

Optimal Supply Networks

Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont



Licensed under the *Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License*.

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 1/85

Outline

Introduction

Optimal branching

Murray meets Tokunaga

Single Source

History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources

Facility location
Size-density law
Cartograms

References

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 2/85

Optimal supply networks

What's the best way to distribute stuff?

- ▶ Stuff = medical services, energy, people,
- ▶ **Some** fundamental network problems:
 1. Distribute stuff from a **single source** to **many sinks**
 2. Distribute stuff from **many sources** to many sinks
 3. **Redistribute** stuff between nodes that are both sources and sinks
- ▶ Supply and Collection are equivalent problems

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 3/85

River network models

Optimality:

- ▶ Optimal channel networks^[10]
- ▶ Thermodynamic analogy^[11]

versus...

Randomness:

- ▶ Scheidegger's directed random networks
- ▶ Undirected random networks

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 4/85

Optimization approaches

Cardiovascular networks:

- ▶ Murray's law (1926) connects branch radii at forks: [8]

$$r_0^3 = r_1^3 + r_2^3$$

where r_0 = radius of main branch
and r_1 and r_2 are radii of sub-branches

- ▶ Calculation assumes Poiseuille flow
- ▶ Holds up well for outer branchings of blood networks
- ▶ Also found to hold for trees
- ▶ Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

where Δp = pressure difference, Φ = flux

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 5/85

Optimization approaches

Cardiovascular networks:

- ▶ Fluid mechanics: Poiseuille impedance for smooth flow in a tube of radius r and length ℓ :

$$Z = \frac{8\eta\ell}{\pi r^4}$$

where η = dynamic viscosity

- ▶ Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z$$

- ▶ Also have rate of energy expenditure in maintaining blood:

$$P_{\text{metabolic}} = cr^2\ell$$

where c is a metabolic constant.

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 6/85

Optimization approaches

Aside on P_{drag}

- ▶ Work done = $F \cdot d$ = energy transferred by force F
- ▶ Power = rate work is done = $F \cdot v$
- ▶ ΔP = Force per unit area
- ▶ Φ = Volume per unit time
= cross-sectional area · velocity
- ▶ So $\Phi \Delta P$ = Force · velocity

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 7/85

Optimization approaches

Murray's law:

- ▶ Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

- ▶ Observe power increases linearly with ℓ
- ▶ But r 's effect is nonlinear:
 - ▶ increasing r makes flow easier **but increases metabolic cost** (as r^2)
 - ▶ decreasing r decrease metabolic cost **but impedance goes up** (as r^{-4})

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 8/85

Optimization

Murray's law:

- ▶ Minimize P with respect to r :

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left(\phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$

$$= -4\phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- ▶ Rearrange/cancel/slap:

$$\phi^2 = \frac{c\pi r^6}{16\eta} = k^2 r^6$$

where $k = \text{constant}$.

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 9/85

Optimization

Murray's law:

- ▶ So we now have:

$$\phi = kr^3$$

- ▶ Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- ▶ All of this means we have a groovy cube-law:

$$r_0^3 = r_1^3 + r_2^3$$

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 10/85

Optimization

Murray meets Tokunaga:

- ▶ $\Phi_\omega = \text{volume rate of flow into an order } \omega \text{ vessel segment}$
- ▶ Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- ▶ Using $\phi_\omega = kr_\omega^3$

$$r_\omega^3 = 2r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

- ▶ Find Horton ratio for vessel radius $R_r = r_\omega / r_{\omega-1} \dots$

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 12/85

Optimization

Murray meets Tokunaga:

- ▶ Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$R_r^3 = R_n = R_v = R_n^3$$

- ▶ Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 13/85

Optimization

Murray meets Tokunaga:

- ▶ Isometry: $V_w \propto l_w^3$
- ▶ Gives $R_l^3 = R_v = R_n$
- ▶ We need one more constraint...
- ▶ West et al (1997)^[16] achieve similar results following Horton's laws.
- ▶ So does Turcotte et al. (1998)^[15] using Tokunaga (sort of).

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 14/85

Optimization approaches

The bigger picture:

- ▶ Rashevsky (1960's)^[9] showed using a network story that power output of heart should scale as $M^{2/3}$
- ▶ West et al. (1997 on)^[16, 2] managed to find $M^{3/4}$ (a mess—super long story—see previous course...)
- ▶ Banavar et al.^[1] attempt to derive a general result for all natural branching networks
- ▶ Again, something of a mess^[2]
- ▶ We'll look at and build on Banavar et al.'s work...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

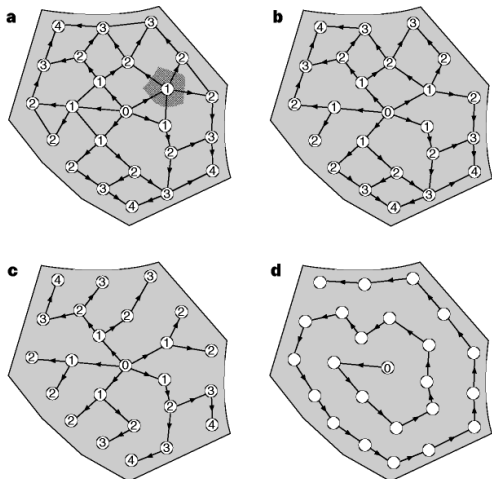
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 16/85

Simple supply networks



- ▶ Banavar et al., Nature, (1999)^[1]
- ▶ Very general attempt to find most efficient transportation networks.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 17/85

Simple supply networks

- ▶ Banavar *et al.* find 'most efficient' networks with

$$P \propto M^{d/(d+1)}$$

- ▶ ... but also find

$$V_{\text{blood}} \propto M^{(d+1)/d}$$

- ▶ Consider a 3 g shrew with $V_{\text{blood}} = 0.1 V_{\text{body}}$
- ▶ \Rightarrow 3000 kg elephant with $V_{\text{blood}} = 10 V_{\text{body}}$
- ▶ Such a pachyderm would be rather miserable.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 18/85

Pachydermal sadness

Checking that last statement:

- ▶ For $d = 3$, we have $V_{\text{blood}} = cV^{(d+1)/d} = cV^{4/3}$
- ▶ If our shrew has $V_{\text{blood}}^{(\text{shrew})} = 0.1 V^{(\text{shrew})}$ then $c = 0.1(V^{(\text{shrew})})^{-1/3}$.
- ▶ Assuming $V^{(\text{elephant})} = 10^6 V^{\text{shrew}}$, we have

$$\begin{aligned}
 V_{\text{blood}}^{(\text{elephant})} &= c(V^{(\text{elephant})})^{4/3} \\
 &= \underbrace{0.1(V^{(\text{shrew})})^{-1/3}}_c \underbrace{(10^6 V^{(\text{shrew})})^{4/3}}_{V^{(\text{elephant})}} \\
 &= 10^7 V^{(\text{shrew})} = 10V^{(\text{elephant})}.
 \end{aligned}$$

▶ Oops.

