Optimal Supply Networks

Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

Introduction

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Optimal branching

Murray meets Tokunaga

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- Murray meets Tokunaga

Single Source
- History
- Reframing the question
- Minimal volume calculation
- Blood networks
- River networks

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Distributed Sources
- Facility location
- Size-density law
- Cartograms

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- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
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- Supply and Collection are equivalent problems

River network models

Optimality:

- Optimal channel networks [10]
- Thermodynamic analogy [11]

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versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

Optimization approaches

Cardiovascular networks:

- Murray's law (1926) connects branch radii at forks:[8]

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

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- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux

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- Fluid mechanics: Poiseuille impedance for smooth flow in a tube of radius r and length ℓ:

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where η = dynamic viscosity

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- Also have rate of energy expenditure in maintaining blood:

$$P_{\mathrm{metabolic}} = c r^2 \ell$$

  where c is a metabolic constant.

Optimization approaches

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- So $\Phi \Delta P$ = Force · velocity

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  - increasing r makes flow easier but increases metabolic cost (as r^2)
  - decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

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$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

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- Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where k = constant.

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Murray's law:

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$$\Phi = kr^3$$

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- Flow rates at each branching have to add up (else our organism is in serious trouble...):

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where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

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- Flow rates at each branching have to add up (else our organism is in serious trouble...):

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- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

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- Find Horton ratio for vessell radius $R_r = r_\omega / r_{\omega-1}$...

Optimization

Murray meets Tokunaga:

- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

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- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

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- West et al (1997) [16] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [15] using Tokunaga (sort of).

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The bigger picture:

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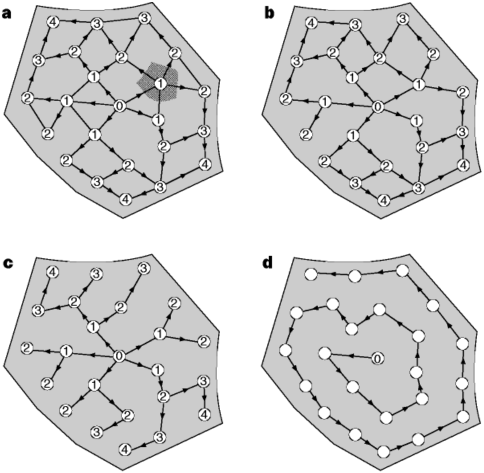
- We'll look at and build on Banavar et al.'s work...

Simple supply networks



- Banavar et al., Nature, (1999) [1]
- Very general attempt to find most efficient transportation networks.

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- Such a pachyderm would be rather miserable.

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Checking that last statement:

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- Assuming $V^{(\text{elephant})} = 10^6\,V^{\text{shrew}}$, we have

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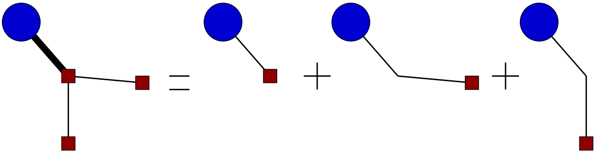

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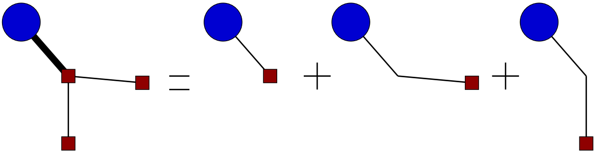

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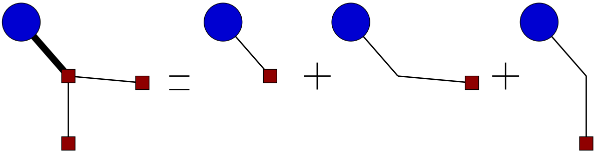
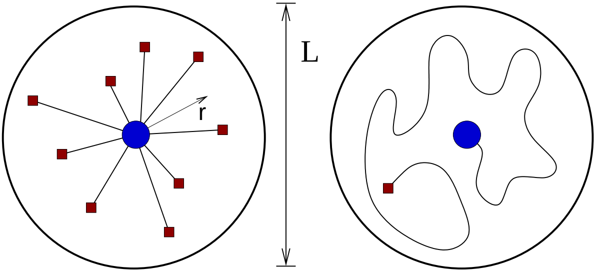


- The right question: how does number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- Or: what is highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

Geometric argument



- Best case: lengths of virtual vessels $\propto r$.
- Worst case: lengths of virtual vessels $\propto L^d$.

