

Scale-Free Networks

Complex Networks, Course 295A, Spring, 2008

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- ▶ Networks with power-law degree distributions have become known as **scale-free** networks.

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- ▶ Networks with power-law degree distributions have become known as **scale-free** networks.
- ▶ Scale-free refers specifically to the **degree distribution** having a **power-law decay** in its tail:

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- ▶ Networks with power-law degree distributions have become known as **scale-free** networks.
- ▶ Scale-free refers specifically to the **degree distribution** having a **power-law decay** in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

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- ▶ One of the seminal works in complex networks:
Laszlo Barabási and Reka Albert, Science, 1999:
“Emergence of scaling in random networks” [2]

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- ▶ One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: **“Emergence of scaling in random networks”** [2]
- ▶ Somewhat misleading nomenclature...

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- ▶ Scale-free networks are **not fractal** in any sense.

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- ▶ Scale-free networks are **not fractal** in any sense.
- ▶ Usually talking about networks whose links are **abstract, relational, informational, . . .** (non-physical)

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- ▶ Scale-free networks are **not fractal** in any sense.
- ▶ Usually talking about networks whose links are **abstract, relational, informational**, . . . (non-physical)
- ▶ Primary example: hyperlink network of the Web

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- ▶ Scale-free networks are **not fractal** in any sense.
- ▶ Usually talking about networks whose links are **abstract, relational, informational**, . . . (non-physical)
- ▶ Primary example: hyperlink network of the Web
- ▶ Much arguing about whether or networks are 'scale-free' or not. . .

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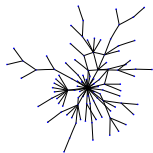
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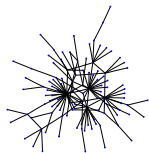
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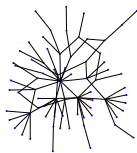
Random networks: largest components



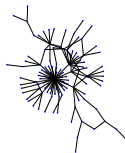
$$\gamma = 2.5 \\ \langle k \rangle = 1.8$$



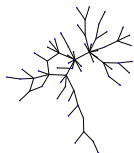
$$\gamma = 2.5 \\ \langle k \rangle = 2.05333$$



$$\gamma = 2.5 \\ \langle k \rangle = 1.66667$$



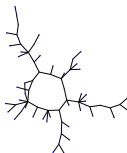
$$\gamma = 2.5 \\ \langle k \rangle = 1.92$$



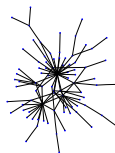
$$\gamma = 2.5 \\ \langle k \rangle = 1.6$$



$$\gamma = 2.5 \\ \langle k \rangle = 1.50667$$



$$\gamma = 2.5 \\ \langle k \rangle = 1.62667$$



$$\gamma = 2.5 \\ \langle k \rangle = 1.8$$

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The big deal:

- ▶ We move beyond describing of networks to finding **mechanisms** for why certain networks are the way they are.

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The big deal:

- ▶ We move beyond describing of networks to finding **mechanisms** for why certain networks are the way they are.

A big deal for scale-free networks:

- ▶ How does the exponent γ depend on the mechanism?

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The big deal:

- ▶ We move beyond describing of networks to finding **mechanisms** for why certain networks are the way they are.

A big deal for scale-free networks:

- ▶ How does the exponent γ depend on the mechanism?
- ▶ Do the mechanism details matter?

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Work that presaged scale-free networks

- ▶ 1924: **G. Udney Yule** [9]:
Species per Genus

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- ▶ 1924: **G. Udny Yule** ^[9]:
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- ▶ 1926: **Lotka** ^[4]:
Scientific papers per author

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- ▶ 1926: **Lotka**^[4]:
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- ▶ 1953: **Mandelbrot**^[5]:
Zipf's law for word frequency through optimization

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- ▶ 1953: **Mandelbrot** [5]:
Zipf's law for word frequency through optimization
- ▶ 1955: **Herbert Simon** [8, 10]:
Zipf's law, city size, income, publications, and species per genus

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- ▶ 1955: **Herbert Simon** [8, 10]:
Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: **Derek de Solla Price** [6, 7]:
Network of Scientific Citations

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- ▶ Barabási-Albert model = BA model.

