Branching Networks

Complex Networks, Course 295A, Spring, 2008

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Outline

Introduction

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Introduction
River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton $\Leftrightarrow$ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models

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- Cardiovascular networks
- Plants
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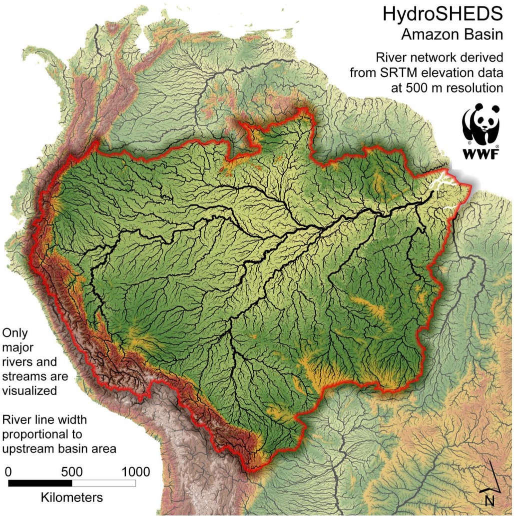
- Organizations (only in theory...)

Branching networks are everywhere...



http://hydrosheds.cr.usgs.gov/ (⊞)

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http://en.wikipedia.org/wiki/Image:Applebox.JPG (⊞)

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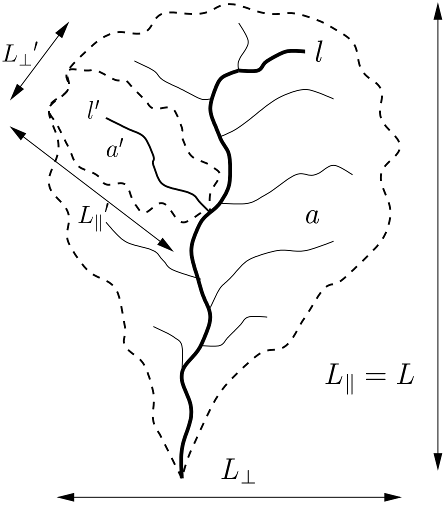
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...

Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:



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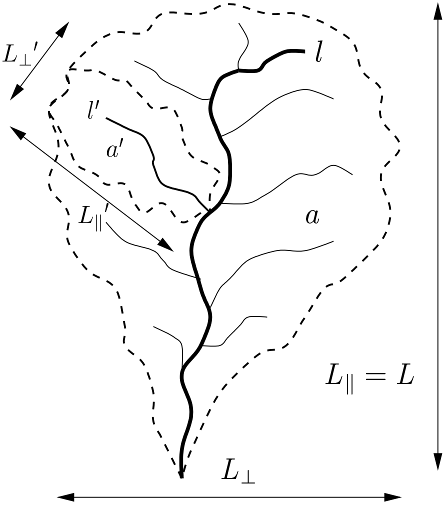

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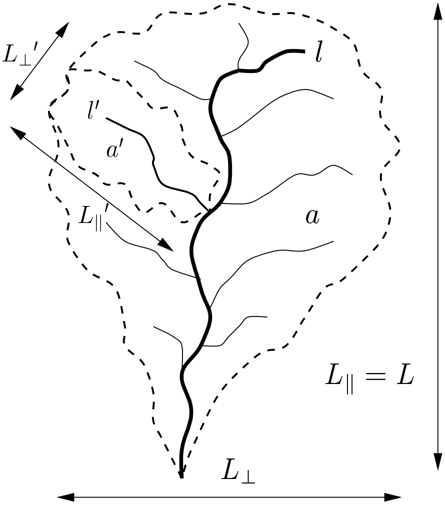

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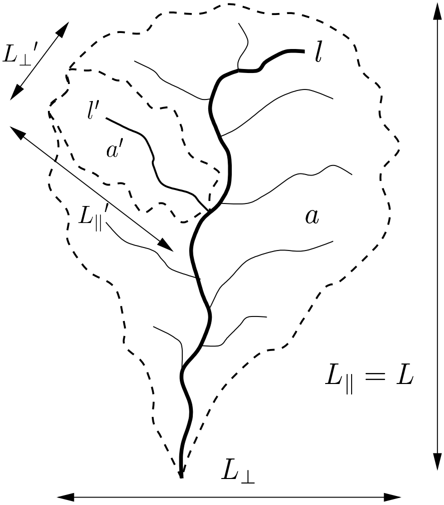