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- $\Rightarrow$ capillary density must decrease as M increases (observed).

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- Claim: Less Efficient networks have lower exponents too (b/c they must have lower densities of sinks).
- We'll work through these claims in detail...

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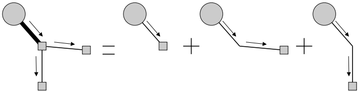
- Reminder: we break network up into virtual vessels:



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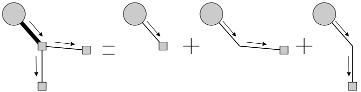

- Assume flow rate at each sink is independent of system size.

Geometric argument

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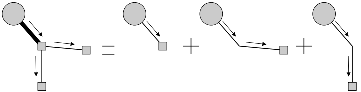

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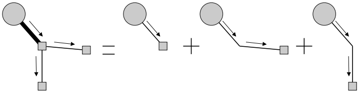

- Assume flow rate at each sink is independent of system size.
- Take the cross-sectional area *a* of virtual vessels to be constant.
- Minimizing the volume of the network is then equivalent to minimizing the sum of the path lengths from the source to all sinks.

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- Changes in impedance (e.g., due to combining of flows) may change material speed but not overall flow rate
- Scaling of material volume must be $\propto$ system volume—it's a 0th order concern.

Outline

Geometric argument

- Consider families of systems that grow allometrically.

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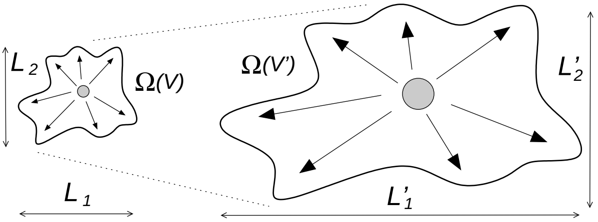
- Family = a basic shape Ω indexed by volume V.



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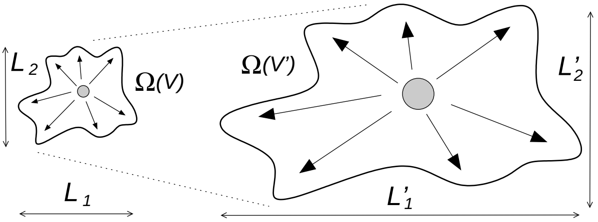

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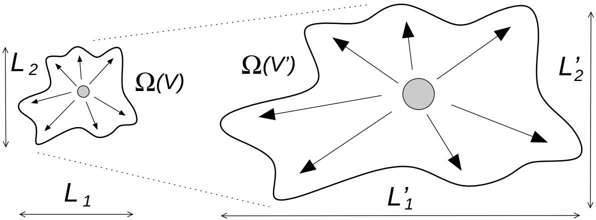

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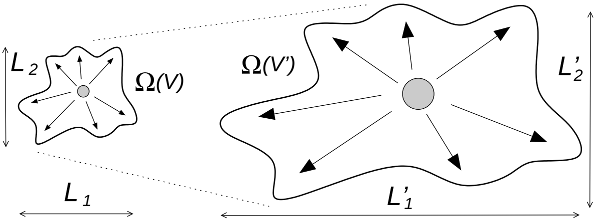

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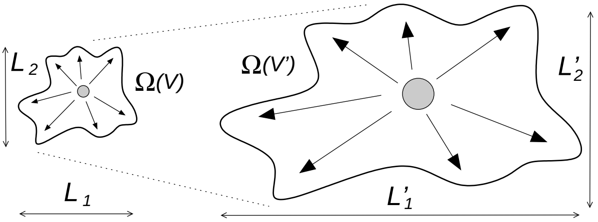

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- For above example, width grows faster than height: $\gamma_1 > \gamma_2$.

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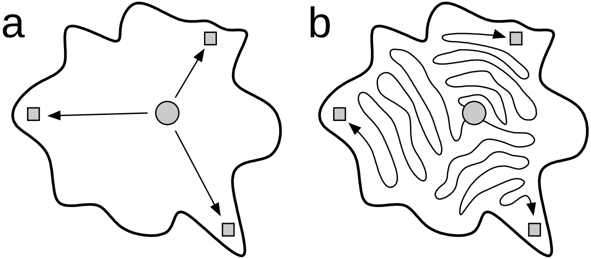
- We can generalize to a much broader class of shapes...

