Complex Networks

Principles of Complex Systems
Course 300, Fall, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

Basic definitions

Network: (net + work, 1500's)

Noun:

1. Any interconnected group or system
2. Multiple computers and other devices connected together to share information

Verb:

1. To interact socially for the purpose of getting connections or personal advancement
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- Opens door to mathematical and numerical analysis
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- links
  - may be real and fixed (rivers),
  - real and dynamic (airline routes),
  - abstract with physical impact (hyperlinks),
  - or purely abstact (semantic connections between concepts).
- Links may be directed or undirected.
- Links may be binary or weighted.

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- Notation: Node i's degree = k_i.
- $k_i = 0,1,2,\ldots$.
- Notation: the average degree of a network = $\langle k \rangle$

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Adjacency matrix:

- We represent a graph or network by a matrix A with link weight a_{ij} for nodes i and j in entry (i, j).
- e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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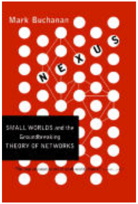
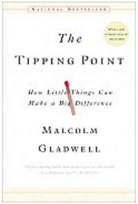


Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan



The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell

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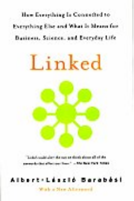
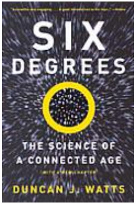


Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási



Six Degrees: The Science of a Connected Age—Duncan Watts

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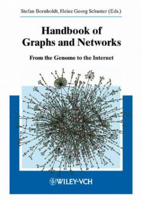
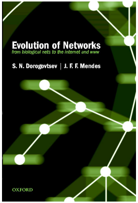


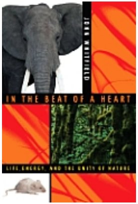
Handbook of Graphs and Networks—editors: Stefan Bornholdt and H. G. Schuster



Evolution of Networks—S. N. Dorogovtsev and J. F. F. Mendes.

Books

[Social Network Analysis]—Stanley Wasserman and Kathleen Faust



[In the Beat of a Heart: Life, Energy, and the Unity of Nature]—John Whitfield

Books

Numerous others:

- Complex Social Networks—F. Vega-Redondo
- Fractal River Basins: Chance and Self-Organization—I. Rodríguez-Iturbe and A. Rinaldo
- Random Graph Dynamics—R. Durette
- Scale-Free Networks—Guido Caldarelli
- Evolution and Structure of the Internet: A Statistical Physics Approach—Romu Pastor-Satorras and Alessandro Vespignani
- Complex Graphs and Networks—Fan Chung

Examples

What passes for a complex network?

- Complex networks are large (in node number)
- Complex networks are sparse (low edge to node ratio)
- Complex networks are usually dynamic and evolving
- Complex networks can be social, economic, natural, informational, abstract, ...

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Physical networks

- River networks
- Neural networks
- Trees and leaves
- Blood networks
- The Internet
- Road networks
- Power grids

- Distribution (branching) versus redistribution (cyclical)

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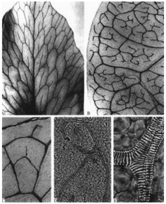

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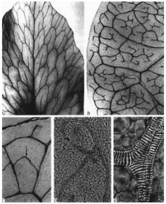

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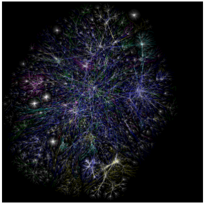

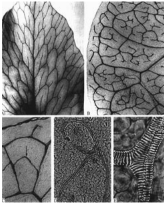

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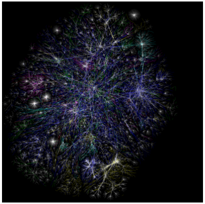

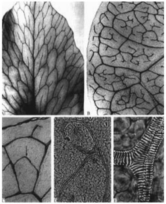

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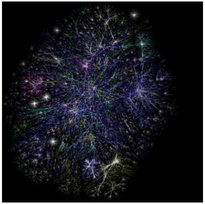

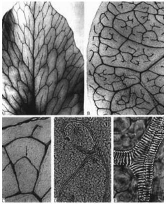

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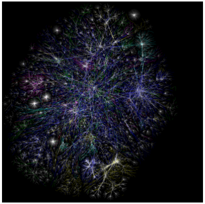