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- ▶ Barabási-Albert model = BA model.
- ▶ Key ingredients:
Growth and **Preferential Attachment** (PA).

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- ▶ Barabási-Albert model = BA model.
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- ▶ **Step 1**: start with m_0 disconnected nodes.

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- ▶ **Step 2**:

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- ▶ **Step 2**:
 1. **Growth**—a new node appears at each time step $t = 0, 1, 2, \dots$

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- ▶ **Step 1**: start with m_0 disconnected nodes.
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 1. **Growth**—a new node appears at each time step $t = 0, 1, 2, \dots$
 2. Each new node makes m links to nodes already present.

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 2. Each new node makes m links to nodes already present.
 3. **Preferential attachment**—Probability of connecting to i th node is $\propto k_i$.

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- ▶ Barabási-Albert model = BA model.
- ▶ Key ingredients:
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- ▶ **Step 1**: start with m_0 disconnected nodes.
- ▶ **Step 2**:
 1. **Growth**—a new node appears at each time step $t = 0, 1, 2, \dots$
 2. Each new node makes m links to nodes already present.
 3. **Preferential attachment**—Probability of connecting to i th node is $\propto k_i$.
- ▶ In essence, we have a **rich-gets-richer** scheme.

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- ▶ **Definition:** A_k is the **attachment kernel** for a node with degree k .

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- ▶ **Definition:** A_k is the **attachment kernel** for a node with degree k .
- ▶ For the original model:

$$A_k = k$$

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- ▶ **Definition:** A_k is the **attachment kernel** for a node with degree k .
- ▶ For the original model:

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- ▶ **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.

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- ▶ For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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- ▶ For the original model:

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where $N(t) = m_0 + t$ is # nodes at time t

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- ▶ **Definition:** A_k is the **attachment kernel** for a node with degree k .
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- ▶ **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.
- ▶ For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\max}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t

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where $N(t) = m_0 + t$ is # nodes at time t
and $N_k(t)$ is # degree k nodes at time t .

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Approximate analysis

- ▶ When $(N + 1)$ th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

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- ▶ Assumes probability of being connected to is **small**.

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- ▶ Assumes probability of being connected to is **small**.
- ▶ Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.

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- ▶ Assumes probability of being connected to is **small**.
- ▶ Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{d}{dt} k_{i,t}$:

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$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- ▶ Assumes probability of being connected to is **small**.
- ▶ Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{d}{dt} k_{i,t}$:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

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Approximate analysis

- ▶ Deal with denominator: each added node brings m new edges.

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Approximate analysis

- ▶ Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

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Approximate analysis

- ▶ Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- ▶ The node degree equation now simplifies:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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References

Approximate analysis

- ▶ Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- ▶ The node degree equation now simplifies:

$$\frac{d}{dt} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt}$$

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- ▶ Rearrange and solve:

$$\frac{dk_i(t)}{k_i(t)} = \frac{dt}{2t}$$

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- Rearrange and solve:

$$\frac{dk_i(t)}{k_i(t)} = \frac{dt}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}.}$$

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- ▶ Rearrange and solve:

$$\frac{dk_i(t)}{k_i(t)} = \frac{dt}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}.}$$

- ▶ Next find $c_i \dots$

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References

- ▶ Know i th node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

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$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \leq m_0 \end{cases}$$

- ▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

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$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- ▶ All node degrees grow as $t^{1/2}$

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$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- ▶ All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which **flattens out** growth curve.

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$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \quad \text{for } t \geq t_{i,\text{start}}.$$

- ▶ All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which **flattens out** growth curve.
- ▶ Early nodes do **best** (First-mover advantage).