Geometric argument

- Reminder of best and worst configurations



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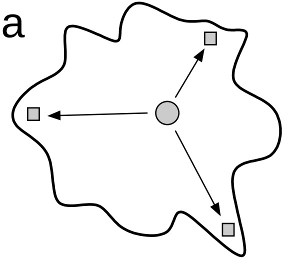

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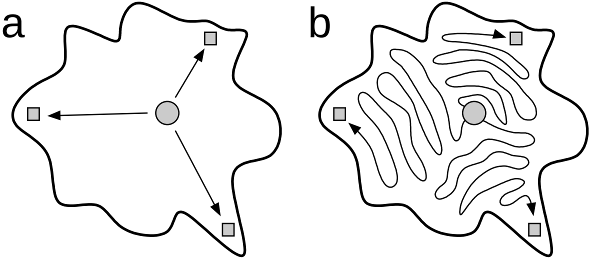

- Basic idea: Minimum volume of material in system $V_{\text{net}} \propto$ sum of distance from the source to the sinks.
- See what this means for sink density ρ if sinks do not change their feeding habits with overall size.

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  where we have rescaled to a volume of size $c < 1$ where c is the shape factor.

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$$\propto \rho V^{1+\gamma_*}$$

since integral is now nice and friendly and small.

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- We see that isometrically scaling volumes require less network volume than allometrically scaling volumes:

$$\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \rightarrow 0 \text{ as } V \rightarrow \infty$$

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Blood networks

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- Since we have shown $V_{\mathrm{net}} \propto \rho L^{d+1}$, sink density must also decrease as volume increases:

$$\rho \propto L^{-1} \propto V^{-1/d}.$$

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- We assume, reasonably, that $V \propto M$ where M is mass.
- It next follows that P, the rate of overall energy use in Ω, can at most scale with volume as

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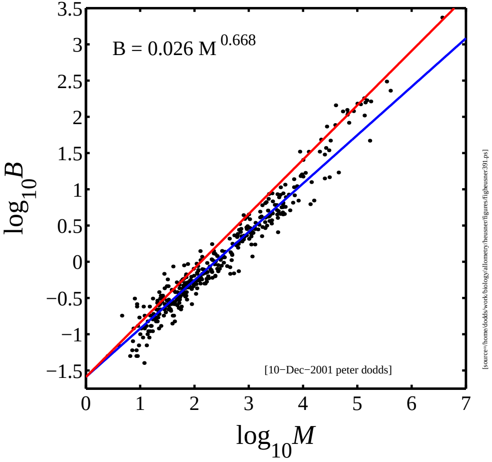
- Much controversy about all this [2] but for small mammals and birds, 2/3 scaling looks good for resting metabolic rate.

Some data on metabolic rates



- Heusner's data (1991) [5]
- 391 Mammals
- blue line: 2/3
- red line: 3/4.
- $B = P$ = power

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  where T is absolute temperature, S is surface area, ε = emissivity < 1 and σ depends on Planck's constant, speed of light, π^5, these sorts of things.
- Rough estimates of these constants give

$$P \simeq 10^5 M^{2/3} \text{erg/sec.}$$

Not bad...

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- Also: apparently some capillaries are dormant during rest.

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- It's all okay:
  Landscapes are 2-d surfaces living in 3-d.

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- It's all okay:
  Landscapes are 2-d surfaces living in 3-d.
- $D = 3$ and $d = 2$.

Geometric argument

- River networks can be seen as collection networks.
- Many sources and one sink.
- For river networks, we know ρ is constant so

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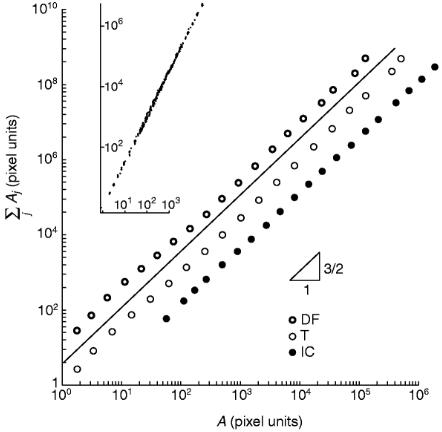