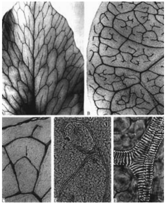

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- Biochemical networks
- Gene-protein networks
- Food webs: who eats whom
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- Friendships
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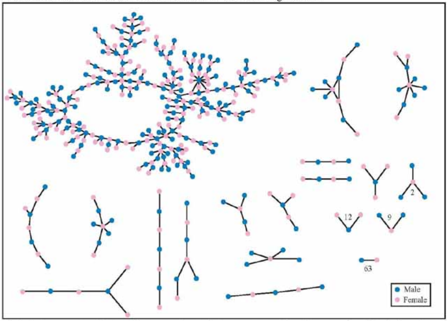
The Structure of Romantic and Sexual Relations at "Jefferson High School"



Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

(Bearman *et al.*, 2004)

- 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).

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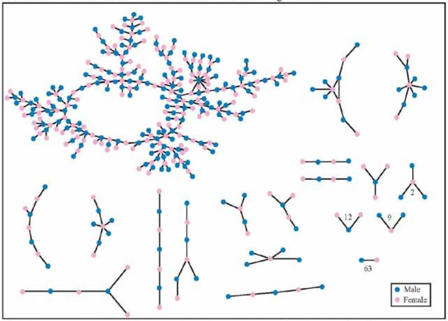

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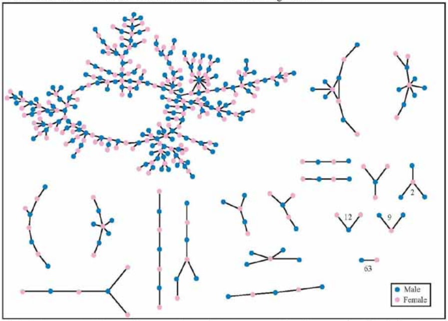

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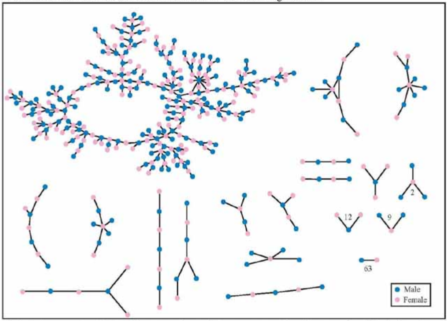

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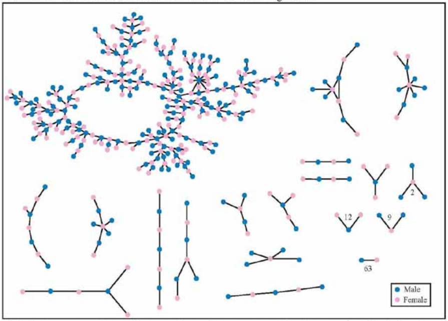

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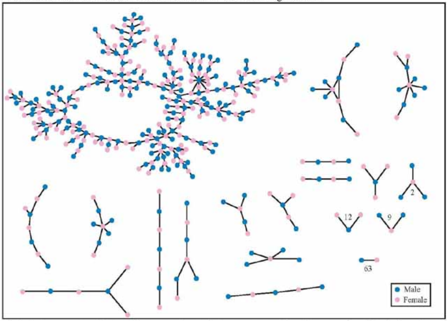

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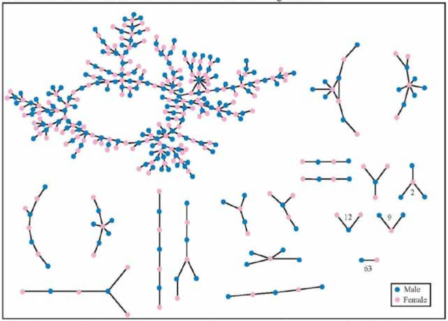

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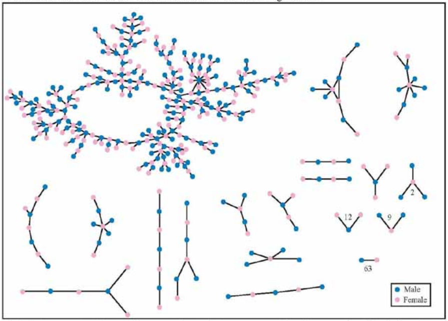