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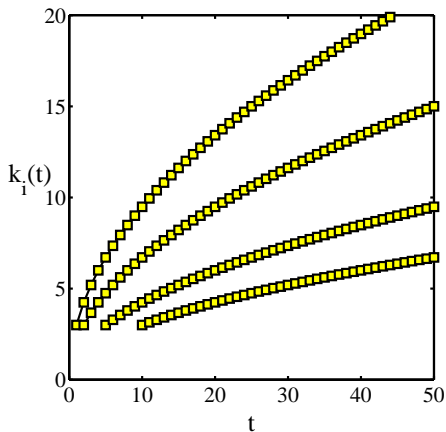
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References

Approximate analysis



- ▶ $m = 3$
- ▶ $t_{i,start} = 1, 2, 5, \text{ and } 10.$

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- ▶ So what's the **degree distribution** at time t ?

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References

Degree distribution

- ▶ So what's the **degree distribution** at time t ?
- ▶ Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}})dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

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$$P(t_{i,\text{start}})dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

- ▶ Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

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- ▶ Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}} \right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2}).$$

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References

- ▶ Using the uniformity of start times:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

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- ▶ Using the uniformity of start times:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

- ▶ Differentiate to find $\Pr(k)$:

$$\Pr(k) = \frac{d}{dk} \Pr(k_i < k)$$

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$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

- ▶ Differentiate to find $\Pr(k)$:

$$\Pr(k) = \frac{d}{dk} \Pr(k_i < k) = \frac{2m^2 t}{(t + m_0)k^3}$$

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- ▶ Using the uniformity of start times:

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- ▶ Differentiate to find $\Pr(k)$:

$$\Pr(k) = \frac{d}{dk} \Pr(k_i < k) = \frac{2m^2 t}{(t + m_0)k^3}$$

$$\sim 2m^2 k^{-3} \text{ as } m \rightarrow \infty.$$

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References

- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.

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References

- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.

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- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.

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References

- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.
- ▶ $2 < \gamma < 3$: finite mean and 'infinite' variance

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References

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- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.
- ▶ $2 < \gamma < 3$: finite mean and 'infinite' variance
- ▶ In practice, $\gamma < 3$ means variance is governed by upper cutoff.

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- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.
- ▶ $2 < \gamma < 3$: finite mean and 'infinite' variance
- ▶ In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- ▶ $\gamma > 3$: finite mean and variance

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References

- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.
- ▶ $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- ▶ In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- ▶ $\gamma > 3$: finite mean and variance

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- ▶ We thus have a very specific prediction of $\Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- ▶ Range true more generally for events with size distributions that have power-law tails.
- ▶ $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- ▶ In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- ▶ $\gamma > 3$: finite mean and variance (mild)

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WWW	$\gamma \simeq 2.1$ for in-degree
WWW	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

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Examples

WWW	$\gamma \simeq 2.1$ for in-degree
WWW	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

The Internet_s is a different business...

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From Barabási and Albert's original paper [2]:

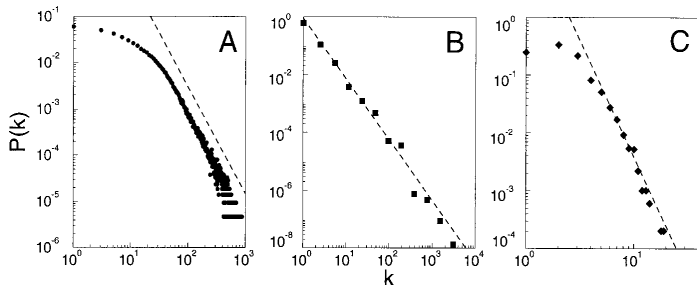


Fig. 1. The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes **(A)** $\gamma_{\text{actor}} = 2.3$, **(B)** $\gamma_{\text{www}} = 2.1$ and **(C)** $\gamma_{\text{power}} = 4$.

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Things to do and questions

- ▶ Vary attachment kernel.

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion

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- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
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 2. Add node deletion
 3. Add edge rewiring

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion
 3. Add edge rewiring
- ▶ Deal with directed versus undirected networks.