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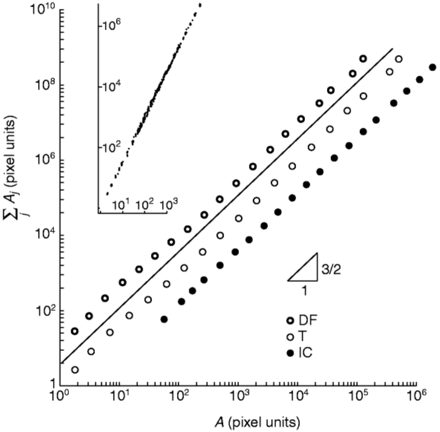

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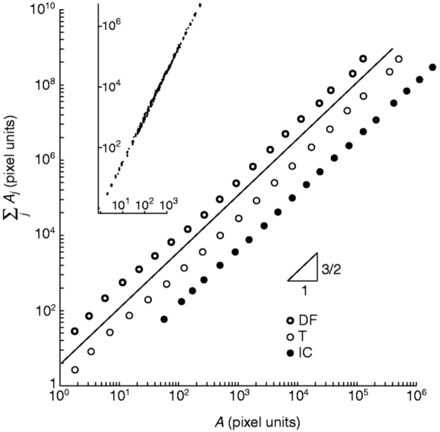

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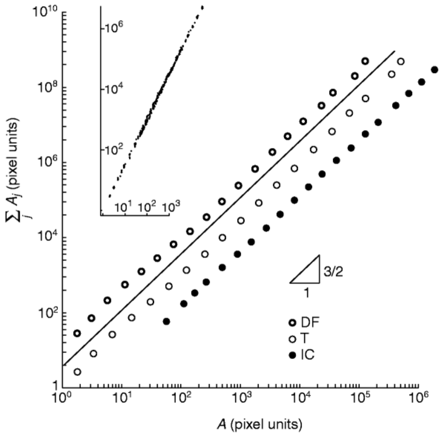

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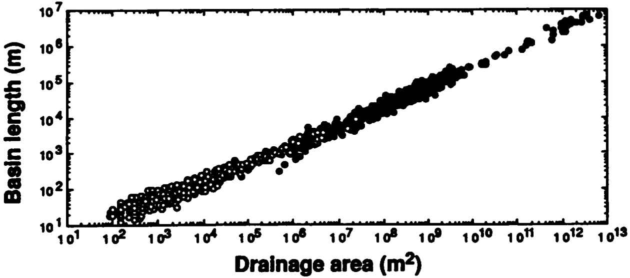
Montgomery and Dietrich [7]



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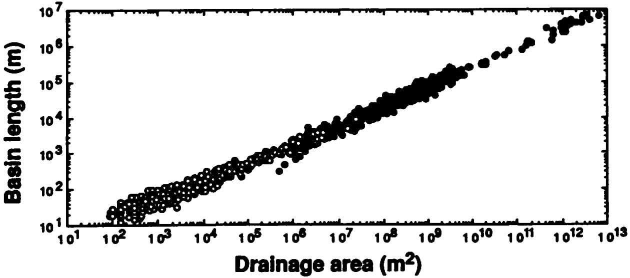
- 

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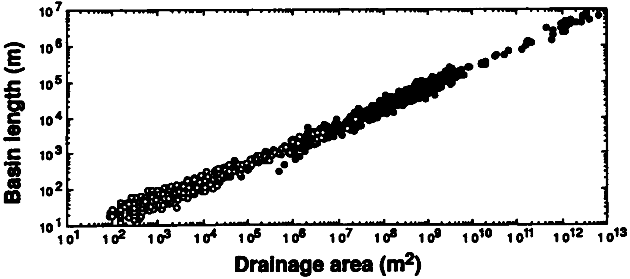
- 

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- N.b., data is a mixture of basin and main stream lengths.

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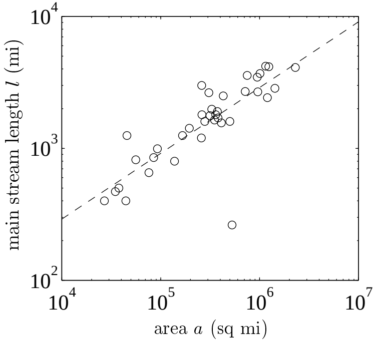
- 

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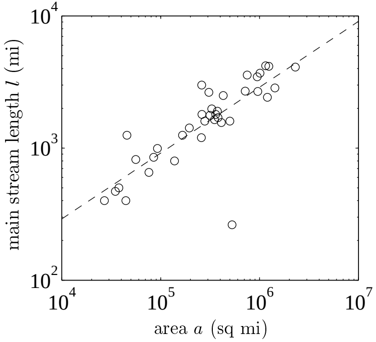

