# Overview of Complex Networks

## Complex Networks, Course 295A, Spring, 2008

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# Outline



Overview

Class admin
Basic definitions
Books
Examples of Complex Networks
Properties of Complex Networks
Modelling Complex Networks
References



# Class Admin

- Office hours: Tuesday 10:45 am–12:30 pm
- Course outline
- Projects

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# Basic definitions

Complex: (Latin = with + fold/weave (com + plex))

Adjective

- Made up of multiple parts; intricate or detailed.
- Not simple or straightforward.

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# Basic definitions

## Complex System—Basic ingredients:

- Relationships are nonlinear
- Relationships contain feedback loops
- Complex systems are open (out of equilibrium)
- Memory
- Modular (nested)/multiscale structure
- Opaque boundaries
- May produce emergent phenomena





# Basic definitions

Network: (net + work, 1500's)

Noun:

1. Any interconnected group or system
2. Multiple computers and other devices connected together to share information

Verb:

1. To interact socially for the purpose of getting connections or personal advancement
2. To connect two or more computers or other computerized devices

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# Observation

- Many complex systems can be regarded as complex networks of physical or abstract interactions
- Opens door to mathematical and numerical analysis
- Dominant approach of last decade of a theoretical-physics/stat-mechish flavor.

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# Basic definitions

Nodes = A collection of entities which have properties that are somehow related to each other

- e.g., people, forks in rivers, proteins, webpages, organisms,...

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# Basic definitions

## Links = Connections between nodes

- links
  - may be real and fixed (rivers),
  - real and dynamic (airline routes),
  - abstract with physical impact (hyperlinks),
  - or purely abstact (semantic connections between concepts).
- Links may be directed or undirected.
- Links may be binary or weighted.




# Basic definitions

## Node degree = Number of links per node

- Notation: Node $i$'s degree = $k_i$.
- $k_i = 0,1,2,\ldots$.
- Notation: the average degree of a network = $\langle k \rangle$ (and sometimes as $z$)




# Basic definitions

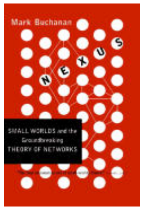
## Adjacency matrix:

- We represent a graph or network by a matrix $A$ with link weight $a_{ij}$ for nodes $i$ and $j$ in entry $(i, j)$.
- e.g.,

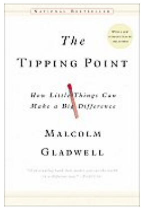
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$




# Books

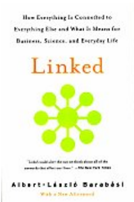



Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan

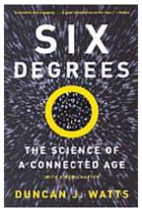



The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell

# Books




Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási

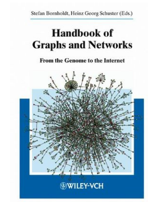



Six Degrees: The Science of a Connected Age—Duncan Watts

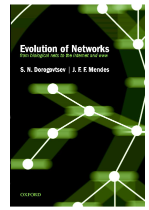




# Books




Handbook of Graphs and Networks—editors: Stefan Bornholdt and H. G. Schuster

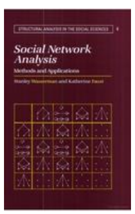



Evolution of Networks—S. N. Dorogovtsev and J. F. F. Mendes.





# Books




Social Network Analysis—Stanley Wasserman and Kathleen Faust


In the Beat of a Heart: Life, Energy, and the Unity of Nature—John Whitfield





# Books

Numerous others:

- Complex Social Networks—F. Vega-Redondo
- Fractal River Basins: Chance and Self-Organization—I. Rodríguez-Iturbe and A. Rinaldo
- Random Graph Dynamics—R. Durette
- Scale-Free Networks—Guido Caldarelli
- Evolution and Structure of the Internet: A Statistical Physics Approach—Romu Pastor-Satorras and Alessandro Vespignani
- Complex Graphs and Networks—Fan Chung

## Examples

### What passes for a complex network?

- Complex networks are large (in node number)
- Complex networks are sparse (low edge to node ratio)
- Complex networks are usually dynamic and evolving
- Complex networks can be social, economic, natural, informational, abstract, ...