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion
 3. Add edge rewiring
- ▶ Deal with directed versus undirected networks.
- ▶ **Important Q.:** Are there distinct universality classes for these networks?

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- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion
 3. Add edge rewiring
- ▶ Deal with directed versus undirected networks.
- ▶ **Important Q.:** Are there distinct universality classes for these networks?
- ▶ **Q.:** How does changing the model affect γ ?

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
 1. Add edge deletion
 2. Add node deletion
 3. Add edge rewiring
- ▶ Deal with directed versus undirected networks.
- ▶ **Important Q.:** Are there distinct universality classes for these networks?
- ▶ **Q.:** How does changing the model affect γ ?
- ▶ **Q.:** Do we need preferential attachment and growth?

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- ▶ Vary attachment kernel.
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- ▶ Deal with directed versus undirected networks.
- ▶ **Important Q.:** Are there distinct universality classes for these networks?
- ▶ **Q.:** How does changing the model affect γ ?
- ▶ **Q.:** Do we need preferential attachment and growth?
- ▶ **Q.:** Do model details matter?

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Things to do and questions

- ▶ Vary attachment kernel.
- ▶ Vary mechanisms:
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 2. Add node deletion
 3. Add edge rewiring
- ▶ Deal with directed versus undirected networks.
- ▶ **Important Q.:** Are there distinct universality classes for these networks?
- ▶ **Q.:** How does changing the model affect γ ?
- ▶ **Q.:** Do we need preferential attachment and growth?
- ▶ **Q.:** Do model details matter?
- ▶ The answer is (surprisingly) **yes**.

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- ▶ Let's look at preferential attachment (PA) a little more closely.

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Preferential attachment

- ▶ Let's look at preferential attachment (**PA**) a little more closely.
- ▶ PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.

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Preferential attachment

- ▶ Let's look at preferential attachment (**PA**) a little more closely.
- ▶ PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.

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References

- ▶ Let's look at preferential attachment (PA) a little more closely.
- ▶ PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...

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References

- ▶ Let's look at preferential attachment (**PA**) a little more closely.
- ▶ PA implies arriving nodes have **complete knowledge** of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is \therefore an **outrageous** assumption of node capability.

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- ▶ We need to know what everyone's degree is...
- ▶ PA is \therefore an **outrageous** assumption of node capability.
- ▶ But a **very simple mechanism** saves the day...

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Preferential attachment through randomness

- ▶ Instead of attaching preferentially, allow new nodes to attach randomly.

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Preferential attachment through randomness

- ▶ Instead of attaching preferentially, allow new nodes to attach randomly.
- ▶ Now add an **extra step**: new nodes then connect to some of their friends' friends.

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Preferential attachment through randomness

- ▶ Instead of attaching preferentially, allow new nodes to attach randomly.
- ▶ Now add an **extra step**: new nodes then connect to some of their friends' friends.
- ▶ Can also do this **at random**.

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Preferential attachment through randomness

- ▶ Instead of attaching preferentially, allow new nodes to attach randomly.
- ▶ Now add an **extra step**: new nodes then connect to some of their friends' friends.
- ▶ Can also do this **at random**.
- ▶ We know that friends are **weird**...

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- ▶ Now add an **extra step**: new nodes then connect to some of their friends' friends.
- ▶ Can also do this **at random**.
- ▶ We know that friends are **weird**...
- ▶ Assuming the existing network is random, we know probability of a **random friend** having degree k is

$$Q_k \propto kP_k$$

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Preferential attachment through randomness

- ▶ Instead of attaching preferentially, allow new nodes to attach randomly.
- ▶ Now add an **extra step**: new nodes then connect to some of their friends' friends.
- ▶ Can also do this **at random**.
- ▶ We know that friends are **weird**...
- ▶ Assuming the existing network is random, we know probability of a **random friend** having degree k is

$$Q_k \propto kP_k$$

- ▶ So **rich-gets-richer** scheme can now be seen to work in a natural way.