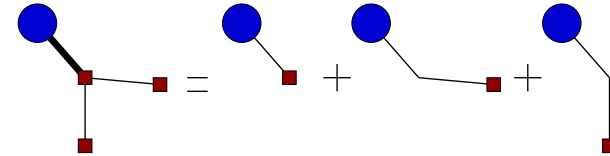
Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 19/85

Geometric argument

- ▶ Consider **one source supplying many sinks** in a d dimensional volume
- ▶ Material draw by sinks is invariant.
- ▶ Assume some cap on flow speed of material, v_{max}
- ▶ See network as a bundle of virtual vessels:



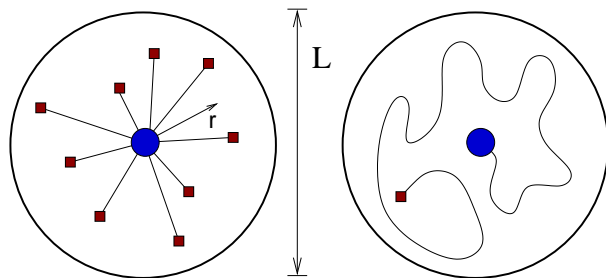
- ▶ **The right question:** how does number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- ▶ Or: what is highest α for $N_{\text{sinks}} \propto V^\alpha$?

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 21/85

Geometric argument



- ▶ Best case: lengths of virtual vessels $\propto r$.
- ▶ Worst case: lengths of virtual vessels $\propto L^d$.

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 22/85

Geometric argument

- ▶ Banavar *et al.* assume sink density ρ is **uniform**
- ▶ If we allow ρ to vary, then we find

$$V_{\text{blood}} \propto \rho L^{d+1}$$

- ▶ Since $V_{\text{blood}} \propto L^d$, we must have $\rho \propto L^{-1}$.
- ▶ \Rightarrow capillary density must decrease as M increases (observed).

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 23/85

Geometric argument

- ▶ $N_{\text{sinks}} \propto \rho L^d \propto L^{-1} L^d \propto M^{(d-1)/d}$
- ▶ so for $d = 3$, we have $\alpha = 2/3$.
- ▶ for $d = 2$, we have $\alpha = 1/2$.
- ▶ Claim: If volume shapes change allometrically, the exponent **decreases**.
- ▶ Claim: Less Efficient networks have **lower** exponents too (b/c they must have lower densities of sinks).
- ▶ We'll work through these claims in detail...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

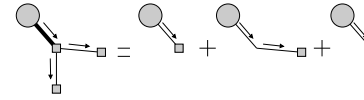
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 24/85

Geometric argument

- ▶ Reminder: we break network up into virtual vessels:



- ▶ Assume flow rate at each sink is independent of system size.
- ▶ Take the cross-sectional area a of virtual vessels to be constant.
- ▶ Minimizing the volume of the network is then equivalent to minimizing the sum of the path lengths from the source to all sinks.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 25/85

Geometric argument

- ▶ Note: we are ignoring issues such as impedance.
- ▶ Changes in impedance (e.g., due to combining of flows) may change material speed **but not overall flow rate**
- ▶ Scaling of material volume must be \propto system volume—it's a 0th order concern.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

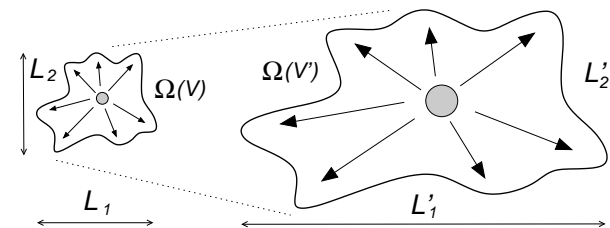
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 26/85

Geometric argument

- ▶ Consider families of systems that grow allometrically.
- ▶ Family = a basic shape Ω indexed by volume V .



- ▶ Orient shape to have dimensions $L_1 \times L_2 \times \dots \times L_d$
- ▶ In 2-d, $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ where $A = \text{area}$.
- ▶ In general, have d lengths which scale as $L_i \propto V^{\gamma_i}$.
- ▶ For above example, width grows faster than height:
 $\gamma_1 > \gamma_2$.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 28/85

Geometric argument

Some generality:

- ▶ Consider d dimensional spatial regions living in D dimensional ambient spaces. Notation: $\Omega_{d,D}(V)$.
- ▶ River networks: $d = 2$ and $D = 3$
- ▶ Cardiovascular networks: $d = 3$ and $D = 3$
- ▶ **Star-convexity of $\Omega_{d,D}(V)$:** A spatial region is star-convex if from at least one point, all other points in the region can be reached by travelling along straight lines while remaining within the region.
- ▶ Assume source can be located at a point which has direct line of sight to all sources.
- ▶ We can generalize to a much broader class of shapes...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

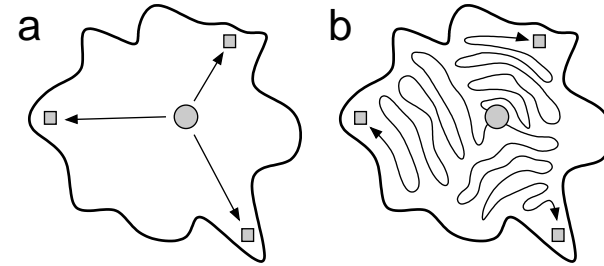
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 29/85