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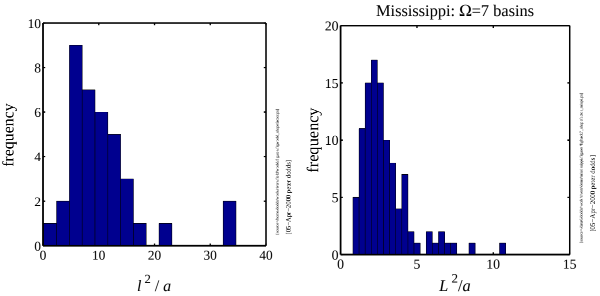
- Estimate of Hack exponent: $h = 0.50 \pm 0.06$

Optimal river networks

Large scale deviations in Hack's law



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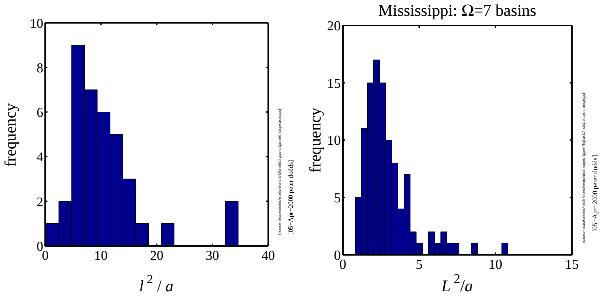
- 

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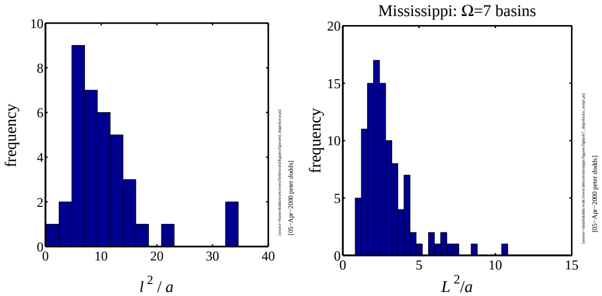
- 

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Outline

Introduction
Optimal branching
Murray meets Tokunaga
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
References

Many sources, many sinks

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- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan [13, 14] and by Gastner and Newman (2006) [4] and work cited by them.

Optimal source allocation

Solidifying the basic problem

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Solidifying the basic problem

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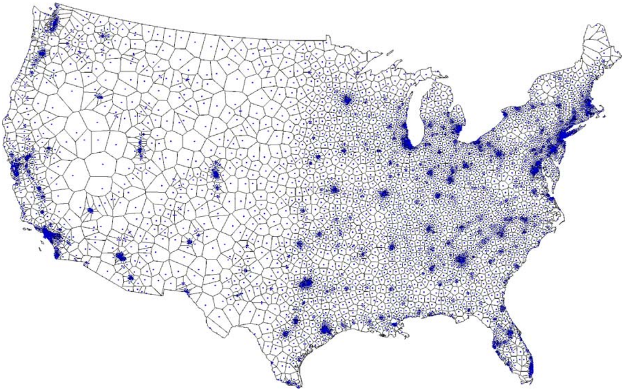
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

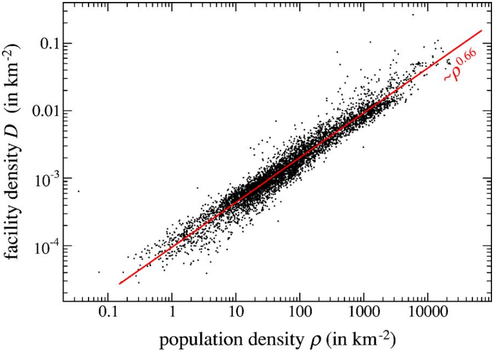
Optimal source allocation



From Gastner and Newman (2006) [4]

- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



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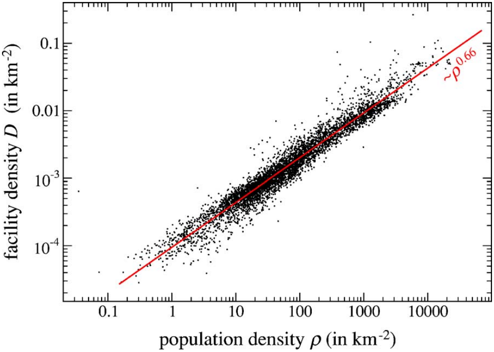