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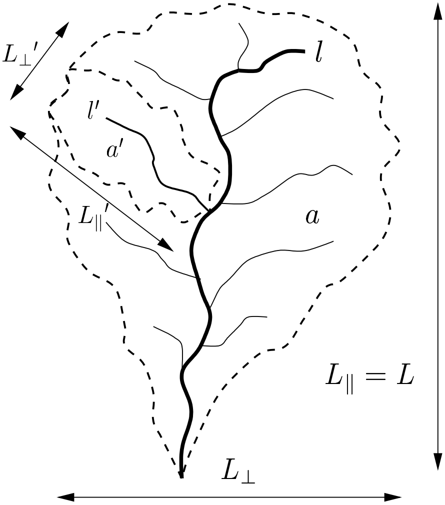

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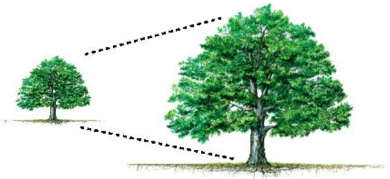

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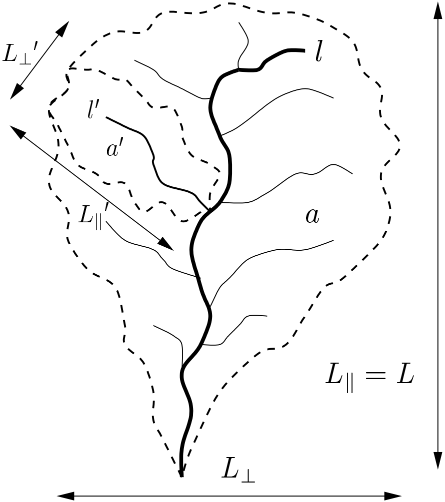
Allometry: dimensions scale nonlinearly.

Basin allometry



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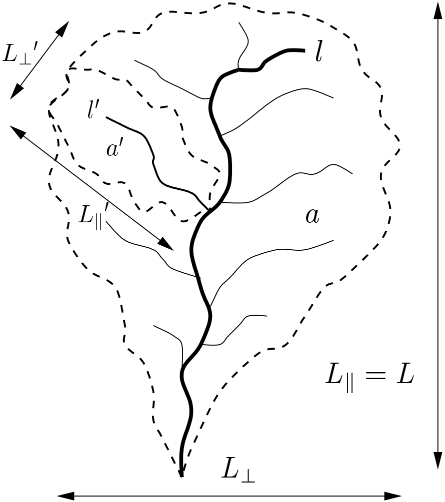

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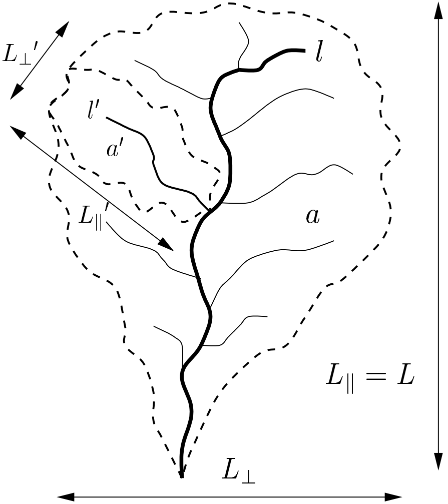

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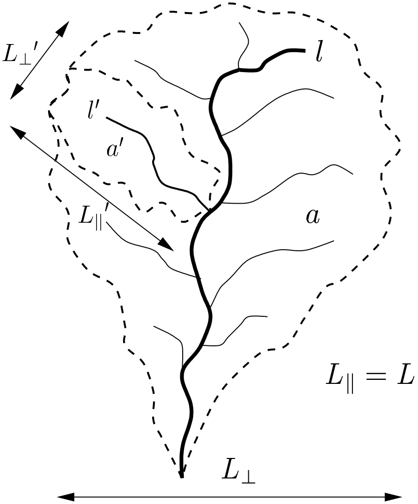

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- $$\ell \propto a^h$$
- $$\ell \propto L^d$$
- Combine above:

$$a \propto L^{d/h} \equiv L^D$$

‘Laws’

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- Basin allometry:

$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

$D < 2 \rightarrow$ basins elongate.

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$\lambda \sim L^\varphi$	variation of Langbein's law

Reported parameter values: [2]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05

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For (3): Many attempts: not yet sorted out...