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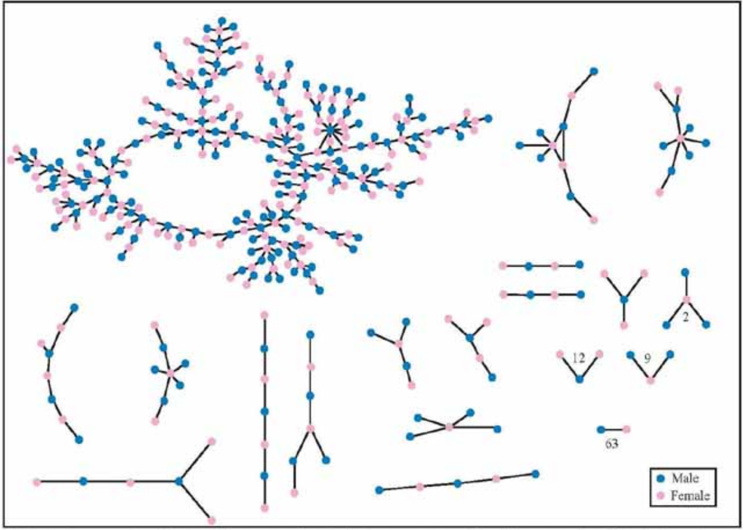

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Relational networks

- Consumer purchases

- Thesauri: Networks of words generated by meanings
- Knowledge/Databases/Ideas
- Metadata—Tagging: del.icio.us (⊞)http://del.icio.usdel.icio.us, flickr (⊞)

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common tags cloud | list

community daily dictionary education **encyclopedia** english free imported info information internet knowledge learning news **reference** research resource resources search tools useful web web2.0 **wiki** **wikipedia**

Observations

A notable features of large-scale networks:

- Graphical renderings of complex networks are often just a big mess.
- Need to be able to extract key patterns
- Science of Description

Properties

Some key aspects of real complex networks:

- degree distribution
- assortativity
- homophily
- clustering
- motifs
- modularity
- concurrency
- hierarchical scaling
- network distances
- centrality
- efficiency
- robustness

- + Coevolution of network structure and processes on networks.

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1. degree distribution P_k

- P_k is the probability that a randomly selected node has degree k
- k = node degree = number of connections
- ex 1: Erdös-Rényi random networks:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- Distribution is Poisson

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- Erdös-Rényi random networks are a *mathematical construct*.
- 'Scale-free' networks are growing networks that form according to a plausible mechanism.
- Randomness is out there, just not to the degree of a completely random network.

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2. assortativity/3. homophily:

- Social networks: Homophily = birds of a feather
- e.g., degree is standard property for sorting: measure degree-degree correlations.
- Assortative network:[10] similar degree nodes connecting to each other.
- Disassortative network: high degree nodes connecting to low degree nodes.

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  Often techological or biological: Internet, WWW, protein interactions, neural networks, food webs.

Clustering

4. clustering:

- Your friends tend to know each other.
- Two measures:
  1. Watts & Strogatz[15]

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

  2. Newman[11]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

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   - e.g., Feed Forward Loop:

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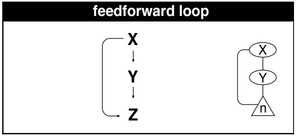
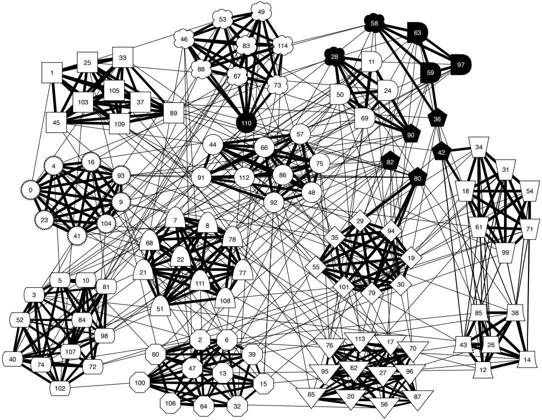


Shen-Orr, Uri Alon, *et al.* [12]

Properties

6. modularity—community detection:



Clauset *et al.*, 2006 [6]: NCAA football

Properties

7. concurrency:

- transmission of a contagious element only occurs during contact
- rather obvious but easily missed in a simple model
- dynamic property—static networks are not enough
- knowledge of previous contacts crucial
- beware cumulated network data
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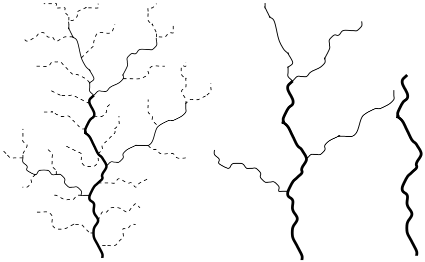
8. Horton-Strahler ratios:

- Metrics for branching networks:
  - Method for ordering streams hierarchically
  - Number: $R_n = N_\omega / N_{\omega+1}$
  - Segment length: $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
  - Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$



(a) (b) (c)

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9. network distances:

(a) shortest path length d_{ij}:

- Fewest number of steps between nodes i and j.
- (Also called the chemical distance between i and j.)

(b) average path length $\langle d_{ij} \rangle$:

- Average shortest path length in whole network.
- Good algorithms exist for calculation.
- Weighted links can be accommodated.

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- Many such measures of a node's 'importance.'
- ex 1: Degree centrality: k_i.
- ex 2: Node i's betweenness
  = fraction of shortest paths that pass through i.
- ex 3: Recursive centrality: Hubs and Authorities (Kleinberg [8])

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2. scale-free networks
3. small-world networks
4. statistical generative models (p^*)
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  Nature, 1998
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- Arbitrary degree distribution P_k.
- Create (unconnected) nodes with degrees sampled from P_k.
- Wire nodes together randomly.
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- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

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- Primary example: hyperlink network of the Web
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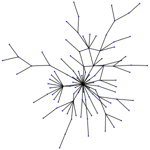
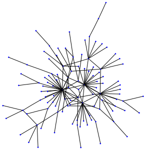
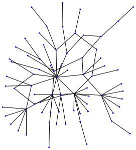
Random networks: largest components



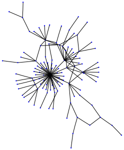
$\gamma = 2.5$
$\langle k \rangle = 1.8$



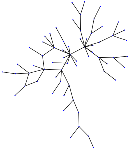
$\gamma = 2.5$
$\langle k \rangle = 2.05333$



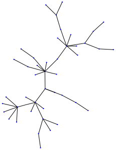
$\gamma = 2.5$
$\langle k \rangle = 1.66667$



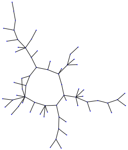
$\gamma = 2.5$
$\langle k \rangle = 1.92$



$\gamma = 2.5$
$\langle k \rangle = 1.6$



$\gamma = 2.5$
$\langle k \rangle = 1.50667$



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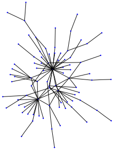

$\gamma = 2.5$
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The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

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- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
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- For the original model:

$$A_k = k$$

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- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

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- For the original model:

$$A_k = k$$

- Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
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- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
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$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

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$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \text{ for } t \ge t_{i,\text{start}}.$$

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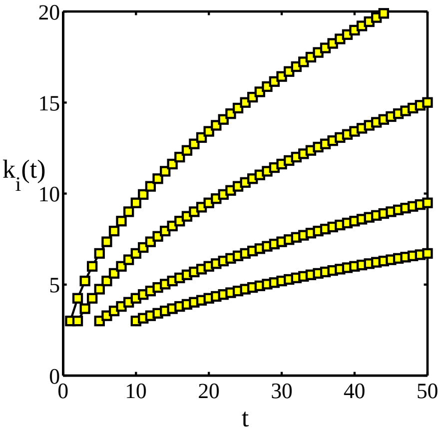

- $m = 3$
- $t_{i,\text{start}} =$ $1, 2, 5,$ and 10.

Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$\mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}} \simeq \frac{\mathrm{d}t_{i,\text{start}}}{t}$$

- Also use

$$k_i(t) = m\left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

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- Range true more generally for events with size distributions that have power-law tails.
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The Internets is a different business...