Geometric argument

▶ Reminder of best and worst configurations



- ▶ **Basic idea:** Minimum volume of material in system $V_{\text{net}} \propto$ sum of distance from the source to the sinks.
- ▶ See what this means for sink density ρ if sinks do not change their feeding habits with overall size.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 30/85

Geometric argument

Assumptions in detail:

- ▶ Each region $\Omega_{d,D}(V)$ has overall dimensions $L_1 \times L_2 \times \dots \times L_d$.
- ▶ Specifically, $V = cL_1L_2 \dots L_d$ where $c \leq 1$ is a shape factor dependent of Ω .
- ▶ We allow for arbitrary shape scaling:

$$L_i = c_i^{-1} V^{\gamma_i}$$

where $\prod_{i=1}^d c_i = c$ and $\sum_{i=1}^d \gamma_i = 1$.

- ▶ For **isometric** growth, $\gamma_i = 1/d$.
- ▶ For **allometric** growth, we must have at least two of the $\{\gamma_i\}$ being different
- ▶ We choose the L_i so that $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_d$

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 31/85

Computing the minimal network volume:

▶

$$\begin{aligned} \min V_{\text{net}} &\propto \int_{\Omega_{d,D}(V)} \rho \|\vec{x}\| d\vec{x} \\ &= \rho \int_{\Omega_{d,D}(V)} (x_1^2 + x_2^2 + \dots + x_d^2)^{1/2} d\vec{x} \end{aligned}$$

▶ Substituting $x_i = L_i u_i$, we have

$$\begin{aligned} \min V_{\text{net}} &\propto \rho L_1 \dots L_d \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + \dots + L_d^2 u_d^2)^{1/2} d\vec{u} \\ &\propto \rho V \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + L_2^2 u_2^2 + \dots + L_d^2 u_d^2)^{1/2} d\vec{u} \end{aligned}$$

where we have rescaled to a volume of size $c < 1$ where c is the shape factor.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 32/85

Computing the minimal network volume:

- ▶ We are here:

$$\min V_{\text{net}} \propto \rho V \int_{\Omega_{d,D}(c)} (L_1^2 u_1^2 + L_2^2 u_2^2 + \dots + L_d^2 u_d^2)^{1/2} d\vec{u}$$

- ▶ **Observe** that the integrand will be dominated by the L_i that scale strongest with V .
- ▶ Assume first $k \leq d$ dimensions scale with equal strength, $L_i = c_i^{-1} V^{\gamma_*}$.
- ▶ Plug in scaling for L_i in terms of V and pull V^{γ_*} out to the front.

$$\min V_{\text{net}} \propto \rho V V^{\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^{-2} u_1^2 + \dots + c_k^{-2} u_k^2 + \dots + c_{k+1}^{-2} V^{2(\gamma_{k+1}-\gamma_*)} u_{k+1}^2 + \dots + c_d^{-2} V^{2(\gamma_d-\gamma_*)} u_d^2)^{1/2} d\vec{u}$$

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 33/85

Computing the minimal network volume:

- ▶ Where we are now:

$$\min V_{\text{net}} \propto \rho V^{1+\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^{-2} u_1^2 + \dots + c_k^{-2} u_k^2 + \dots + c_{k+1}^2 V^{2(\gamma_{k+1}-\gamma_*)} u_{k+1}^2 + \dots + c_d^2 V^{2(\gamma_d-\gamma_*)} u_d^2)^{1/2} d\vec{u}$$

- ▶ Now allow $V \rightarrow \infty$ and see that part of integrand vanishes:

$$\min V_{\text{net}} \rightarrow \rho V^{1+\gamma_*} \int_{\Omega_{d,D}(c)} (c_1^2 u_1^2 + \dots + c_k^2 u_k^2)^{1/2} d\vec{u} \propto \rho V^{1+\gamma_*}$$

since integral is now nice and friendly and small.

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 34/85

Geometric argument

- ▶ Our general result:

$$\min V_{\text{net}} \propto \rho V^{1+\gamma_*}$$

- ▶ For scaling is **isometric**, we have $\gamma_* = \gamma_{\text{iso}} = 1/d$ and all the L_i scale as $V^{1/d}$:

$$\min V_{\text{net/iso}} \propto \rho V^{1+1/d} = \rho V^{(d+1)/d}$$

- ▶ If scaling is **allometric**, we have

$$\gamma_* = \gamma_{\text{allo}} = \max_i \gamma_i > 1/d \text{ and}$$

$$\min V_{\text{net/allo}} \propto \rho V^{1+\gamma_{\text{allo}}}$$

- ▶ We see that isometrically scaling volumes **require less network volume** than allometrically scaling volumes:

$$\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \rightarrow 0 \text{ as } V \rightarrow \infty$$

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 35/85

Geometric argument

Blood networks

- ▶ Material costly \Rightarrow expect lower optimal bound of $V_{\text{net}} \propto \rho V^{(d+1)/d} \propto \rho L^{d+1}$ to be closely followed.
- ▶ For cardiovascular networks, $d = D = 3$.
- ▶ Know that volume of blood scales linearly with blood volume^[12], $V_{\text{net}} \propto V_{\Omega} \propto L^d$.
- ▶ Since we have shown $V_{\text{net}} \propto \rho L^{d+1}$, sink density must also decrease as volume increases:

$$\rho \propto L^{-1} \propto V^{-1/d}.$$

Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 37/85

Geometric argument

Blood networks

- ▶ We assume, reasonably, that $V \propto M$ where M is mass.
- ▶ It next follows that P , the rate of overall energy use in Ω , can at most scale with volume as

$$P \propto \rho V \propto \rho M \propto M^{(d-1)/d}$$

- ▶ For three dimensional organisms, we have $P \propto M^{2/3}$.
- ▶ Much controversy about all this [2] but for small mammals and birds, 2/3 scaling looks good for resting metabolic rate.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation

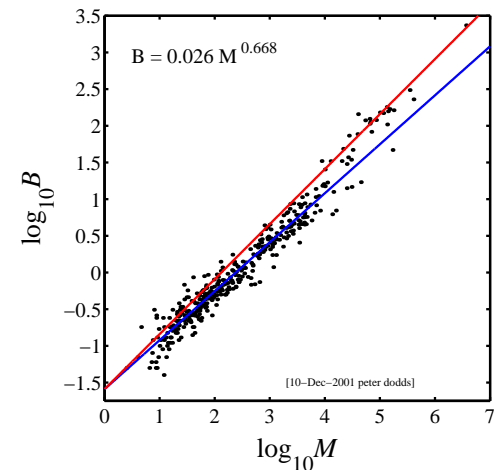
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 38/85

Some data on metabolic rates



- ▶ Heusner's data (1991) [5]
- ▶ 391 Mammals
- ▶ blue line: 2/3
- ▶ red line: 3/4.
- ▶ $B = P =$ power

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation

Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 39/85

Geometric argument

Interesting result from quantum mechanics:

- ▶ Homeothermic organisms need to keep their temperature static
- ▶ A good amount of heat loss is through infra-red radiation (when resting)
- ▶ For mammals with $M \leq 10$ kg:
 $P = 2.57 \times 10^5 M^{2/3}$ erg/sec.
- ▶ Stefan-Boltzmann's law (☼): $\frac{dE}{dt} = \sigma \varepsilon S T^4$
where T is absolute temperature, S is surface area, ε = emissivity < 1 and σ depends on Planck's constant, speed of light, π^5 , these sorts of things.
- ▶ Rough estimates of these constants give

$$P \simeq 10^5 M^{2/3} \text{erg/sec.}$$

Not bad...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation

Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 40/85

Geometric argument

Organisms at work:

- ▶ What about organisms working as hard as possible?
- ▶ For short bursts, power scales closer to mass.
- ▶ Energy is stored locally muscles and we have accounted for this.
- ▶ Also: apparently some capillaries are dormant during rest.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation

Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 41/85

Geometric argument

- ▶ River networks can be seen as collection networks.
- ▶ Many sources and one sink.
- ▶ For river networks, we know ρ is constant so

$$V_{\text{net}} \propto \rho V^{(d+1)/d} = \text{constant} \times V^{3/2}$$

- ▶ **Hmmm:** now network volume is growing faster than basin 'volume' (really area).
- ▶ **It's all okay:** Landscapes are 2-d surfaces living in 3-d.
- ▶ $D = 3$ and $d = 2$.
- ▶ Streams can grow not just in width but in depth...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 43/85

Geometric argument

- ▶ Volume of water in river network can be calculated by adding up basin areas
- ▶ (Discreteness of data means summing instead of integrating)
- ▶ Each site on discrete lattice is a source.
- ▶ Imagine a steady flow from each source to outlet.
- ▶ Flows sum in such a way that

$$V_{\text{net}} = \sum_{\text{all pixels } i} a_{\text{pixel } i}$$

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

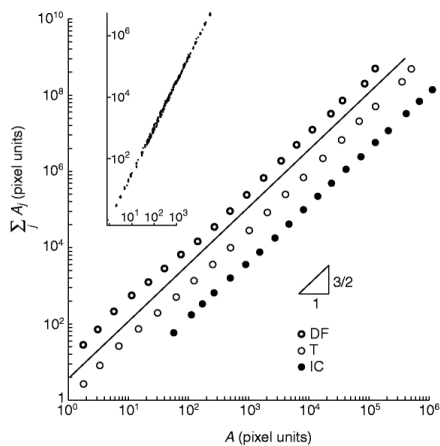
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 44/85

Geometric argument

- ▶ Banavar et al.'s approach [1] is okay because ρ really is constant.
- ▶ **The irony:** shows optimal basins are isometric
- ▶ Optimal Hack's law: $a \sim \ell^h$ with $h = 1/2$
- ▶ (Zzzzz)



From Banavar et al. (1999) [1]

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

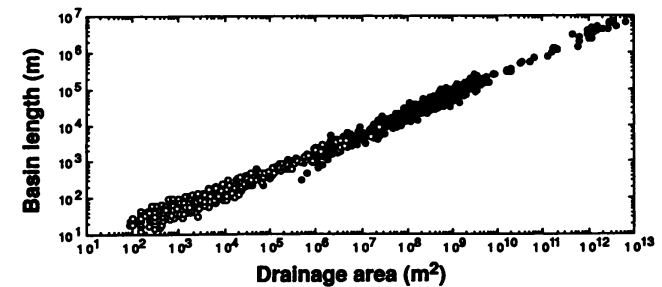
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 45/85

Geometric argument: evidence

Montgomery and Dietrich [7]



- ▶ Composite data set: includes everything from unchanneled valleys up to world's largest rivers.
- ▶ Estimated fit:

$$L \simeq 1.78a^{0.49}$$
- ▶ N.b., data is a mixture of basin and main stream lengths.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

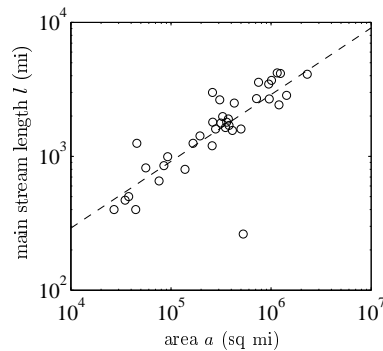
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 46/85

World's largest rivers only:



- ▶ Data from Leopold (1994) [6]
- ▶ Estimate of Hack exponent: $h = 0.50 \pm 0.06$

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

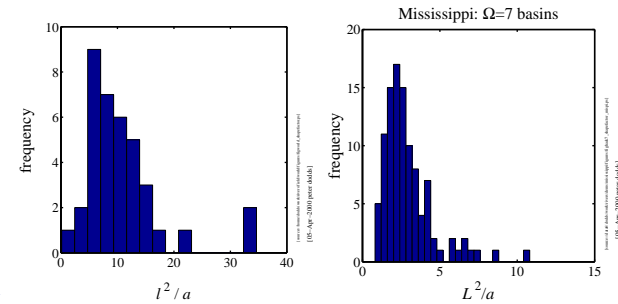
Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 47/85

Optimal river networks

Large scale deviations in Hack's law



- ▶ Rivers seem generally relatively long (but isometric).
- ▶ Measured width/length ratio unexplained.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 48/85

Many sources, many sinks

How do we distribute sources?