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- Term: Horton-Strahler Stream Ordering [11]
- Can be seen as iterative trimming of a network.

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- Use symbol $\omega = 1, 2, 3, ...$ for stream order.

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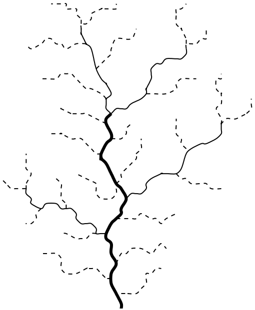

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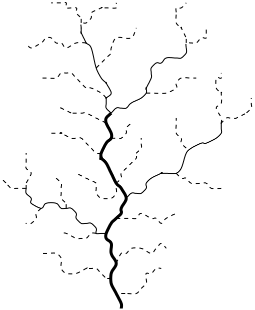

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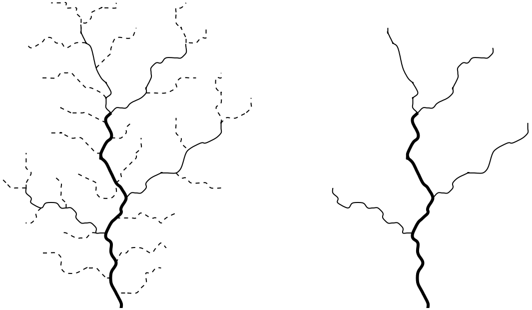

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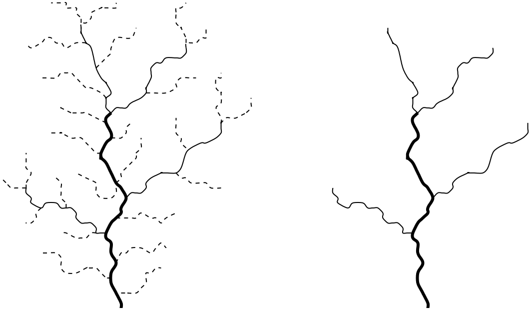

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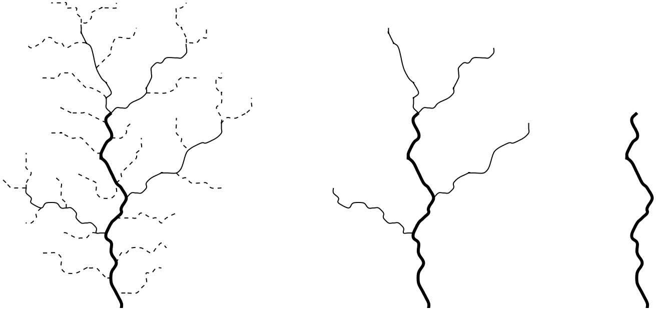

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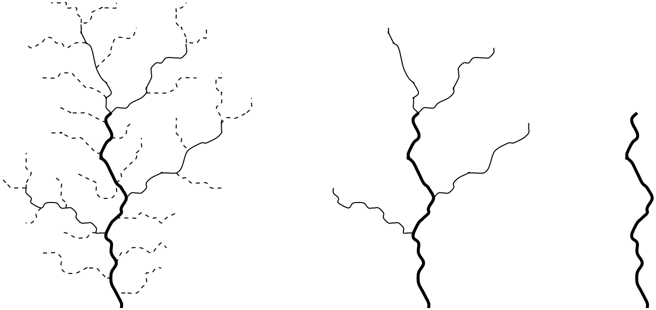

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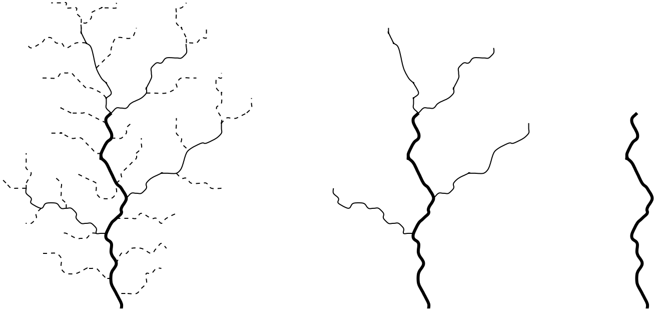