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Real data

From Barabási and Albert's original paper [3]:

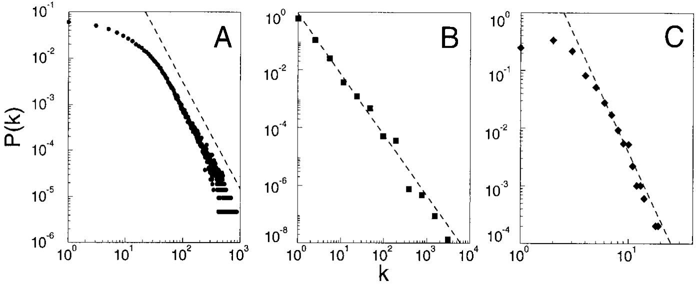

Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.

Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
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- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\mathrm{attach}}(k) \propto k$, we need to determine the constant of proportionality.
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- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

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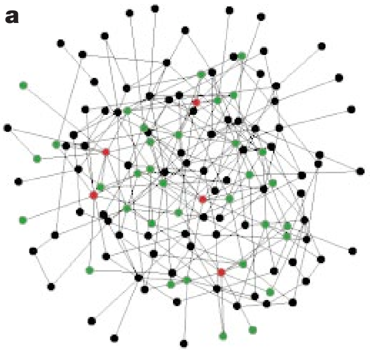
Robustness

- Standard random networks (Erdös-Rényi) versus Scale-free networks

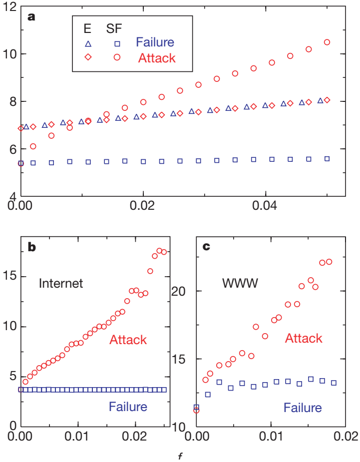


Exponential   Scale-free

from

Albert et al., 2000

Robustness



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- Plots of network diameter as a function of fraction of nodes removed
- Erdös-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
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Outline

The social world appears to be small...

- Connected random networks have short average path lengths:

$$\langle d_{AB} \rangle \sim \log(N)$$

  N = population size,

  d_{AB} = distance between nodes A and B.
- But: social networks aren't random...

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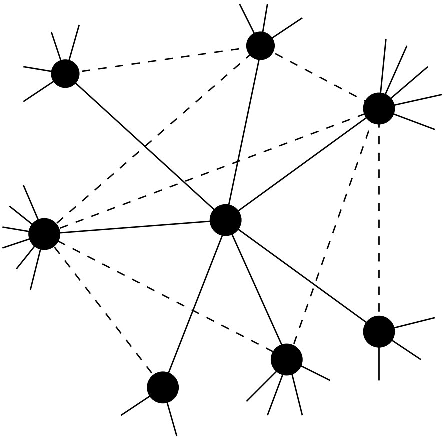
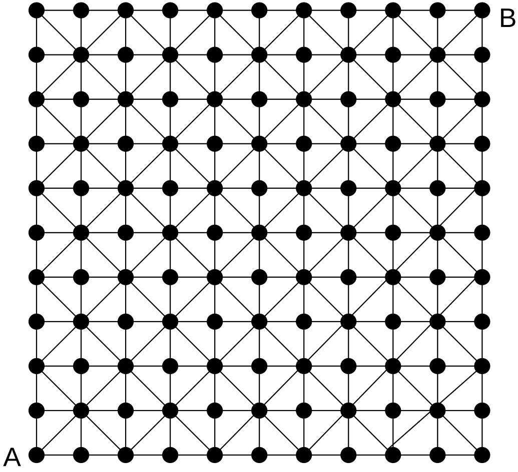
Simple socialness in a network:



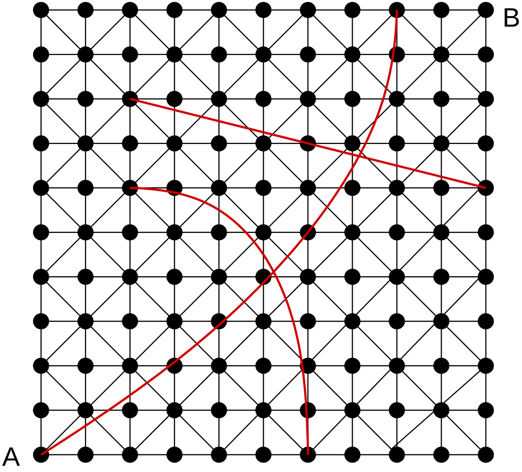
Need "clustering" (your friends are likely to know each other):

Non-randomness gives clustering:



$d_{AB} = 10 \rightarrow$ too many long paths.

