds for continuous space/time walks.
- $$P(t) \propto t^{-3/2}, \; \gamma = 3/2$$
- $P(t)$ is normalizable
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Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

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- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
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Random walks

On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
- Random walking $\equiv$ Diffusion
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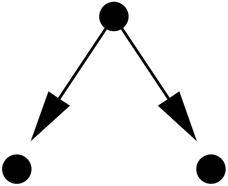
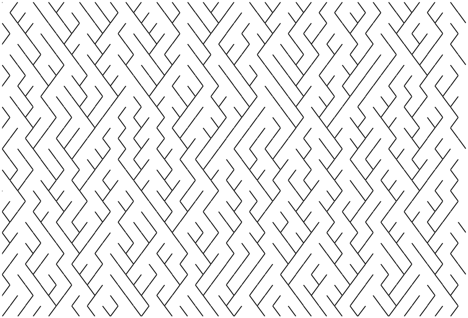
Outline

Scheidegger Networks [10, 2]





- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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- Characterize universality class with independent exponents

Other First Returns

Failure

- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x_0 could be > 0.
- Entity fails when x hits 0.

Streams

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- Long times for clearing.

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Understand power laws as arising from

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1. elementary distributions (e.g., exponentials)
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Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
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- Exponentials arise from randomness...

Example

Exponential distribution

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- Exponentials arise from randomness...
- More later when we cover robustness.

Outline

Gravity

- Select a random point in space $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

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Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2\mathrm{d}r$$

- Assume stars are distributed randomly in space
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

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- $$= F^{-1-3/2}\mathrm{d}F$$
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- Variance = ∞
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We need mechanisms!

Outline

Aggregation

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
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Work of Yore

- 1924: G. Udny Yule [13]:
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- 1926: Lotka [6]:
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- 1953: Mandelbrot [7]:
  Optimality argument for Zipf's law; focus on language.
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Essential Extract of a Growth Model

Random Competitive Replication (RCR):

1. Start with 1 element of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
   - With probability ρ, create a new element with a new flavor
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Random Competitive Replication

Example: Words in a text

- Consider words as they appear sequentially.
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- Competition for replication between elements is random
- Competition for growth between groups is not random
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     (But mechanism for selection is not as simple...)

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- k_i = size of a group i
- $N_k(t)$ = # groups containing k elements at time t.

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Basic question: How does $N_k(t)$ evolve with time?

$$\text{First: } \sum_k kN_k(t) = t = \text{number of elements at time } t$$

Random Competitive Replication

$P_k(t)$ = Probability of choosing an element that belongs to a group of size k:

- $N_k(t)$ size k groups
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$$P_k(t) = \frac{kN_k(t)}{t}$$

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1. An element belonging to a group with k elements is replicated
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Special case for $N_1(t)$:

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Special case for $N_1(t)$:

1. The new element is a new flavor:
   $N_1(t+1) = N_1(t) + 1$
   Happens with probability ρ

2. A unique element is replicated.
   $N_1(t+1) = N_1(t) - 1$
   Happens with probability $(1-\rho)N_1/t$

Random Competitive Replication

Put everything together:

For $k > 1$:

$$\langle N_k(t+1) - N_k(t)\rangle = (1-\rho)\left((k-1)\frac{N_{k-1}(t)}{t} - k\frac{N_k(t)}{t}\right)$$

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For $k = 1$:

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$

Random Competitive Replication

Assume distribution stabilizes: $N_k(t) = n_k t$

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional
- Okay over large time scales
- n_k/ρ = the fraction of groups that have size k.

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Stochastic difference equation:

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$$\Rightarrow n_k\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$

Random Competitive Replication

We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in k large (the tail of the distribution)
- Expand as a series of powers of $1/k$
- [Assignment question occurs]

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- We (okay, you) find

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- Observe $2 < \gamma < \infty$ as ρ varies.
- For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- Recalls Zipf's law: $s_r \sim r^{-\alpha}$
  (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma - 1)$
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- We (roughly) see Zipfian exponent[14] of $\alpha = 1$ for many real systems: city sizes, word distributions, ...
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We had one other equation:

- $$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$
- As before, set $N_1(t) = n_1 t$ and drop expectations
- $$n_1(t+1) - n_1 t = \rho - (1-\rho)1 \cdot \frac{n_1 t}{t}$$
- $$n_1 = \rho - (1-\rho)n_1$$
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$$n_1 + (1-\rho)n_1 = \rho$$
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So... $$N_1(t) = n_1 t = \frac{\rho t}{2-\rho}$$

- Recall number of distinct elements = ρt.
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2-\rho}$$

(also = fraction of groups of size 1)

- For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ρ increases, fraction increases
- Can show fraction of groups with two elements $\sim 1/6$
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Outline

Words

From Simon [11]:

Estimate $\rho_{\text{est}} = \#$ unique words$/\#$ all words

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For Joyce's Ulysses: $\rho_{\text{est}} \simeq 0.115$

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Estimate ρ_{est} = # unique words/# all words

For Joyce's Ulysses: $\rho_{\text{est}} \simeq 0.115$

N_1 (real)	N_1 (est)	N_2 (real)	N_2 (est)
16,432	15,850	4,776	4,870

Evolution of catch phrases

- Yule's paper (1924) [13]:
  "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) [11]:
  "On a class of skew distribution functions" (snore)

From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economoic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only be a similarity in the structure of the underlying probability mechanisms.

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More on Herbert Simon (1916–2001):

- Political scientist
- Involved in Cognitive Psychology, Computer Science, Public Administration, Economics, Management, Sociology
- Coined ‘bounded rationality’ and ‘satisficing’
- Nearly 1000 publications
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
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Evolution of catch phrases

- Derek de Solla Price was the first to study network evolution with these kinds of models.
- Citation network of scientific papers
- Price's term: Cumulative Advantage
- Idea: papers receive new citations with probability proportional to their existing # of citations
- Directed network
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2. role model
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And just to rub it in...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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- Another term: "Preferential Attachment"
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