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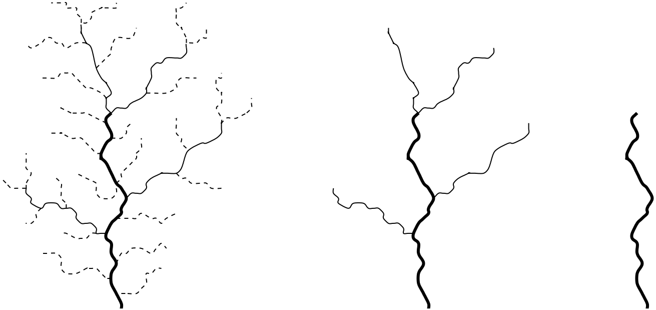

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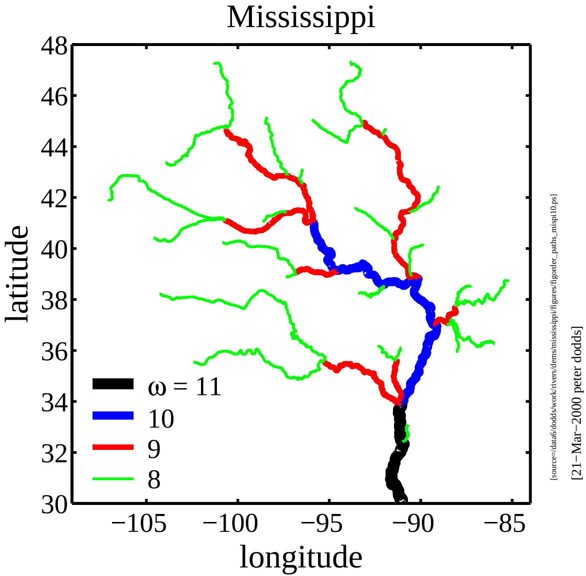
5. Example above is a basin of order $\Omega = 3$.

Stream Ordering—A large example:



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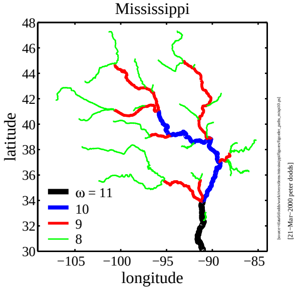

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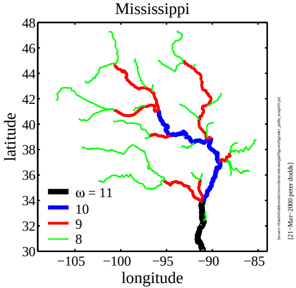

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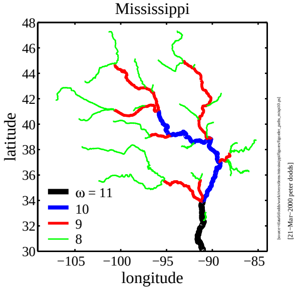

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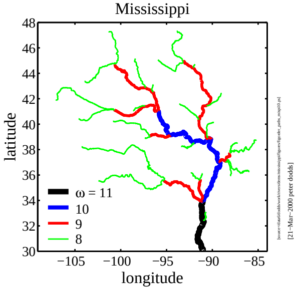
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).



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Another way to define ordering:

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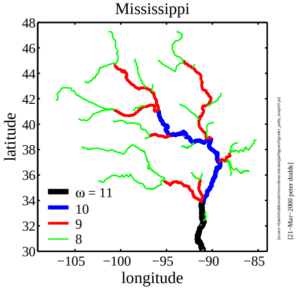

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- Simple rule:

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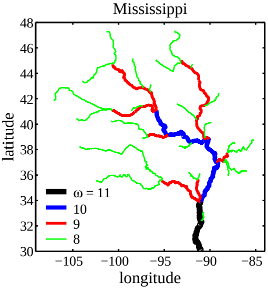
where δ is the Kronecker delta.



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  2. an order ω stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams

Outline

Horton's laws

Self-similarity of river networks

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Horton's laws

Horton's Ratios:

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Similar story for area and length:

- $$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$
- $$\bar{\ell}_\omega = \bar{\ell}_1 e^{(\omega-1)\ln R_\ell}$$
- As stream order increases, number drops and area and length increase.

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- But we need one other piece of information...