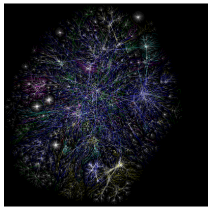


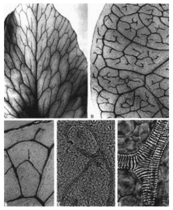


## Examples

### Physical networks

- River networks
- Neural networks
- Trees and leaves
- Blood networks
- The Internet
- Road networks
- Power grids





- Distribution (branching) versus redistribution (cyclical)

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## Examples

### Interaction networks

- The Blogosphere
- Biochemical networks
- Gene-protein networks
- Food webs: who eats whom
- The World Wide Web (?)
- Airline networks
- Call networks (AT&T)
- The Media

datamining.typepad.com (⊞)

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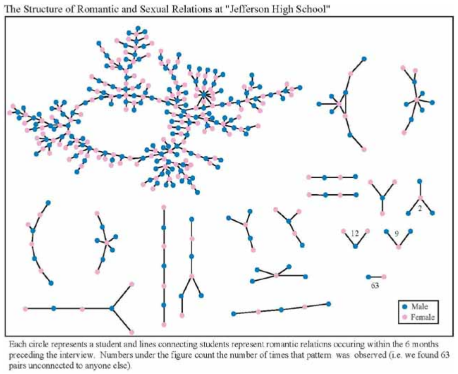
## Examples

### Interaction networks: social networks

- Snogging
- Friendships
- Acquaintances
- Boards and directors
- Organizations
- myspace.com (⊞), facebook.com (⊞)




(Bearman *et al.*, 2004)

- 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).

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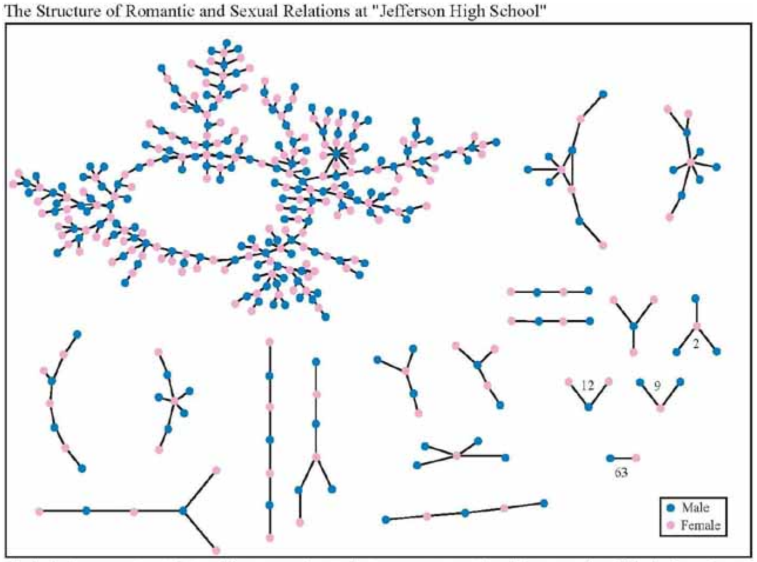
# Examples

The Structure of Romantic and Sexual Relations at "Jefferson High School"




Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).





# Examples

## Relational networks

- Consumer purchases (Wal-Mart: $\approx$ 1 petabyte $= 10^{15}$ bytes)
- Thesauri: Networks of words generated by meanings
- Knowledge/Databases/Ideas
- Metadata—Tagging: del.icio.us (⊞)http://del.icio.usdel.icio.us, flickr (⊞)

**common tags** cloud | list

community daily dictionary education encyclopedia english free imported info information internet knowledge learning news reference research resource resources search tools useful web web2.0 wiki wikipedia





# Observations

## A notable features of large-scale networks:

- Graphical renderings of complex networks are often just a big mess.





# Properties

## Some key aspects of real complex networks:

- degree distribution
- assortativity
- homophily
- clustering
- motifs
- modularity
- concurrency
- hierarchical scaling
- network distances
- centrality
- efficiency
- robustness

- + Coevolution of network structure and processes on networks.

# Properties

## 1. degree distribution $P_k$

- $P_k$ is the probability that a randomly selected node has degree $k$
- $k$ = node degree = number of connections
- ex 1: Erdös-Rényi random networks:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- Distribution is Poisson



# Properties

## 1. degree distribution $P_k$

- ex 2: "Scale-free" networks: $P_k \propto k^{-\gamma} \Rightarrow$ 'hubs'
- link cost controls skew
- hubs may facilitate or impede contagion

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# Properties

## Note:

- Erdös-Rényi random networks are a *mathematical construct*.
- 'Scale-free' networks are growing networks that form according to a plausible mechanism.
- Randomness is out there, just not to the degree of a completely random network.

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# Properties

## 2. assortativity/3. homophily:

- Social networks: Homophily = birds of a feather
- e.g., degree is standard property for sorting: measure degree-degree correlations.
- Assortative network:[7] similar degree nodes connecting to each other. *Often social: company directors, coauthors, actors.*
- Disassortative network: high degree nodes connecting to low degree nodes. *Often techological or biological: Internet, WWW, protein interactions, neural networks, food webs.*

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## Clustering

### 4. clustering:

- Your friends tend to know each other.
- Two measures:
  1. Watts & Strogatz [12]

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i-1)/2} \right\rangle_i$$

  2. Newman [8]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$




## Properties

### First clustering measure:

- $C_1$ is the average fraction of pairs of neighbors who are connected.
- Fraction of pairs of neighbors who are connected is

$$\frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i-1)/2}$$

  where $k_i$ is node $i$'s degree, and $\mathcal{N}_i$ is the set of $i$'s neighbors.
- Averaging over all nodes, we have

$$C_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i-1)/2} = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i-1)/2} \right\rangle_i$$


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## Properties

- For sparse networks, $C_1$ tends to discount highly connected nodes.
- $C_2$ is a useful variant
- In general, $C_1 \neq C_2$.


