Random Networks

Complex Networks, Course 295A, Spring, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

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Outline

Basics

Definitions

How to build

Some visual examples

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How to build

Some visual examples

Structure

Clustering

Degree distributions

Configuration model

Largest component

Outline

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Random networks

Pure, abstract random networks:

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Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdös-Rényi random networks or ER graphs.

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- Limit of $m = 0$: empty graph.
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- Real world: links are usually costly so real networks are almost always sparse.

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   - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
   - Best for adding small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

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- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

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Random networks: examples

Next slides:

Example realizations of random networks

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- Average degree $\langle k \rangle$ runs from 0.4 to 4.

Random networks: examples

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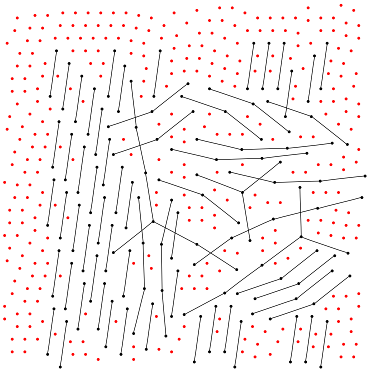
Example realizations of random networks

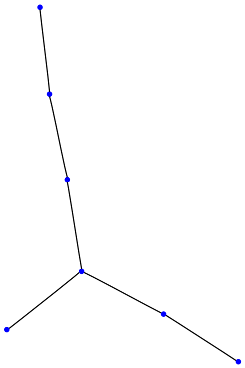
- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples

entire network:



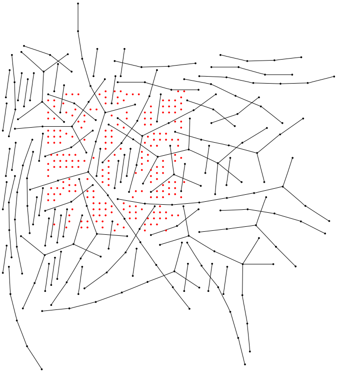
largest component:



$N = 500$, number of edges $m = 100$
average degree $\langle k \rangle = 0.4$

Random networks: examples

entire network:



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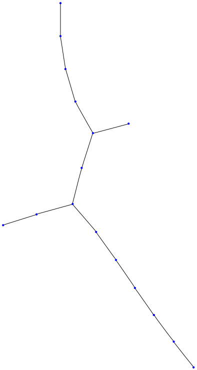
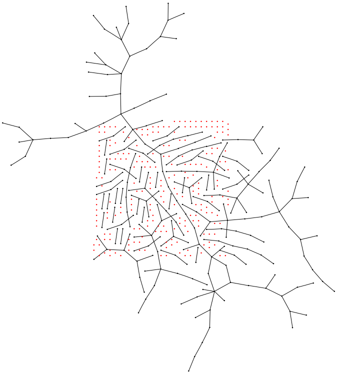


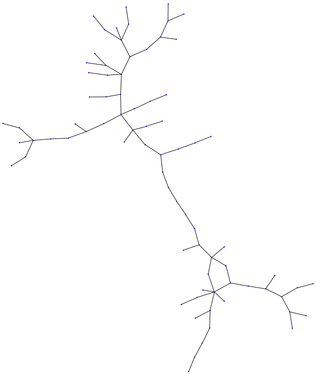
$N = 500$, number of edges $m = 200$
average degree $\langle k \rangle = 0.8$

Random networks: examples

entire network:



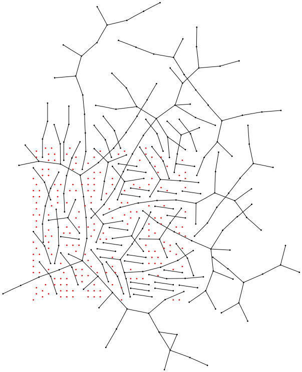
largest component:



$N = 500$, number of edges $m = 230$
average degree $\langle k \rangle = 0.92$

Random networks: examples

entire network:



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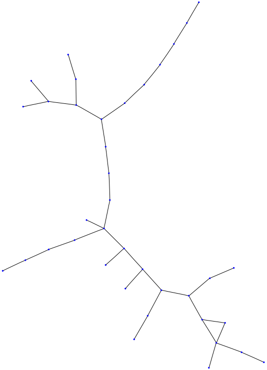
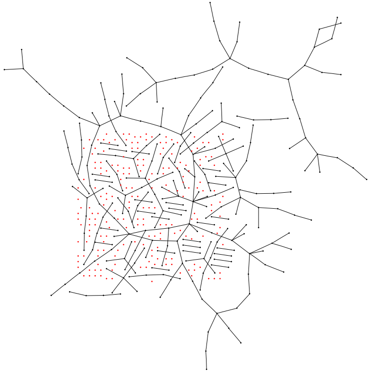
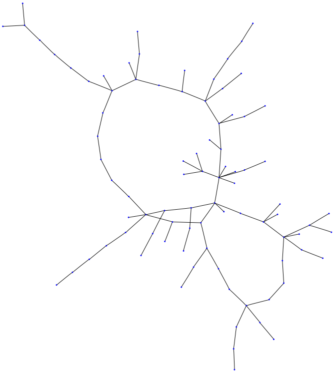


$N = 500$, number of edges $m = 240$
average degree $\langle k \rangle = 0.96$

Random networks: examples

entire network: largest component:

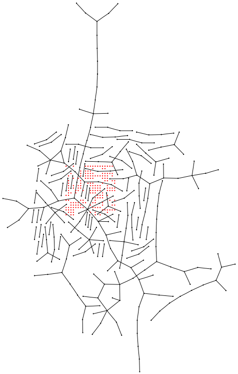




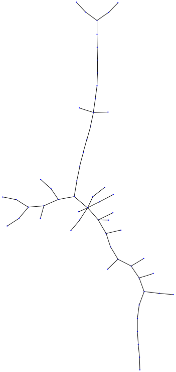
$N = 500$, number of edges $m = 250$
average degree $\langle k \rangle = 1$

Random networks: examples

entire network:



largest component:



$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$

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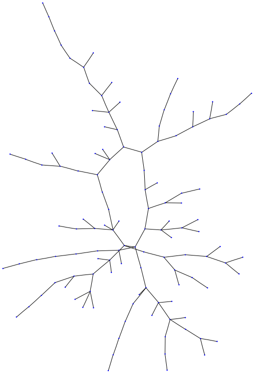
largest component:



$N = 500$, number of edges $m = 280$
average degree $\langle k \rangle = 1.12$

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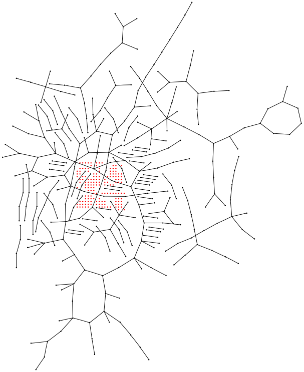
entire network:



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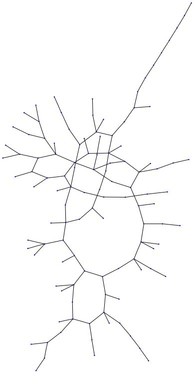

$N = 500$, number of edges $m = 300$
average degree $\langle k \rangle = 1.2$

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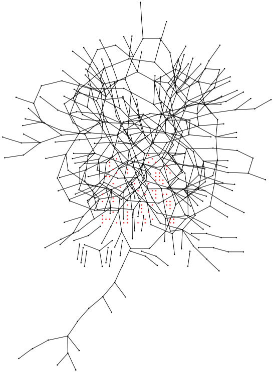
entire network:



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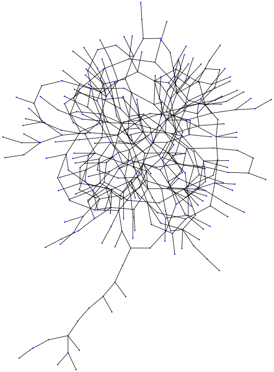
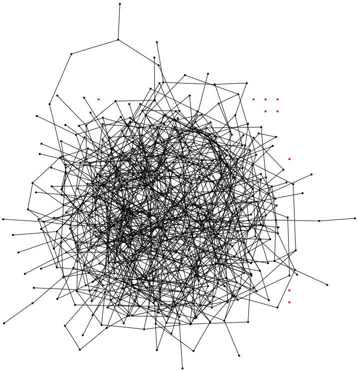


$N = 500$, number of edges $m = 500$
average degree $\langle k \rangle = 2$

Random networks: examples

entire network:



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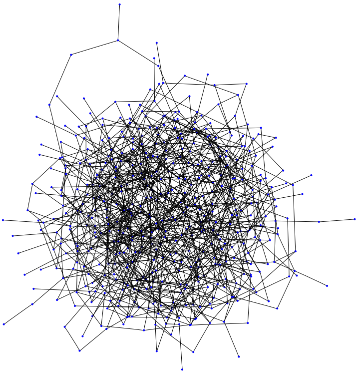
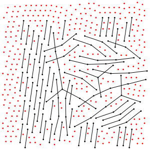