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A bonus law:

- Horton's law of stream segment lengths:

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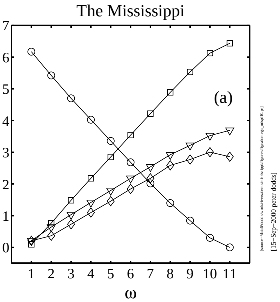
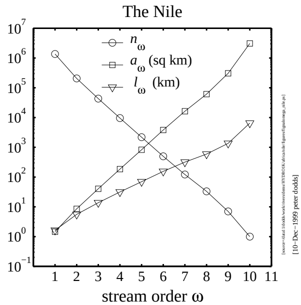
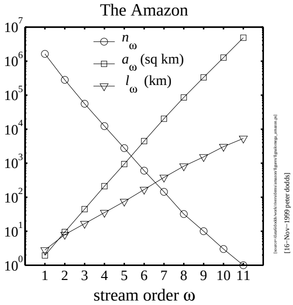
- Can show that $R_s = R_\ell$.

Horton's laws in the real world:







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- Horton's laws hold for sections of cardiovascular networks
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- Vessel diameters obey an analogous Horton's law.

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- Horton's ratios vary:

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- ... but they don't explain how networks are structured.

Outline

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Delving deeper into network architecture:

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Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]
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- Tokunaga's law is also a law of averages.

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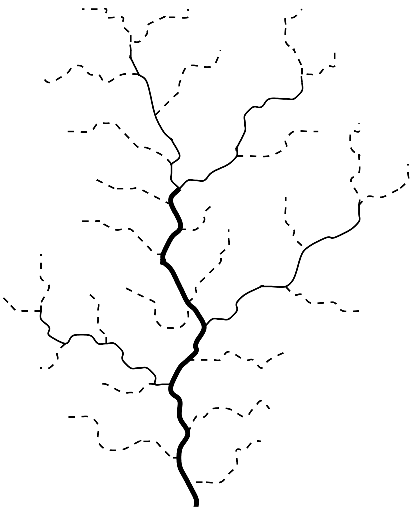
- We usually write Tokunaga's law as:

$$\boxed{T_k = T_1 (R_T)^{k-1}} \quad \text{where } R_T \simeq 2$$

Tokunaga's law—an example:

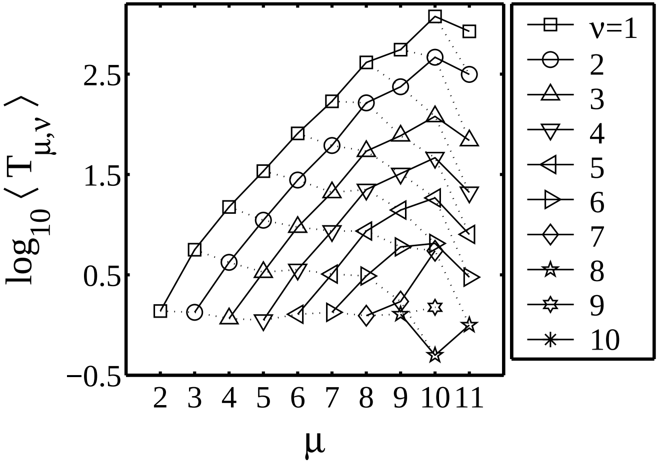
$T_1 \simeq 2$

$R_T \simeq 4$



The Mississippi

A Tokunaga graph:



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- Known result: Tokunaga $\rightarrow$ Horton [21, 22, 23, 10, 2]

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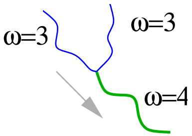

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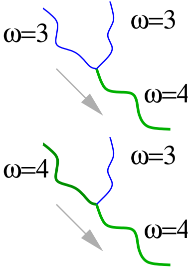

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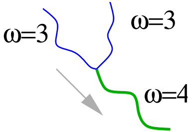

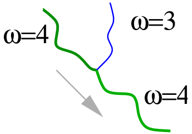

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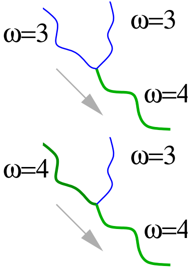

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More with the happy-making thing

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- Recall, we want to show $R_n = n_\omega/n_{\omega+1}$ is a constant, independent of ω...

More with the happy-making thing

Finding Horton ratios:

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$$\frac{n_\omega}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}} \quad (1)$$

More with the happy-making thing

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- Plug into Eq. (1)...

More with the happy-making thing

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More with the happy-making thing

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\begin{aligned}
R_n &= 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)} \\
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&= 2 + T_1 \frac{1}{1 - R_T/R_n}
\end{aligned}
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More with the happy-making thing

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- Rearrange to find:

$$(R_n - 2)(1 - R_T/R_n) = T_1$$

More with the happy-making thing

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- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density $\rho_{\rm dd}$.