- ▶ Focus on 2-d (results generalize to higher dimensions)
- ▶ Sources = hospitals, post offices, pubs, ...
- ▶ **Key problem:** How do we cope with uneven population densities?
- ▶ Obvious: if density is uniform then sources are best distributed **uniformly**
- ▶ Which lattice is optimal? The **hexagonal lattice**
- ▶ **Q1:** How big should the hexagons be?
- ▶ **Q2:** Given population density is uneven, what do we do?
- ▶ We'll follow work by Stephan [13, 14] and by Gastner and Newman (2006) [4] and work cited by them.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 50/85

Optimal source allocation

Solidifying the basic problem

- ▶ Given a region with some population distribution ρ , most likely uneven.
- ▶ Given resources to build and maintain N facilities.
- ▶ **Q:** How do we locate these N facilities so as to **minimize the average distance** between an **individual's residence** and the **nearest facility**?

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

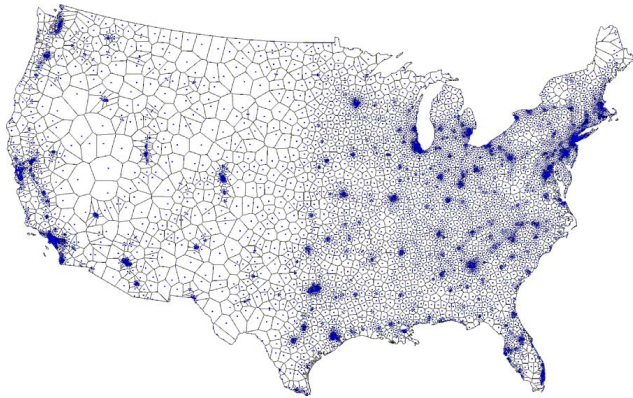
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 51/85

Optimal source allocation



From Gastner and Newman (2006) [4]

- ▶ Approximately optimal location of 5000 facilities.
- ▶ Based on 2000 Census data.
- ▶ Simulated annealing + Voronoi tessellation.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

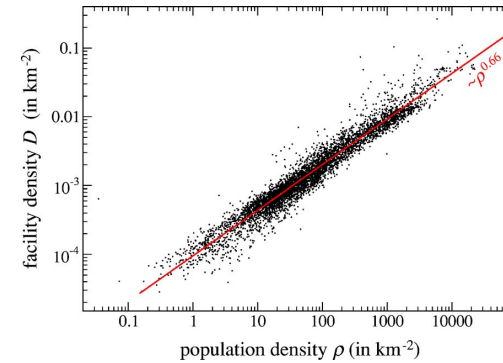
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 52/85

Optimal source allocation



From Gastner and Newman (2006) [4]

- ▶ Optimal facility density D vs. population density ρ .
- ▶ Fit is $D \propto \rho^{0.66}$ with $r^2 = 0.94$.
- ▶ Looking good for a 2/3 power...

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 53/85

Optimal source allocation

Size-density law:

- ▶
- ▶ Why?
- ▶ Again: Different story to branching networks where there was either one source or one sink.
- ▶ Now sources/sinks are distributed throughout region...

$$D \propto \rho^{2/3}$$

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 55/85

Optimal source allocation

- ▶ We first examine Stephan's treatment (1977) [13, 14]
- ▶ "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- ▶ Zipf-like approach: invokes principle of minimal effort.
- ▶ Also known as the Homer principle.

Supply Networks

Introduction

Optimal branching
Murray meets Tokunaga

Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks

Distributed Sources
Facility location
Size-density law
Cartograms

References

Frame 56/85

Optimal source allocation

- ▶ Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- ▶ Build up a general cost function based on time expended to **access and maintain center**.
- ▶ Write **average travel distance** to center is \bar{d} and assume **average speed of travel** is \bar{v} .
- ▶ Note that average travel distance will be on the length scale of the region which is $A^{1/2}$
- ▶ Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 57/85

Optimal source allocation

- ▶ Next assume facility requires regular maintenance (person-hours per day)
- ▶ Call this quantity τ
- ▶ If burden of maintenance is shared then average cost per person is τ/P .
- ▶ Replace P by ρA where ρ is density.
- ▶ Total average time cost per person:

$$T = \bar{d}/\bar{v} + \tau/(\rho A) = gA^{1/2}/\bar{v} + \tau/(\rho A).$$

- ▶ Now Minimize with respect to A ...

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 58/85

Optimal source allocation

- ▶ Differentiating...

$$\begin{aligned}\frac{\partial T}{\partial A} &= \frac{\partial}{\partial A} \left(cA^{1/2}/\bar{v} + \tau/(\rho A) \right) \\ &= c/(2\bar{v}A^{1/2}) - \tau/(\rho A^2) = 0\end{aligned}$$

- ▶ Rearrange:

$$A = (2\bar{v}\tau/c\rho)^{2/3} \propto \rho^{-2/3}$$

- ▶ # facilities per unit area \propto

$$A^{-1} \propto \rho^{2/3}$$

- ▶ Groovy...

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 59/85

Optimal source allocation

An issue:

- ▶ Maintenance (τ) is assumed to be **independent** of population and area (P and A)

Supply Networks

- Introduction
- Optimal branching
Murray meets Tokunaga
- Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
- Distributed Sources
Facility location
Size-density law
Cartograms
- References

Frame 60/85

Optimal source allocation

Stephan's online book

"The Division of Territory in Society" is [here](#) (田).