Randomness + regularity



Now have $d_{AB} = 3$

$\langle d \rangle$ decreases overall

Small-world networks

Introduced by Watts and Strogatz (Nature, 1998) [15]
"Collective dynamics of 'small-world' networks."

Small-world networks were found everywhere:

- neural network of C. elegans,
- semantic networks of languages,
- actor collaboration graph,
- food webs,
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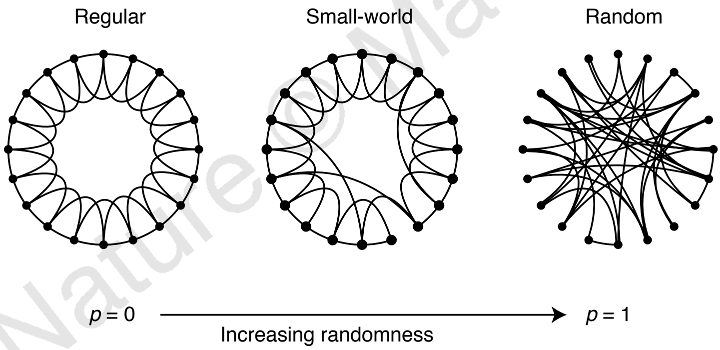
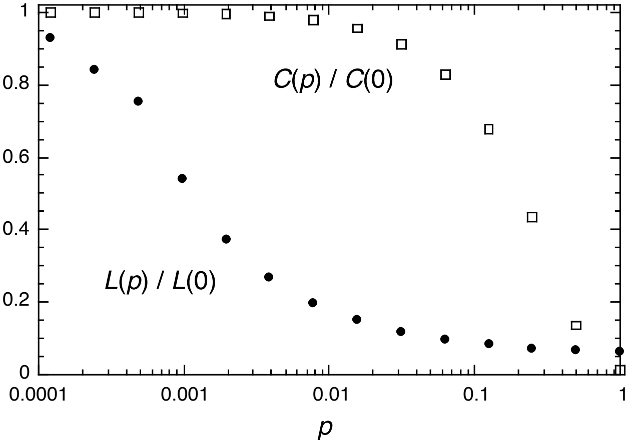
Toy model:



The structural small-world property:



- $L(p)$ = average shortest path length as a function of p
- $C(p)$ = average clustring as a function of p

Previous work—finding short paths

But are these short cuts findable?

Nope.

Nodes cannot find each other quickly with any local search method.

Need a more sophisticated model...

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- Target's identity
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Kleinberg's Network:

1. Start with regular d-dimensional cubic lattice.
2. Add local links so nodes know all nodes within a distance q.
3. Add m short cuts per node.
4. Connect i to j with probability

$$p_{ij} \propto x_{ij}{}^{-\alpha}.$$

- $\alpha = 0$: random connections.
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But: social networks aren't lattices plus links.

Previous work—finding short paths

- If networks have hubs can also search well: Adamic et al. (2001) [1]

$$P(k_i) \propto k_i^{-\gamma}$$

where k = degree of node i (number of friends).

- Basic idea: get to hubs first (airline networks).
- But: hubs in social networks are limited.

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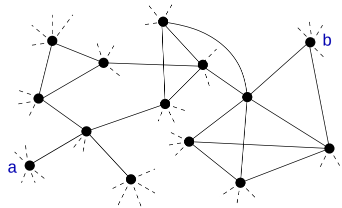
Outline

The problem

If there are no hubs and no underlying lattice, how can search be efficient?



Which friend of a is closest to the target b?

What does ‘closest’ mean?

What is ‘social distance’?

Models

One approach: incorporate identity.