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Horton and Tokunaga are happy

Altogether then:

- $$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T$$

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Horton and Tokunaga are happy

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$$\boxed{R_\ell = R_T}$$
- And from before:
$$\boxed{R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}}$$

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Some observations:

- R_n and R_ℓ depend on T_1 and R_T.

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- Suggests Horton's laws must contain some redundancy
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between statistical distributions. [3, 4]

Horton and Tokunaga are happy

The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.

Horton and Tokunaga are happy

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Horton and Tokunaga are happy

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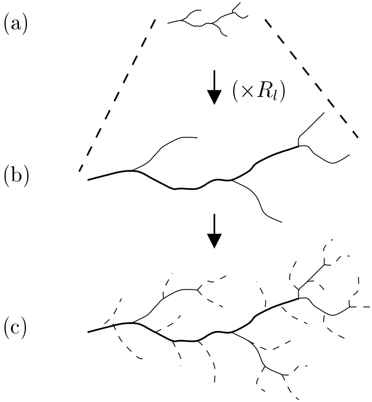
- $$R_T = R_\ell,$$
- $$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform)...

Horton and Tokunaga are friends

From Horton to Tokunaga [2]



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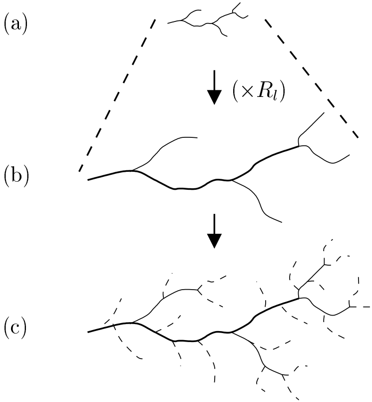

- Assume Horton's laws hold for number and length

Horton and Tokunaga are friends

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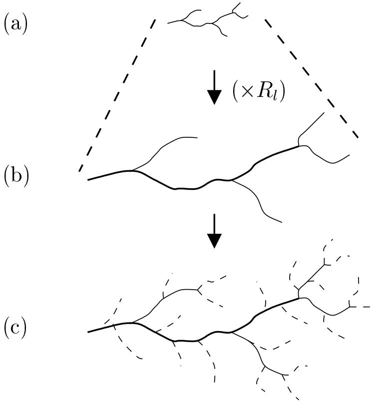

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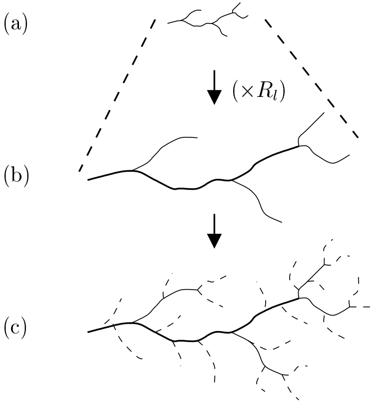

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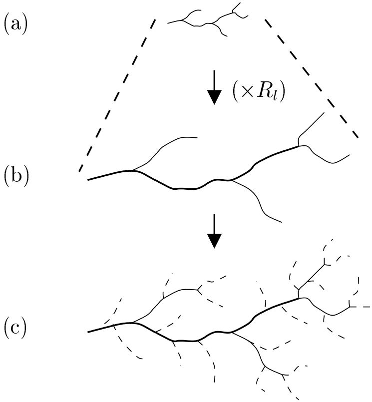

- Assume Horton's laws hold for number and length
- Start with an order ω stream
- Scale up by a factor of R_ℓ, orders increment
- Maintain drainage density by adding new order 1 streams

Horton and Tokunaga are friends

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- For large ω, Tokunaga's law is the solution—let's check...