Supply Networks

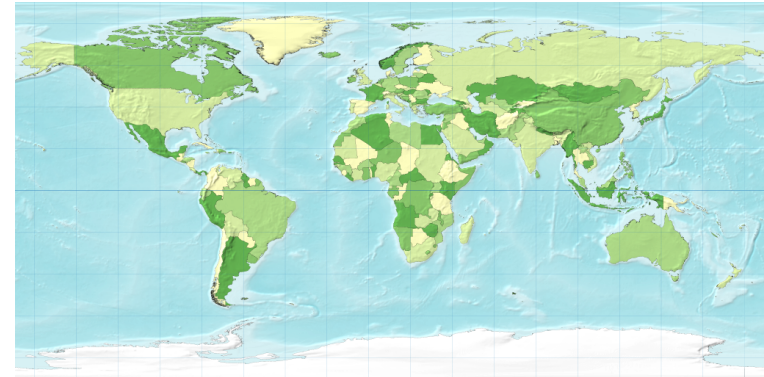
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 61/85



Cartograms

Standard world map:



Supply Networks

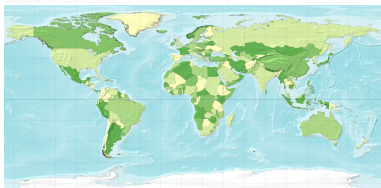
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 63/85



Cartograms

Cartogram of countries 'rescaled' by population:



Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 64/85



Cartograms

Diffusion-based cartograms:

- ▶ Idea of cartograms is to **distort areas** to more accurately represent some local density ρ (e.g. population).
- ▶ Many methods put forward—typically involve some kind of physical analogy to **spreading or repulsion**.
- ▶ Algorithm due to Gastner and Newman (2004) [3] is based on **standard diffusion**:

$$\nabla^2 \rho - \frac{\partial \rho}{\partial t} = 0.$$

- ▶ Allow density to diffuse and trace the movement of individual elements and boundaries.
- ▶ Diffusion is constrained by boundary condition of surrounding area having density $\bar{\rho}$.

Supply Networks

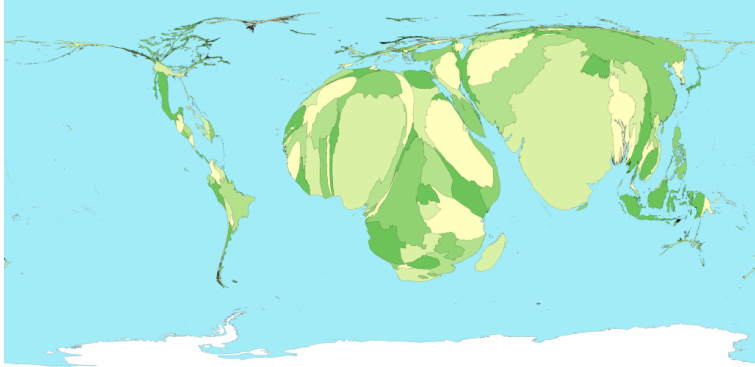
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 65/85



Cartograms

Child mortality:



Supply Networks

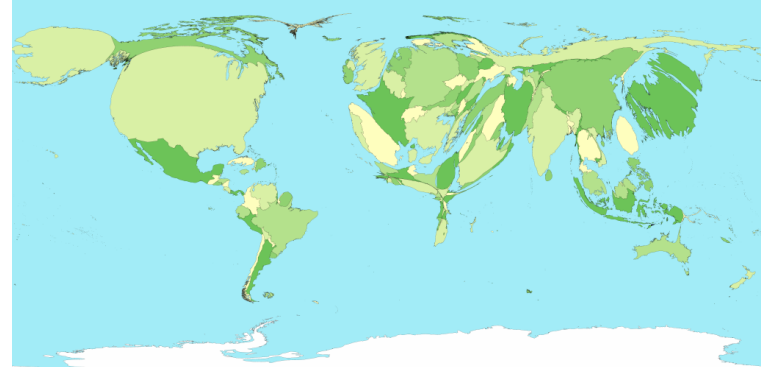
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
- Cartograms
- References

Frame 66/85



Cartograms

Energy consumption:



Supply Networks

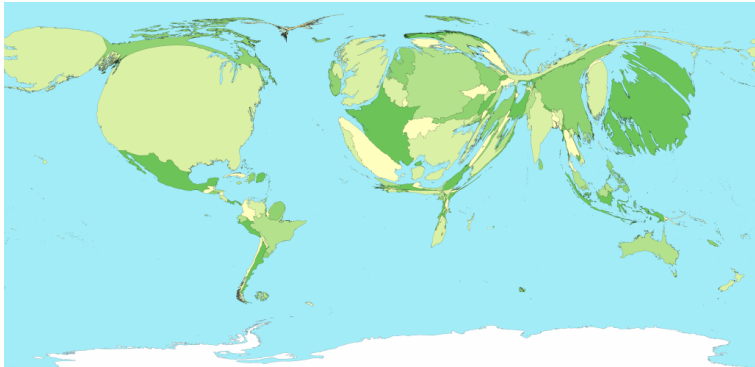
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
- Cartograms
- References

Frame 67/85



Cartograms

Gross domestic product:



Supply Networks

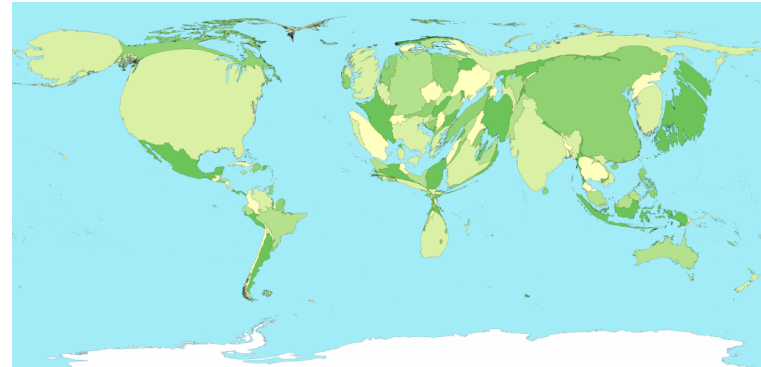
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
- Cartograms
- References

Frame 68/85



Cartograms

Greenhouse gas emissions:



Supply Networks

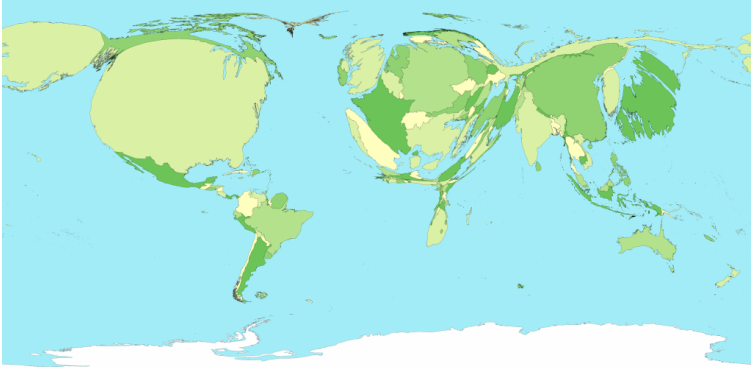
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
- Cartograms
- References

Frame 69/85



Cartograms

Spending on healthcare:



Supply Networks

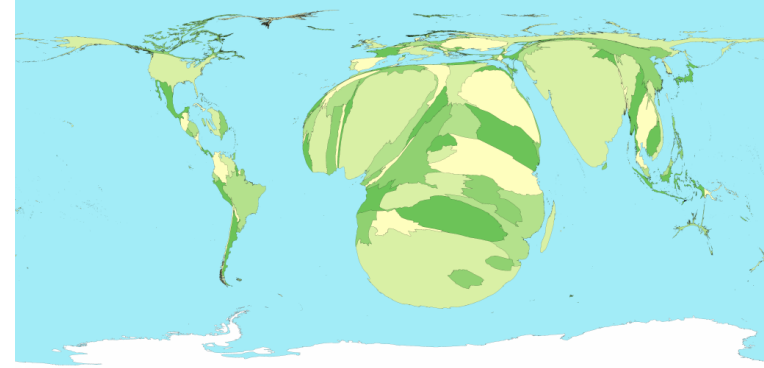
- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 70/85

🖨️ ↶ 🔍 ↷

Cartograms

People living with HIV:



Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 71/85

🖨️ ↶ 🔍 ↷

Cartograms

- ▶ The preceding sampling of Gastner & Newman's cartograms lives [here](#) (田).
- ▶ A larger collection can be found at worldmapper.org (田).



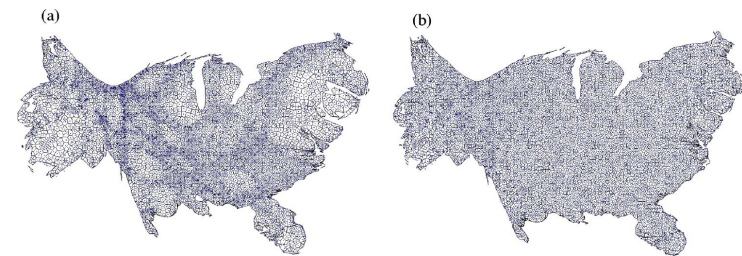
Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 72/85

🖨️ ↶ 🔍 ↷

Size-density law



- ▶ **Left:** population density-equalized cartogram.
- ▶ **Right:** (population density)^{2/3}-equalized cartogram.
- ▶ Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [4]

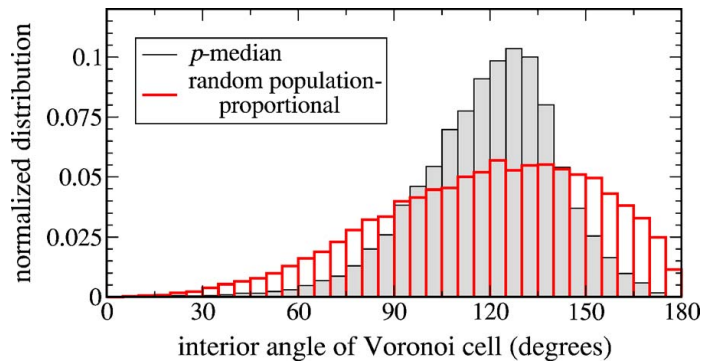
Supply Networks

- Introduction
- Optimal branching
 - Murray meets Tokunaga
- Single Source
 - History
 - Reframing the question
 - Minimal volume calculation
 - Blood networks
 - River networks
- Distributed Sources
 - Facility location
 - Size-density law
 - Cartograms
- References

Frame 73/85

🖨️ ↶ 🔍 ↷

Size-density law



From Gastner and Newman (2006) [4]

- ▶ Cartogram's Voronoi cells are somewhat hexagonal.

Size-density law

Deriving the optimal source distribution:

- ▶ **Basic idea:** Minimize the average distance from a random individual to the nearest facility. [3]
- ▶ Assume given a fixed population density ρ defined on a spatial region Ω .
- ▶ Formally, we want to find the locations of n sources $\{\vec{x}_1, \dots, \vec{x}_n\}$ that minimizes the **cost function**

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_{\Omega} \rho(\vec{x}) \min_i \|\vec{x} - \vec{x}_i\| d\vec{x}.$$

- ▶ Also known as the p-median problem.
- ▶ Not easy... in fact this one is an NP-hard problem. [3]

Size-density law

Approximations:

- ▶ For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (田), one per source.
- ▶ Define $A(\vec{x})$ as the **area** of the Voronoi cell containing \vec{x} .
- ▶ As per Stephan's calculation, estimate typical distance from \vec{x} to the nearest source (say i) as

$$c_i A(\vec{x})^{1/2}$$

where c_i is a shape factor for the i th Voronoi cell.

- ▶ Approximate c_i as a constant c .

Size-density law

Carrying on:

- ▶ The cost function is now

$$F = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} d\vec{x}.$$

- ▶ We also have that the **constraint** that Voronoi cells divide up the overall area of Ω : $\sum_{i=1}^n A(\vec{x}_i) = A_{\Omega}$.
- ▶ Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{d\vec{x}}{A(\vec{x})} = n.$$

- ▶ Within each cell, $A(\vec{x})$ is constant.
- ▶ So... integral over each of the n cells equals 1.