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- ▶ We've looked at some aspects of contagion on scale-free networks:

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- ▶ We've looked at some aspects of contagion on scale-free networks:
 1. Facilitate disease-like spreading.

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- ▶ We've looked at some aspects of contagion on scale-free networks:
 1. Facilitate disease-like spreading.
 2. Inhibit threshold-like spreading.

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References

- ▶ We've looked at some aspects of contagion on scale-free networks:
 1. Facilitate disease-like spreading.
 2. Inhibit threshold-like spreading.
- ▶ Another simple story concerns **system robustness**.

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References

- ▶ We've looked at some aspects of contagion on scale-free networks:
 1. Facilitate disease-like spreading.
 2. Inhibit threshold-like spreading.
- ▶ Another simple story concerns **system robustness**.
- ▶ Albert et al., Nature, 2000:
“Error and attack tolerance of complex networks”^[1]

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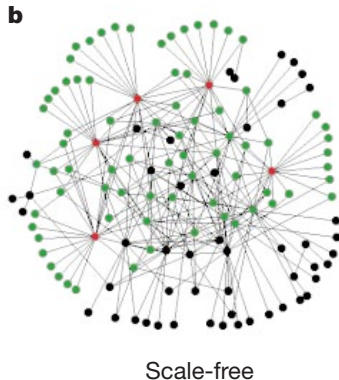
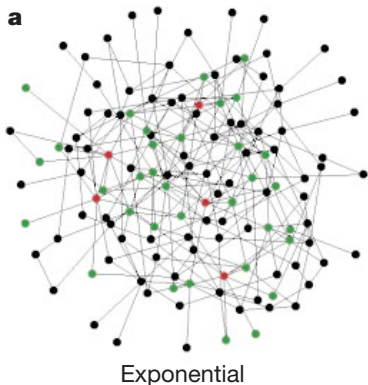
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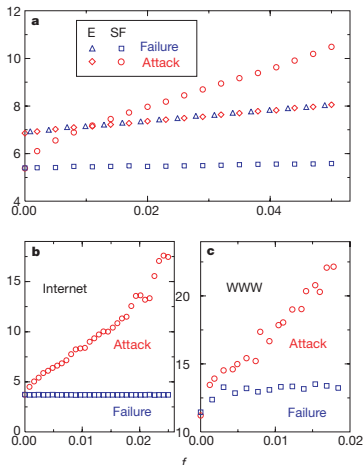
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References

- ▶ Standard random networks (Erdős-Rényi)
versus
Scale-free networks





- ▶ Plots of network diameter as a function of fraction of nodes removed
- ▶ Erdős-Rényi versus scale-free networks
- ▶ **blue symbols** = random removal
- ▶ **red symbols** = targeted removal (most connected first)

from Albert et al., 2000

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- ▶ All very reasonable: **Hubs** are a big deal.

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
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- ▶ **But**: next issue is whether hubs are vulnerable or not.

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- ▶ All very reasonable: **Hubs** are a big deal.
- ▶ **But**: next issue is whether hubs are vulnerable or not.
- ▶ Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)

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- ▶ Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- ▶ Most connected nodes are either:

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- ▶ All very reasonable: **Hubs** are a big deal.
- ▶ **But**: next issue is whether hubs are vulnerable or not.
- ▶ Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- ▶ Most connected nodes are either:
 1. Physically larger nodes that may be harder to 'target'

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- ▶ **But**: next issue is whether hubs are vulnerable or not.
- ▶ Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- ▶ Most connected nodes are either:
 1. Physically larger nodes that may be harder to 'target'
 2. or subnetworks of smaller, normal-sized nodes.

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- ▶ Scale-free networks are thus **robust to random failures** yet **fragile to targeted ones**.
- ▶ All very reasonable: **Hubs** are a big deal.
- ▶ **But**: next issue is whether hubs are vulnerable or not.
- ▶ Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- ▶ Most connected nodes are either:
 1. Physically larger nodes that may be harder to 'target'
 2. or subnetworks of smaller, normal-sized nodes.
- ▶ Need to explore cost of various targeting schemes.