Horton and Tokunaga are friends

Just checking:

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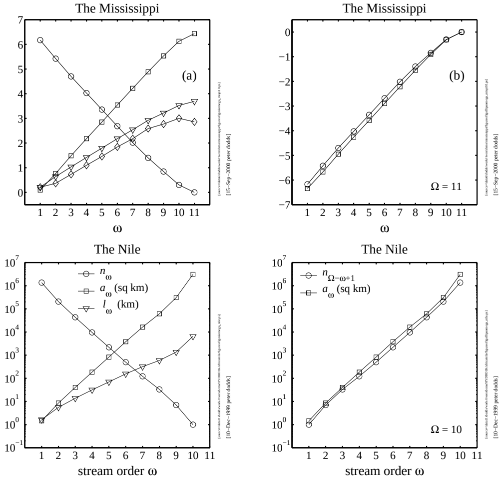
$$\simeq (R_\ell - 1)\,T_1 \frac{R_\ell^{\,k}}{R_\ell - 1} = T_1 R_\ell^{k} \quad \text{... yep.}$$

Outline

Horton's laws of area and number:



- In right plots, stream number graph has been flipped vertically.

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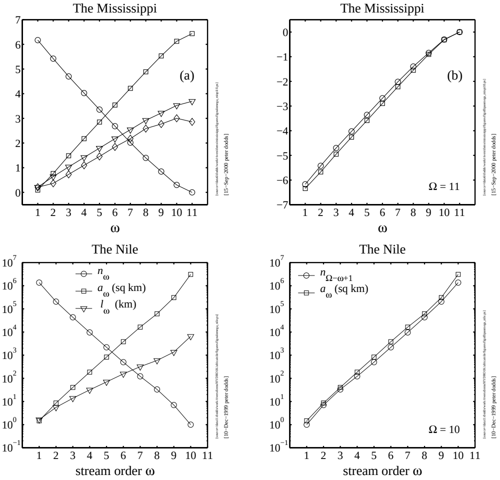

- In right plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$...

Measuring Horton ratios is tricky:

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- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	5.27	5.26	2.48	2.30	1.00
$[2,5]$	4.86	4.96	2.42	2.31	1.02
$[2,7]$	4.77	4.88	2.40	2.31	1.02
$[3,4]$	4.72	4.91	2.41	2.34	1.04
$[3,6]$	4.70	4.83	2.40	2.35	1.03
$[3,8]$	4.60	4.79	2.38	2.34	1.04
$[4,6]$	4.69	4.81	2.40	2.36	1.02
$[4,8]$	4.57	4.77	2.38	2.34	1.05
$[5,7]$	4.68	4.83	2.36	2.29	1.03
$[6,7]$	4.63	4.76	2.30	2.16	1.03
$[7,8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024

Amazon:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

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Continued ...

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$$\sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \nearrow$$

Reducing Horton's laws:

Continued ...

- $$a_\Omega \propto \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^\omega$$

$$= \frac{R_n^\Omega}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1 - (R_s/R_n)^\Omega}{1 - (R_s/R_n)}$$

$$\sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1 - (R_s/R_n)} \text{ as } \Omega \nearrow$$

- So, a_Ω is growing like R_n^Ω and therefore:

$$\boxed{R_n \equiv R_a}$$

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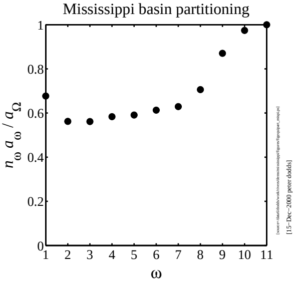
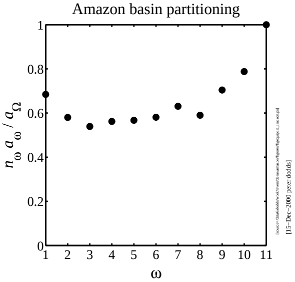
$$n_\omega \propto (R_n)^{-\omega}$$

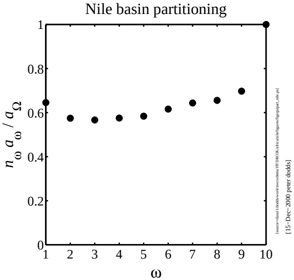
$$\bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1}$$

Equipartitioning:

Some examples:







Outline

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- Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$

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- Also known as the exceedance probability.

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- Use Horton's law of stream segments: $s_\omega / s_{\omega-1} = R_s$...