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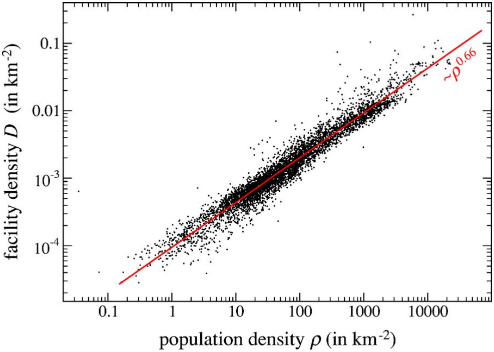

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- Now sources sinks are distributed throughout region...

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- Also known as the Homer principle.

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- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

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- Now Minimize with respect to A...

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- Groovy...

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An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)

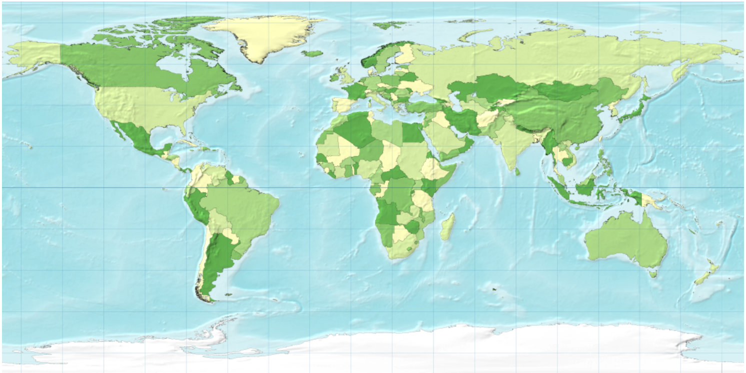
Optimal source allocation

Stephan's online book
"The Division of Territory in Society" is here (⊞).

Outline

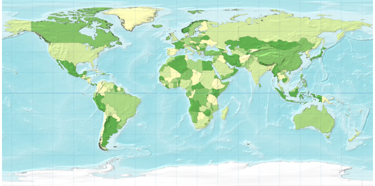
Cartograms

Standard world map:



Cartograms

Cartogram of countries 'rescaled' by population:



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- Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}$.

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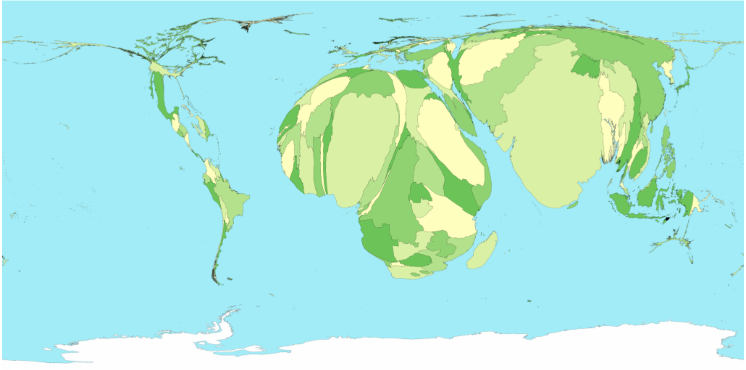
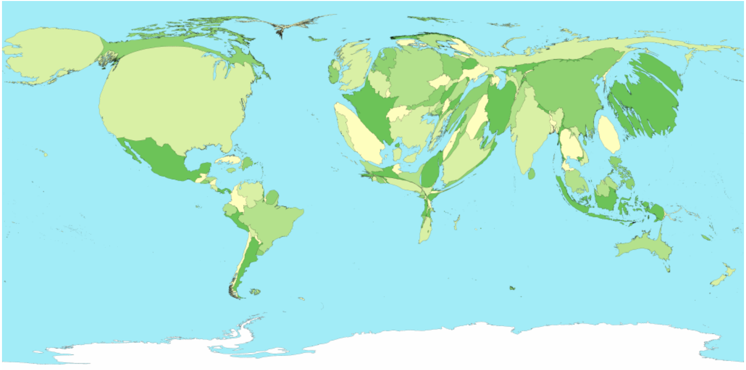
Child mortality:



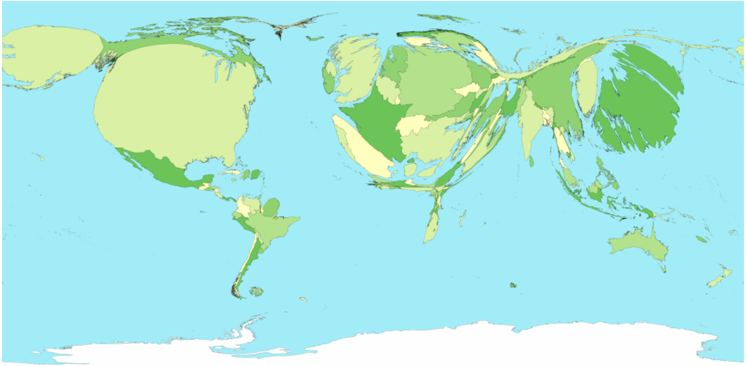
Cartograms

Energy consumption:



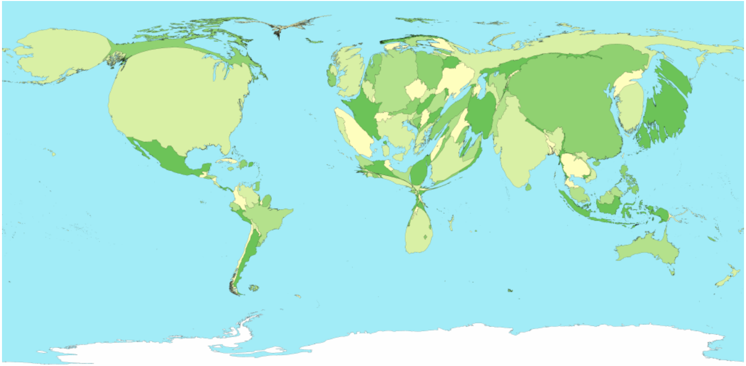
Cartograms

Gross domestic product:



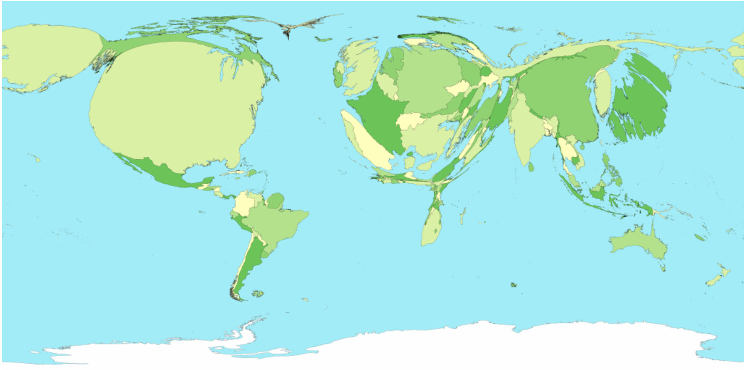
Cartograms

Greenhouse gas emissions:



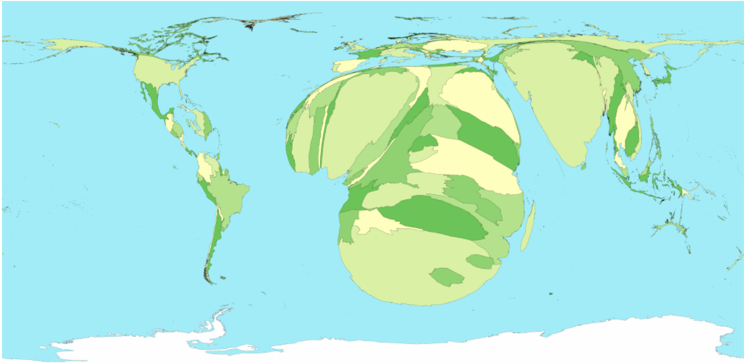
Cartograms

Spending on healthcare:



Cartograms

People living with HIV:



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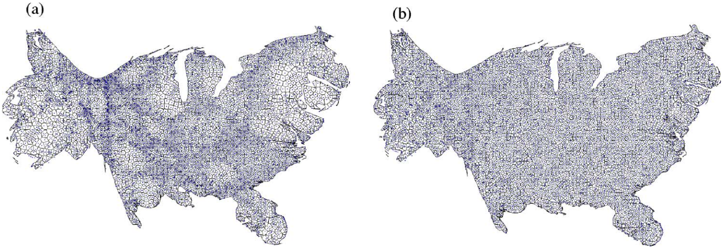
- A larger collection can be found at worldmapper.org (⊞).