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Fooling with the mechanism:

- ▶ 2001: Redner & Krapivsky (RK) [3] explored the **general attachment kernel**:

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Fooling with the mechanism:

- ▶ 2001: Redner & Krapivsky (RK) [3] explored the **general attachment kernel**:

$$\Pr(\text{attach to node } i) \propto A_k = k_i^\nu$$

where A_k is the attachment kernel and $\nu > 0$.

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Fooling with the mechanism:

- ▶ 2001: Redner & Krapivsky (RK) [3] explored the **general attachment kernel**:

$$\Pr(\text{attach to node } i) \propto A_k = k_i^\nu$$

where A_k is the attachment kernel and $\nu > 0$.

- ▶ RK also looked at changing the details of the attachment kernel.

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Fooling with the mechanism:

- ▶ 2001: Redner & Krapivsky (RK) [3] explored the **general attachment kernel**:

$$\Pr(\text{attach to node } i) \propto A_k = k_i^\nu$$

where A_k is the attachment kernel and $\nu > 0$.

- ▶ RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (⊞).

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- ▶ Here's the set up:

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1} N_{k-1} - A_k N_k] + \delta_{k1}$$

where N_k is the number of nodes of degree k .

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1. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.

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where N_k is the number of nodes of degree k .

1. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
2. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.

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where N_k is the number of nodes of degree k .

1. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
2. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
3. Detail: $A_0 = 0$

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1. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
2. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
3. Detail: $A_0 = 0$
4. One node is added per unit time.

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2. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
3. Detail: $A_0 = 0$
4. One node is added per unit time.
5. Seed with some initial network

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where N_k is the number of nodes of degree k .

1. The **first term** corresponds to degree $k - 1$ nodes becoming degree k nodes.
2. The **second term** corresponds to degree k nodes becoming degree $k - 1$ nodes.
3. Detail: $A_0 = 0$
4. One node is added per unit time.
5. Seed with some initial network (e.g., a connected pair)

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- ▶ In general, probability of attaching to a **specific node** of degree k at time t is

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- ▶ In general, probability of attaching to a **specific node** of degree k at time t is

$$\Pr(\text{attach to node } i) = \frac{A_k}{A(t)}$$

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- ▶ In general, probability of attaching to a **specific node** of degree k at time t is

$$\Pr(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

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where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.

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- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- ▶ For $A_k = k$, we have

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- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t)$$

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- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

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where $A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

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- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} k N_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

- ▶ Detail: we are ignoring initial seed network's edges.

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- ▶ So now

$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1} N_{k-1} - A_k N_k] + \delta_{k1}$$

becomes

$$\frac{dN_k}{dt} = \frac{1}{2t} [(k-1)N_{k-1} - kN_k] + \delta_{k1}$$

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- ▶ As for BA method, look for steady-state growing solution:

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- ▶ As for BA method, look for steady-state growing solution: $N_k = n_k t$.

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- ▶ As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- ▶ We replace dN_k/dt with $dn_k t/dt = n_k$.

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$$\frac{dN_k}{dt} = \frac{1}{A} [A_{k-1}N_{k-1} - A_kN_k] + \delta_{k1}$$

becomes

$$\frac{dN_k}{dt} = \frac{1}{2t} [(k-1)N_{k-1} - kN_k] + \delta_{k1}$$

- ▶ As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- ▶ We replace dN_k/dt with $dn_k t/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_k t] + \delta_{k1}$$

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- ▶ Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

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- ▶ Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

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- ▶ Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

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- ▶ Two cases:

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$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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- ▶ Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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- ▶ Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)(k-2)}{k+2} n_{k-2}$$

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► Now find n_k :

$$\begin{aligned}k > 1: n_k &= \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)(k-2)}{k+2} n_{k-2} \\ &= \frac{(k-1)(k-2)(k-3)}{k+2} n_{k-3}\end{aligned}$$