$N = 500$, number of edges $m = 1000$
average degree $\langle k \rangle = 4$

Random networks: examples for N=500



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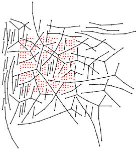

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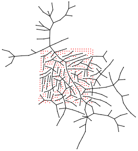

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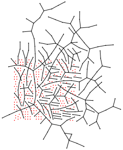
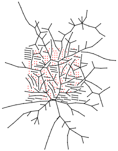


$m = 240$
$\langle k \rangle = 0.96$

$m = 250$
$\langle k \rangle = 1$

$m = 260$
$\langle k \rangle = 1.04$



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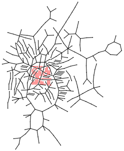

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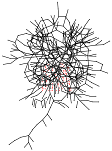

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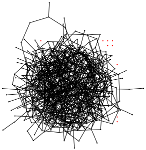
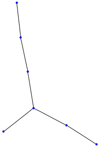


$m = 1000$
$\langle k \rangle = 4$

Random networks: largest components



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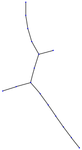

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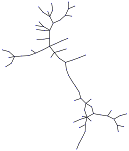

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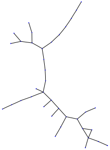

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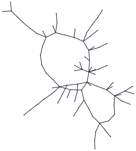

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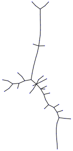

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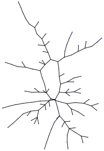

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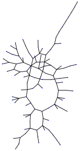

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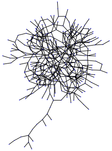

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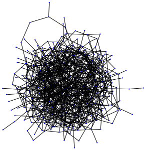

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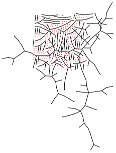
Random networks: examples for N=500



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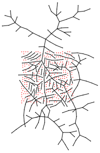

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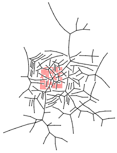

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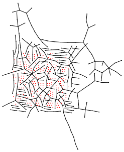

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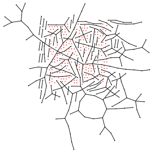

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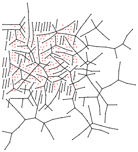

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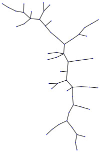
Random networks: largest components



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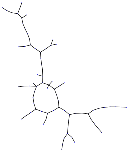

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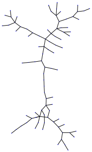

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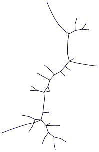

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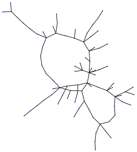

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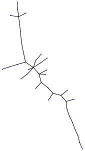

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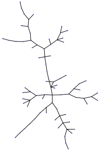

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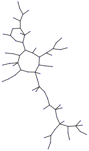

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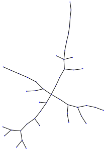

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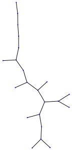

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Outline

Basics
Definitions
How to build
Some visual examples
Structure
Clustering
Degree distributions
Configuration model
Largest component
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size
References

Random networks

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- For standard random networks, we have simply that

$$C_2 = p.$$

Random networks

Clustering:

- So for large random networks ($N \longrightarrow \infty$), clustering drops to zero.

Random networks

Clustering:

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- Key structural feature of random networks is that they locally look like branching networks (no loops).

Outline

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- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

Random networks

Limiting form of $P(k;p,N)$:

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- If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \rightarrow \infty$.
- But we want to keep $\langle k \rangle$ fixed...
- So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N-1) =$ constant.

Limiting form of $P(k; p, N)$:

- Substitute $p = \frac{\langle k \rangle}{N-1}$ into $P(k; p, N)$ and hold k fixed:

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- This is a Poisson distribution (⊞) with mean $\langle k \rangle$.

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  2. Examining mechanisms that lead to networks with certain degree distributions.

Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

Random networks: examples

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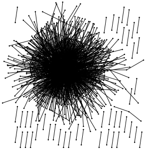
- Apart from degree distribution, wiring is random.