WORLDMAPPER The world as you've never seen it before

Size-density law



- Left: population density-equalized cartogram.

From Gastner and Newman (2006) [4]

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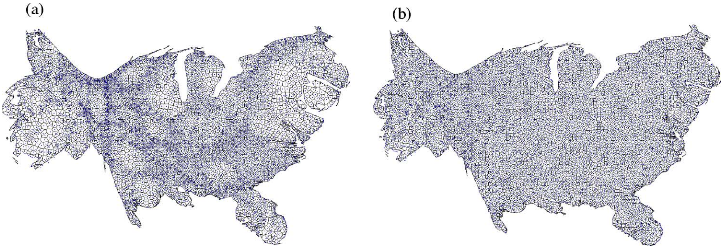

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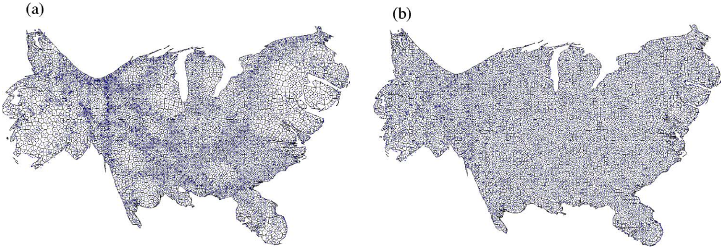

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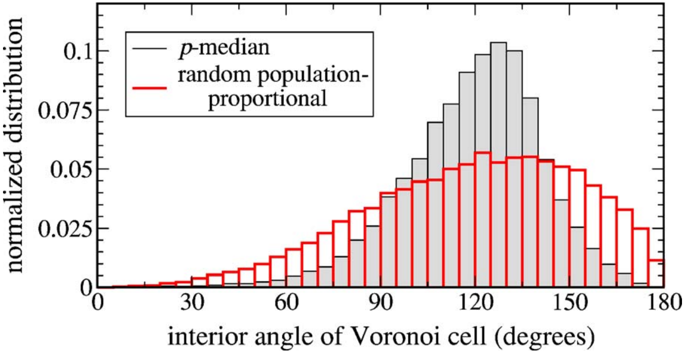
- Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [4]

Size-density law



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- Cartogram's Voronoi cells are somewhat hexagonal.

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- Also known as the p-median problem.
- Not easy... in fact this one is an NP-hard problem. [3]

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- Approximate c_i as a constant c.

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Carrying on:

- The cost function is now

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- Within each cell, $A(\vec{x})$ is constant.
- So... integral over each of the n cells equals 1.

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Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_{\Omega} \left[A(\vec{x}) \right]^{-1} \mathrm{d}\vec{x} \right)$$

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- Setting the integrand to be zilch, we have:

$$\rho(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

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- Normalizing (or solving for λ):

$$D(\vec{x}) = n\frac{[\rho(\vec{x})]^{2/3}}{\int_{\Omega}[\rho(\vec{x})]^{2/3}\mathrm{d}\vec{x}} \propto [\rho(\vec{x})]^{2/3}.$$

Global redistribution networks

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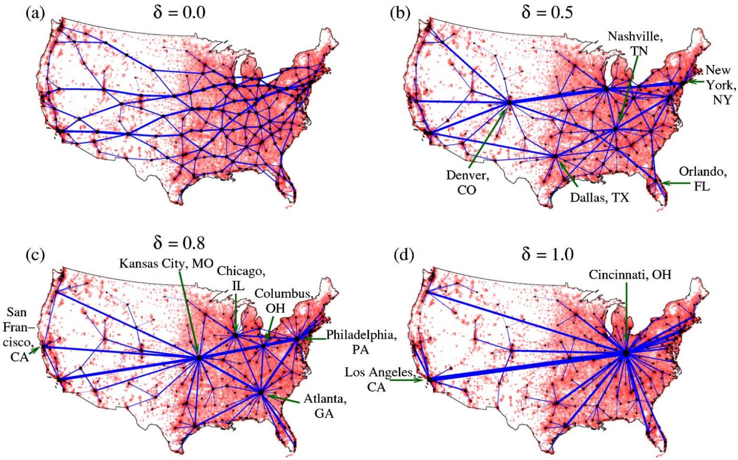
- When $\delta = 1$, only number of hops matters.

Global redistribution networks



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