Size-density law

Now a Lagrange multiplier story:

- ▶ By varying $\{\vec{x}_1, \dots, \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} d\vec{x} - \lambda \left(n - \int_{\Omega} [A(\vec{x})]^{-1} d\vec{x} \right)$$

- ▶ Next compute $\delta G / \delta A$, the functional derivative (田) of the functional $G(A)$.

- ▶ This gives

$$\int_{\Omega} \left[\frac{-c}{2} \rho(\vec{x}) A(\vec{x})^{-1/2} + \lambda [A(\vec{x})]^{-2} \right] d\vec{x}$$

- ▶ Setting the integrand to be zilch, we have:

$$\rho(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}.$$

Size-density law

Now a Lagrange multiplier story:

- ▶ Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho^{-2/3}.$$

- ▶ Finally, we identify $1/A(\vec{x})$ as $D(\vec{x})$, an approximation of the local source density.

- ▶ Substituting $D = 1/A$, we have

$$D(\vec{x}) = \left(\frac{c}{2\lambda} \rho \right)^{2/3}.$$

- ▶ Normalizing (or solving for λ):

$$D(\vec{x}) = n \frac{[\rho(\vec{x})]^{2/3}}{\int_{\Omega} [\rho(\vec{x})]^{2/3} d\vec{x}} \propto [\rho(\vec{x})]^{2/3}.$$

Global redistribution networks

One more thing:

- ▶ How do we supply these facilities?
- ▶ How do we best redistribute mail? People?
- ▶ How do we get beer to the pubs?
- ▶ Gaster and Newman model: cost is a function of basic maintenance and travel time:

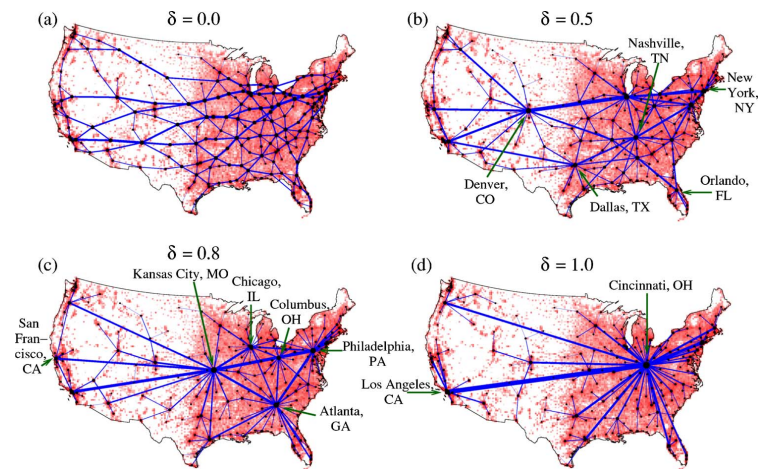
$$C_{\text{maint}} + \gamma C_{\text{travel}}.$$

- ▶ Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1 - \delta)\ell_{ij} + \delta(\#\text{hops}).$$





- ▶ When $\delta = 1$, only number of hops matters.

Global redistribution networks



From Gastner and Newman (2006) [4]

References I

-  [J. R. Banavar, A. Maritan, and A. Rinaldo.](#)
Size and form in efficient transportation networks.
Nature, 399:130–132, 1999. [pdf](#) (田)
-  [P. S. Dodds, D. H. Rothman, and J. S. Weitz.](#)
Re-examination of the “3/4-law” of metabolism.
Journal of Theoretical Biology, 209:9–27, 2001.
[pdf](#) (田)
-  [M. T. Gastner and M. E. J. Newman.](#)
Diffusion-based method for producing
density-equalizing maps.
Proc. Natl. Acad. Sci., 101:7499–7504, 2004. [pdf](#) (田)
-  [M. T. Gastner and M. E. J. Newman.](#)
Optimal design of spatial distribution networks.
Phys. Rev. E, 74:Article # 016117, 2006. [pdf](#) (田)





Supply Networks

Introduction
Optimal branching
Murray meets Tokunaga
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
References

Frame 82/85



References II

-  [A. A. Heusner.](#)
Size and power in mammals.
Journal of Experimental Biology, 160:25–54, 1991.
-  [L. B. Leopold.](#)
A View of the River.
Harvard University Press, Cambridge, MA, 1994.
-  [D. R. Montgomery and W. E. Dietrich.](#)
Channel initiation and the problem of landscape
scale.
Science, 255:826–30, 1992. [pdf](#) (田)
-  [C. D. Murray.](#)
The physiological principle of minimum work. I. The
vascular system and the cost of blood volume.
Proc. Natl. Acad. Sci. U.S.A., 12:207–214, 1926.





Supply Networks

Introduction
Optimal branching
Murray meets Tokunaga
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
References

Frame 83/85



References III

-  [N. Rashevsky.](#)
General mathematical principles in biology.
In N. Rashevsky, editor, *Physicomathematical
Aspects of Biology*, Proceedings of the International
School of Physics “Enrico Fermi”; course 16, pages
493–524, New York, 1962. Academic Press.
-  [I. Rodríguez-Iturbe and A. Rinaldo.](#)
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambridge, UK, 1997.
-  [A. E. Scheidegger.](#)
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.
-  [W. R. Stahl.](#)
Scaling of respiratory variables in mammals.
Journal of Applied Physiology, 22:453–460, 1967.





Supply Networks

Introduction
Optimal branching
Murray meets Tokunaga
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
References

Frame 84/85



References IV

-  [G. E. Stephan.](#)
Territorial division: The least-time constraint behind
the formation of subnational boundaries.
Science, 196:523–524, 1977. [pdf](#) (田)
-  [G. E. Stephan.](#)
Territorial subdivision.
Social Forces, 63:145–159, 1984. [pdf](#) (田)
-  [D. L. Turcotte, J. D. Pelletier, and W. I. Newman.](#)
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998.
-  [G. B. West, J. H. Brown, and B. J. Enquist.](#)
A general model for the origin of allometric scaling
laws in biology.
Science, 276:122–126, 1997. [pdf](#) (田)

Supply Networks

Introduction
Optimal branching
Murray meets Tokunaga
Single Source
History
Reframing the question
Minimal volume calculation
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
References

Frame 85/85