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## Properties

### Triples and triangles

- Nodes $i_1$, $i_2$, and $i_3$ form a triple around $i_1$ if $i_1$ is connected to $i_2$ and $i_3$.
- Nodes $i_1$, $i_2$, and $i_3$ form a triangle if each pair of nodes is connected
- The definition

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

  measures the fraction of closed triples
- Social Network Analysis (SNA): fraction of transitive triples.
- The '3' appears because for each triangle, we have 3 closed triples.


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# Properties

## Wait, there's more!

- Newman[8]:

$$C_3 = \frac{6 \times \#\text{triangles}}{\#\text{ordered pairs}}$$

- Now count each triple twice
- Same as $C_2$ but interpretation is different
- Probability that a friend of $i$'s friend is also $i$'s friend.





# Properties

## Quick summary:

- $C_1$ = probability that two friends of a randomly chosen node are connected
- $C_2$ = probability that two nodes are connected given they have a friend in common.
- $C_3 (= C_2)$ = probability that a node's friend of a friend is also a friend of that node.





# Properties

- For sparse networks, $C_1$ tends to discount highly connected nodes.
- While $C_1$ is a measure of clustering, it doesn't quite as simple interpretation as $C_2$.
- Some variability in which measure is used in the literature.
- Not always clear which one is being used...

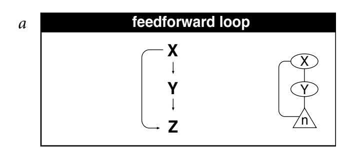




# Properties

## 5. motifs:

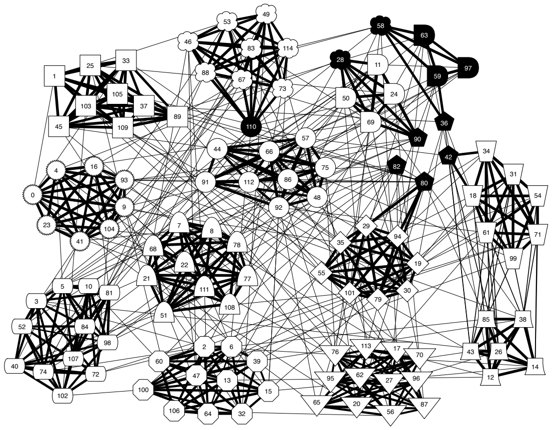
- small, recurring functional subnetworks
- e.g., Feed Forward Loop:




Shen-Orr, Uri Alon, *et al.*[9]

# Properties

## 6. modularity—community detection:



Clauset *et al.*, 2006 [4]: NCAA football





# Properties

## 7. concurrency:

- transmission of a contagious element only occurs during contact
- rather obvious but easily missed in a simple model
- dynamic property—static networks are not enough
- knowledge of previous contacts crucial
- beware cumulated network data
- Kretzschmar and Morris, 1996 [6]

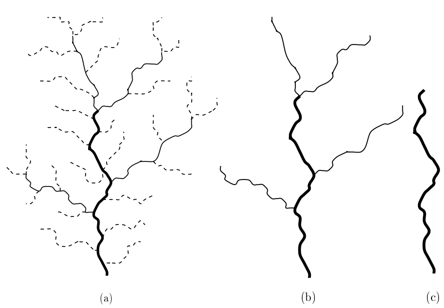




# Properties

## 8. Horton-Strahler ratios:

- Metrics for branching networks:
  - Method for ordering streams hierarchically
  - Number: $R_n = N_\omega / N_{\omega+1}$
  - Segment length: $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
  - Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$








# Properties

## 9. network distances:

### (a) shortest path length $d_{ij}$:

- Fewest number of steps between nodes *i* and *j*.
- (Also called the chemical distance between *i* and *j*.)

### (b) average path length $\langle d_{ij} \rangle$:

- Average shortest path length in whole network.
- Good algorithms exist for calculation.
- Weighted links can be accommodated.

# Properties

9. network distances:

- network diameter $d_{\max}$:
  Maximum shortest path length between any two nodes.
- closeness $d_{\rm cl} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$:
  Average 'distance' between any two nodes.