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- Now find n_k :

$$\begin{aligned}k > 1: n_k &= \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)(k-2)}{k+2} n_{k-2} \\ &= \frac{(k-1)(k-2)(k-3)}{k+2} n_{k-3} \\ &= \frac{(k-1)(k-2)(k-3)(k-4)}{k+2} n_{k-4} \\ &= \frac{\cancel{(k-1)}(k-2)(k-3)(k-4)(k-5)\dots\cancel{5}4321}{k+2} \frac{\cancel{(k-1)}(k-2)\dots\cancel{8}7654}{k+1} n_1\end{aligned}$$

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- Now find n_k :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)(k-2)}{k+2} n_{k-2}$$

$$= \frac{(k-1)(k-2)(k-3)}{k+2} n_{k-3}$$

$$= \frac{(k-1)(k-2)(k-3)(k-4)}{k+2} n_{k-4}$$

$$= \frac{\cancel{(k-1)} \cancel{(k-2)} (k-3) (k-4) (k-5) \dots \cancel{5} \cancel{4} \cancel{3} \cancel{2} \cancel{1}}{k+2} \frac{1}{\cancel{(k-1)} \cancel{(k-2)} \dots \cancel{8} \cancel{7} \cancel{6} \cancel{5} \cancel{4}} n_1$$

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$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1$$

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$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)}$$

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$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

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- ▶ As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3} \text{ for large } k.$$

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- ▶ Now: what happens if we start playing around with the attachment kernel A_k ?

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- ▶ Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (田)?

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- ▶ As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3} \text{ for large } k.$$

- ▶ Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (田)?
- ▶ Natural modification: $A_k = k^\nu$ with $\nu \neq 1$.

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- ▶ Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (田)?
- ▶ Natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]

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- ▶ Again, is the result $\gamma = 3$ universal (田)?
- ▶ Natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ▶ Keep A_k **linear in k** but tweak details.

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$$N_k(t) = n_k(t)t \propto k^{-3} \text{ for large } k.$$

- ▶ Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (田)?
- ▶ Natural modification: $A_k = k^\nu$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ▶ Keep A_k **linear in k** but tweak details.
- ▶ **Idea:** Relax from $A_k = k$ to $A_k \sim k$ as $k \rightarrow \infty$.

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- ▶ Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

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Universality?

- ▶ Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- ▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

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- ▶ Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- ▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

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Universality?

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- ▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- ▶ We assume that $A = \mu t$

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Universality?

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$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- ▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- ▶ We assume that $A = \mu t$
- ▶ We'll find μ later and make sure that our assumption is consistent.

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Universality?

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$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t \text{ for large } t.$$

- ▶ We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- ▶ We assume that $A = \mu t$
- ▶ We'll find μ later and make sure that our assumption is consistent.
- ▶ As before, also assume $N_k(t) = n_k t$.

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- ▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

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Universality?

- ▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

- ▶ This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1}n_{k-1} - A_k n_k] + \delta_{k1}$$

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Universality?

- ▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

- ▶ This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1}n_{k-1} - A_k n_k] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

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- ▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

- ▶ This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1}n_{k-1} - A_k n_k] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

- ▶ Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

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- ▶ Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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- ▶ Dealing with the $k > 1$ case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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Universality?

- ▶ Dealing with the $k > 1$ case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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- ▶ Dealing with the $k > 1$ case:

$$\begin{aligned}n_k &= n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}} \\ &= n_{k-2} \frac{A_{k-2}}{A_{k-1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}\end{aligned}$$

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Universality?

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Universality?

- ▶ Dealing with the $k > 1$ case:

$$\begin{aligned}n_k &= n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}} \\&= n_{k-2} \frac{A_{k-2}}{\cancel{A_{k-1}}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{\cancel{A_{k-1}}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}} \\&= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}} \\&= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1}\right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \\&= \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \text{ since } n_1 = \mu / (\mu + A_1)\end{aligned}$$

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- ▶ Time for pure excitement: Find **asymptotic behavior** of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.