Random networks: examples for N=1000



γ = 2.1
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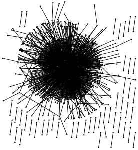

γ = 2.19
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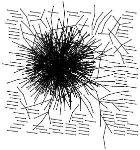

γ = 2.28
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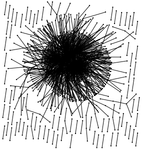

γ = 2.37
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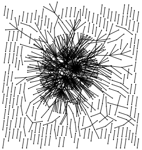
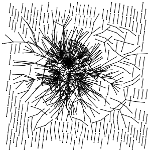


γ = 2.46
$\langle k \rangle$ = 1.856



γ = 2.55
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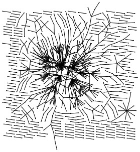
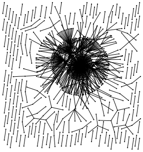
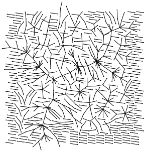


γ = 2.64
$\langle k \rangle$ = 1.6



γ = 2.73
$\langle k \rangle$ = 1.862



γ = 2.82
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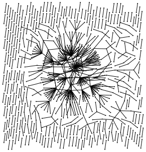

γ = 2.91
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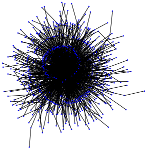
Random networks: largest components



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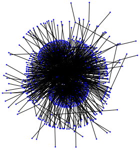

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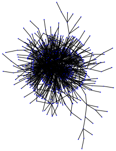

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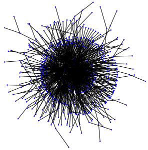

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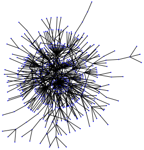
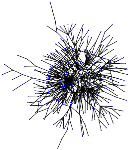


$\gamma = 2.46$
$\langle k \rangle = 1.856$



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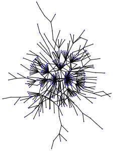
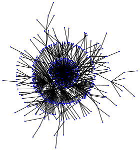
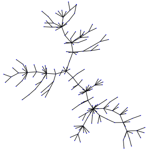


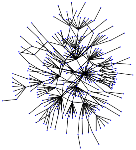
$\gamma = 2.64$
$\langle k \rangle = 1.6$



$\gamma = 2.73$
$\langle k \rangle = 1.862$



$\gamma = 2.82$
$\langle k \rangle = 1.386$



$\gamma = 2.91$
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- We'll get to a better way of doing this...

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- Note: This is a special property of Poisson distribution and can trip us up...

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- Natural question: what's the expected number of other friends that one friend has?

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- So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...

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- Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- Comparison:

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- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.

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- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

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- Note: Component = Cluster

Outline

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- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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- We say $\langle k \rangle = 1$ marks the critical point of the system.

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- Cutoff scaling is k^{-3}: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.

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- Substitute in Poisson distribution...

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- Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

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- Really a transcritical bifurcation [2].

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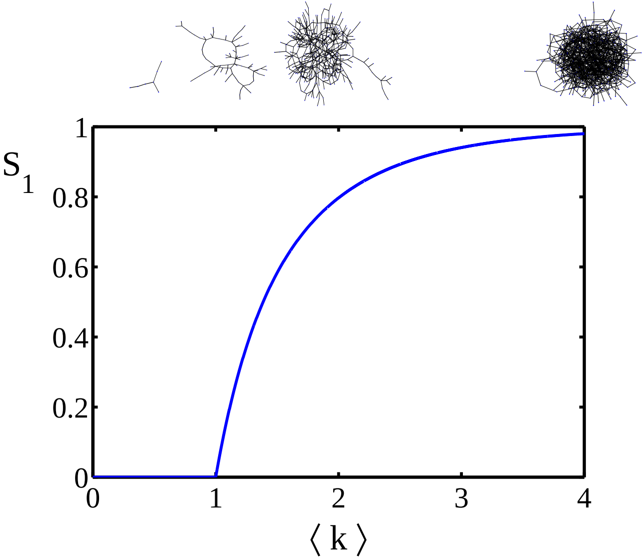

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- We can do this but we need to enhance our toolkit with Generatingfunctionology... [3]

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- Related to Fourier, Laplace, Mellin, …

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Definitions
How to build
Some visual examples
Structure
Clustering
Degree distributions
Configuration model
Largest component
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size
References

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$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \bigg|_{x=1}$$

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- kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \bigg|_{x=0}$$

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- Now find how F_R is related to F_P...

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Finally, since $\langle k \rangle = F_P'(1)$,

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- Setting $x = 1$, our condition becomes

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Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors $\Leftrightarrow$ components

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Our mission, which we accept:

- Find the four generating functions

$$F_P, F_R, F_\pi, \text{ and } F_\rho.$$

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$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
-

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x). \checkmark$$

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- The first equation then immediately gives us F_π in terms of F_ρ and F_R.

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- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.

Average component size

- Limits of $\langle k \rangle = 0$ and ∞ make sense for

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Average component size

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- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

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