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Identity is formed from attributes such as:

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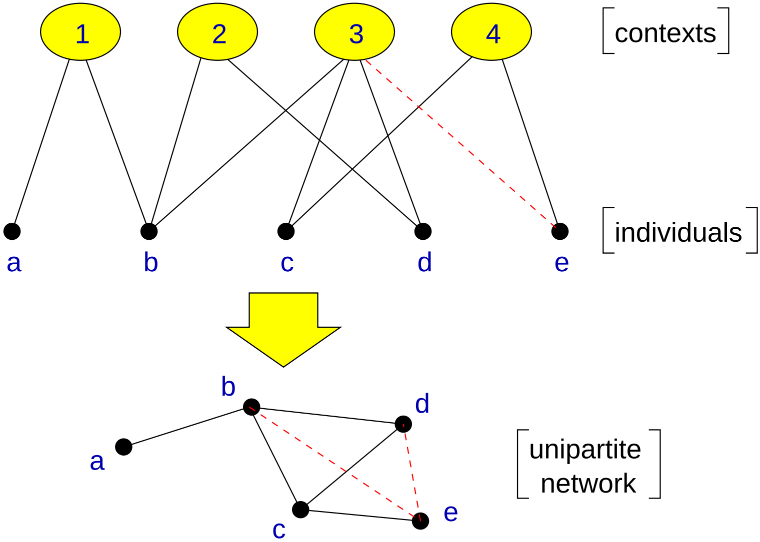
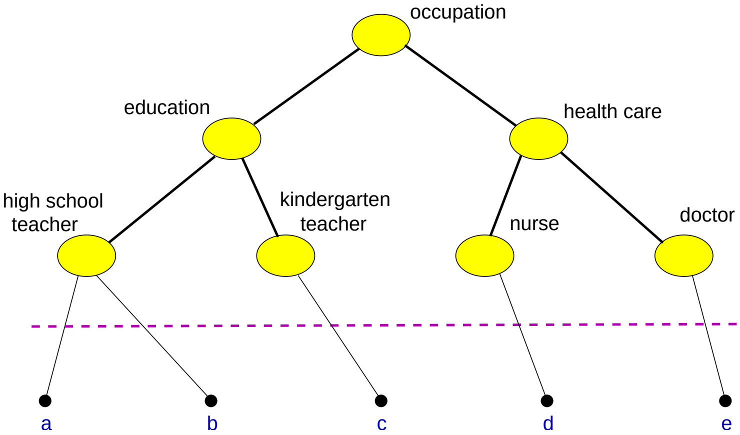
Attributes $\Leftrightarrow$ Contexts $\Leftrightarrow$ Interactions $\Leftrightarrow$ Networks.

Social distance—Bipartite affiliation networks



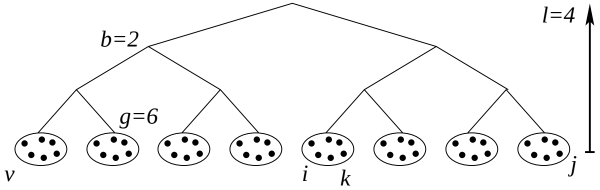
Bipartite affiliation networks: boards and directors, movies and actors.

Social distance—Context distance



Models

Distance between two individuals x_{ij} is the height of lowest common ancestor.



$x_{ij} = 3$, $x_{ik} = 1$, $x_{iv} = 4$.

Models

- Individuals are more likely to know each other the closer they are within a hierarchy.
- Construct z connections for each node using

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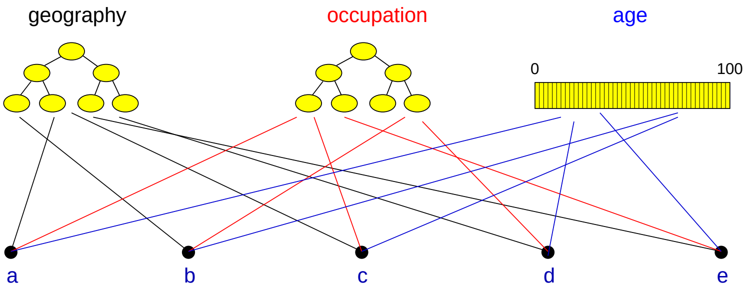
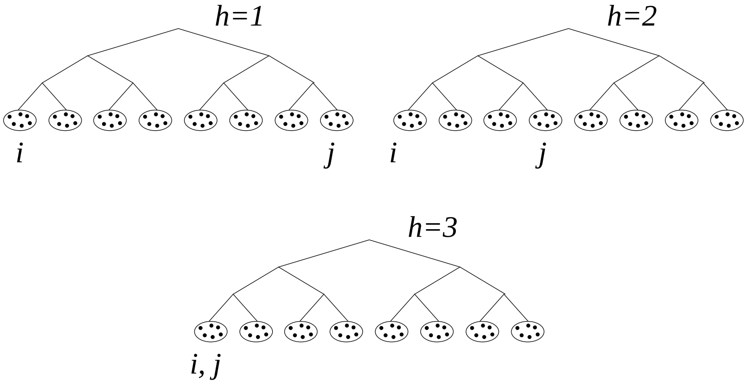
Models