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- ▶ Time for pure excitement: Find **asymptotic behavior** of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.
- ▶ For large k :

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Time for pure excitement: Find **asymptotic behavior** of n_k given $A_k \rightarrow k$ as $k \rightarrow \infty$.
- ▶ For large k :

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{A_j}{A_j + \mu}$$

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- ▶ Since μ depends on A_k , **details matter...**

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- ▶ Now we need to find μ .

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- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$

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- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$

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$$\mu = \sum_{k=1}^{\infty} n_k A_k$$
- ▶ Now substitute in our expression for n_k :

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- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$
- ▶ Now substitute in our expression for n_k :

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$
- ▶ Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$
- ▶ Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .

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- ▶ Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.

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- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- ▶ Since $N_k = n_k t$, we have the simplification
$$\mu = \sum_{k=1}^{\infty} n_k A_k$$
- ▶ Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

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Universality?

- ▶ Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.

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Universality?

- ▶ Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.
- ▶ $\gamma = 2$ is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

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- ▶ Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.
- ▶ $\gamma = 2$ is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

- ▶ Let's now look at a specific example of A_k to see this range of γ is possible.

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- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.

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- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .

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Universality?

- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- ▶ Expression for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- ▶ Expression for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Expression for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \geq 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- ▶ Expression for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

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► Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\cancel{1 + \frac{\mu}{\alpha}}} = \frac{1}{\cancel{1 + \frac{\mu}{\alpha}}} \sum_{k=2}^{\infty} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

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$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

- ▶ Now use result that^[3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with $a = 1$ and $b = \mu + 1$.

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with $a = 1$ and $b = \mu + 1$.



$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

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with $a = 1$ and $b = \mu + 1$.



$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

$$\Rightarrow \mu(\mu - 1) = 2\alpha$$

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▶

$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

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$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

▶ Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

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$$\mu(\mu - 1) = 2\alpha \Rightarrow \mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

▶ Since $\gamma = \mu + 1$, we have

$$0 \leq \alpha < \infty \Rightarrow 2 \leq \gamma < \infty$$

▶ Crazyiness...

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- ▶ Rich-get-somewhat-richer:

$$A_k \sim k^\nu \text{ with } 0 < \nu < 1.$$

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- ▶ Rich-get-somewhat-richer:

$$A_k \sim k^\nu \text{ with } 0 < \nu < 1.$$

- ▶ General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

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- ▶ General finding by Krapivsky and Redner:^[3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

- ▶ Stretched exponentials (truncated power laws).

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- ▶ General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

- ▶ Stretched exponentials (truncated power laws).
- ▶ aka Weibull distributions.

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- ▶ General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

- ▶ Stretched exponentials (truncated power laws).
- ▶ aka Weibull distributions.
- ▶ **Universality**: now details of kernel **do not** matter.

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- ▶ Rich-get-somewhat-richer:

$$A_k \sim k^\nu \text{ with } 0 < \nu < 1.$$

- ▶ General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu}} + \text{correction terms}.$$

- ▶ Stretched exponentials (truncated power laws).
- ▶ aka Weibull distributions.
- ▶ **Universality**: now details of kernel **do not** matter.
- ▶ Distribution of degree is universal providing $\nu < 1$.

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Details:

- ▶ For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu} - 2^{1-\nu}}{1-\nu} \right)}$$

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- ▶ For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

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- ▶ For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

- ▶ And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

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- ▶ Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

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- ▶ Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

- ▶ Now a **winner-take-all** mechanism.

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References

- ▶ Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

- ▶ Now a **winner-take-all** mechanism.
- ▶ One single node ends up being connected to almost all other nodes.

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References

- ▶ Rich-get-much-richer:

$$A_k \sim k^\nu \text{ with } \nu > 1.$$

- ▶ Now a **winner-take-all** mechanism.
- ▶ One single node ends up being connected to almost all other nodes.
- ▶ For $\nu > 2$, all but a finite # of nodes connect to one node.

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