# Properties

10. centrality:

- Many such measures of a node's 'importance.'
- ex 1: Degree centrality: $k_i$.
- ex 2: Node $i$'s betweenness
  = fraction of shortest paths that pass through $i$.
- ex 3: Recursive centrality: Hubs and Authorities (Kleinberg [5])





# Models

Some important models:

1. generalized random networks
2. scale-free networks
3. small-world networks
4. statistical generative models ($p^*$)
5. generalized affiliation networks





# Models

1. generalized random networks:

- Arbitrary degree distribution $P_k$.
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

# Models

## 2. 'scale-free networks':

- Introduced by Barabasi and Albert [1]
- Generative model
- Preferential attachment model with growth:
- $P$[attachment to node i] $\propto \mathrm{k}_\mathrm{i}^{\alpha}$.
- Produces $P_k \sim k^{-\gamma}$ when $\alpha = 1$.
- Trickiness: other models generate skewed degree distributions.





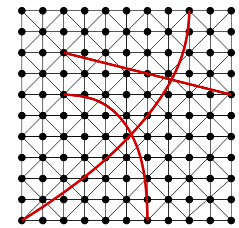
# Models

## 3. small-world networks

- Introduced by Watts and Strogatz [12]

Two scales:

- local regularity (an individual's friends know each other)
- global randomness (shortcuts).



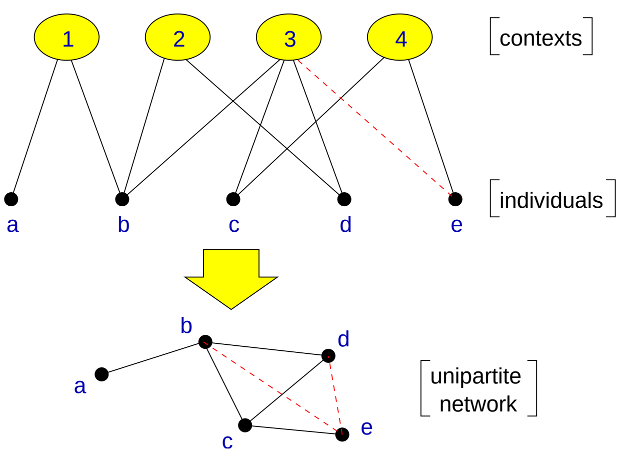
- Shortcuts allow disease to jump
- Number of infectives increases exponentially in time





# Models

## 5. generalized affiliation networks



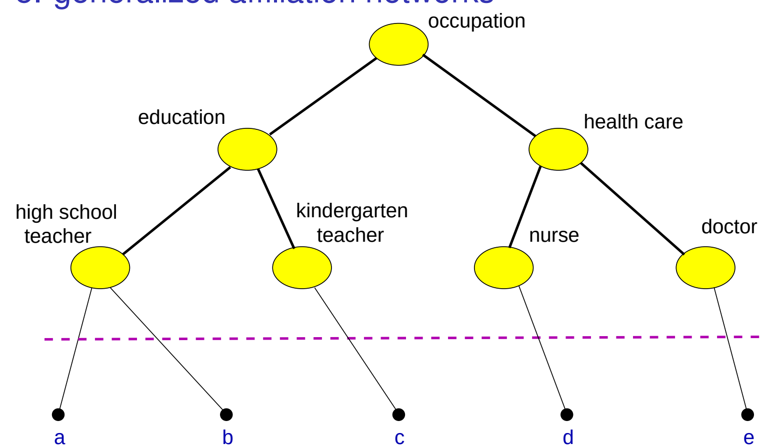

Bipartite affiliation networks: boards and directors, movies and actors.





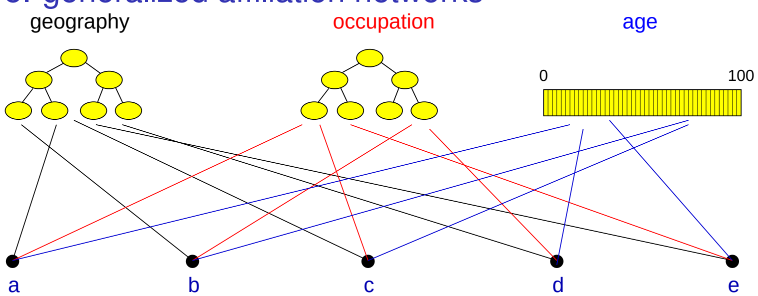
# Models

## 5. generalized affiliation networks



# Models

## 5. generalized affiliation networks




- Blau & Schwartz [2], Simmel [10], Breiger [3], Watts *et al.* [11]



# Popularity

"Collective dynamics of 'small-world' networks" [12]

- Watts and Strogatz
  Nature, 1998
- ≈ 2400 citations (as of Jan 14, 2008)

"Emergence of scaling in random networks" [1]

- Barabási and Albert
  Science, 1999
- ≈ 2300 citations (as of Jan 14, 2008)



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