Generalized affiliation networks



- Blau & Schwartz [4], Simmel [13], Breiger [5], Watts *et al.* [14]

The model



$\vec{v}_i = [1\ 1\ 1]^T$, $\vec{v}_j = [8\ 4\ 1]^T$

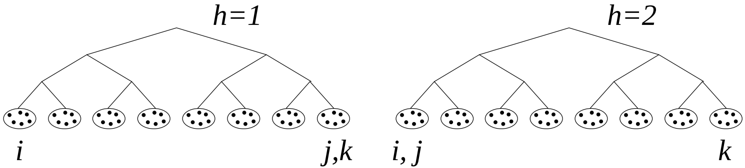
$x_{ij}^1 = 4,\ x_{ij}^2 = 3,\ x_{ij}^3 = 1.$

Social distance:

$$y_{ij} = \min_h x_{ij}^h.$$

The model

Triangle inequality doesn't hold:



$$y_{ik} = 4 > y_{ij} + y_{jk} = 1 + 1 = 2.$$

The model

- Individuals know the identity vectors of
  1. themselves,
  2. their friends,
     and
  3. the target.
- Individuals can estimate the social distance between their friends and the target.
- Use a greedy algorithm + allow searches to fail randomly.

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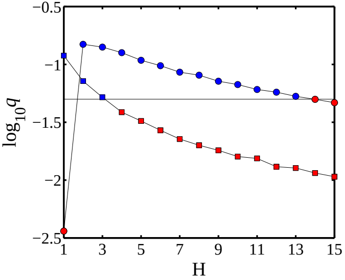
The model-results—searchable networks

$\alpha = 0$ versus $\alpha = 2$ for $N \simeq 10^5$:



$q \geq r$

$q < r$

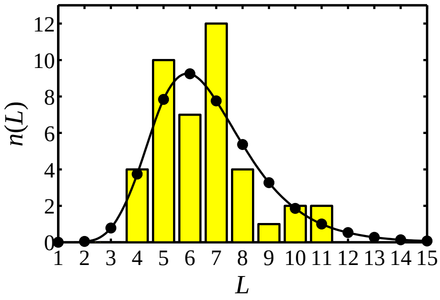
$r = 0.05$

q = probability an arbitrary message chain reaches a target.

- A few dimensions help.
- Searchability decreases as population increases.
- Precise form of hierarchy largely doesn't matter.

The model-results

Milgram's Nebraska-Boston data:



Model parameters:

- $N = 10^8$,
- $z = 300$, $g = 100$,
- $b = 10$,
- $\alpha = 1$, $H = 2$;

- $\langle L_{\mathrm{model}} \rangle \simeq 6.7$
- $L_{\mathrm{data}} \simeq 6.5$

Social search—Data

Adamic and Adar (2003)

- For HP Labs, found probability of connection as function of organization distance well fit by exponential distribution.
- Probability of connection as function of real distance $\propto 1/r$.

Social search—Data

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Social Search—Real world uses

- Tags create identities for objects
- Website tagging: `http://www.del.icio.us`
- (e.g., Wikipedia)
- Photo tagging: `http://www.flickr.com`
- Dynamic creation of metadata plus links between information objects.
- Folksonomy: collaborative creation of metadata

Social Search—Real world uses

Recommender systems:

- Amazon uses people's actions to build effective connections between books.
- Conflict between 'expert judgments' and tagging of the hoi polloi.

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Conclusions

- Bare networks are typically unsearchable.
- Paths are findable if nodes understand how network is formed.
- Importance of identity (interaction contexts).
- Improved social network models.
- Construction of peer-to-peer networks.
- Construction of searchable information databases.

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