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- $$\ell_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$

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- Let's connect to one last relationship: Hack's law

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Hack's law: [6]

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Only 3 parameters are independent:
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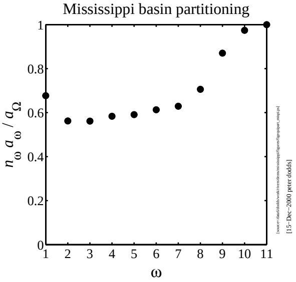
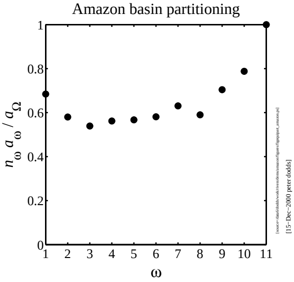
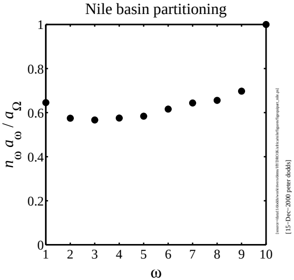
$\lambda \sim L^\varphi$	$\varphi = d$

Equipartitioning reexamined:

Recall this story:







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- while stream ordering separates basins...

Outline

Fluctuations

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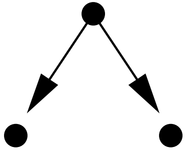
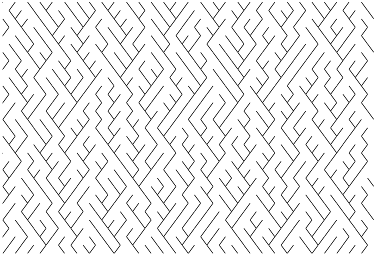
- Yields rich and full description of branching network structure
- See into the heart of randomness...

A toy model—Scheidegger's model

Directed random networks [12, 13]





-
-

$$P(\searrow) = P(\swarrow) = 1/2$$

- Flow is directed downwards
- Useful and interesting test case—more later...

Generalizing Horton's laws

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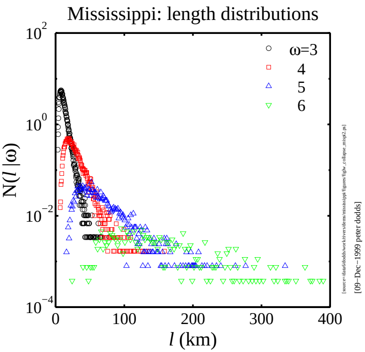

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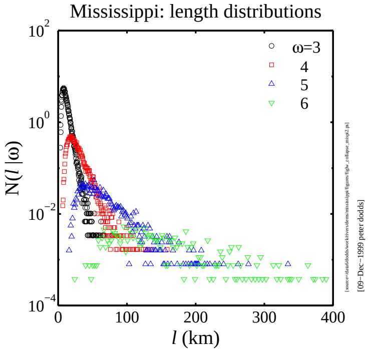

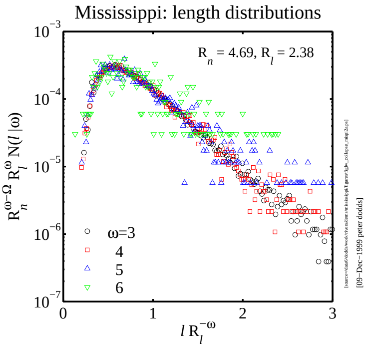

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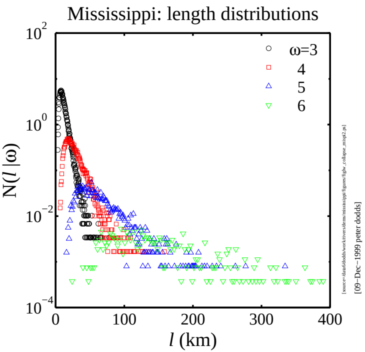
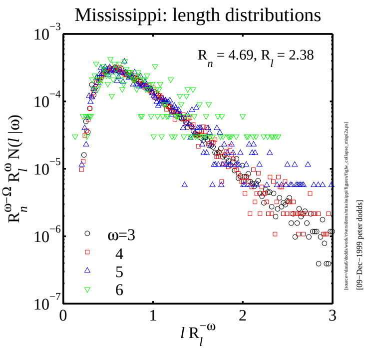
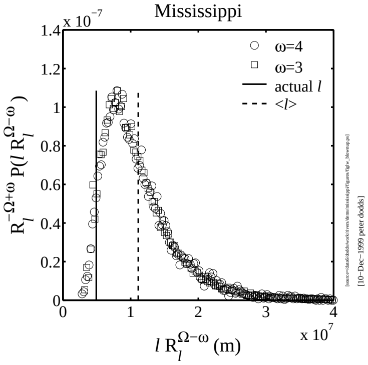




- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

Generalizing Horton's laws

- How well does overall basin fit internal pattern?



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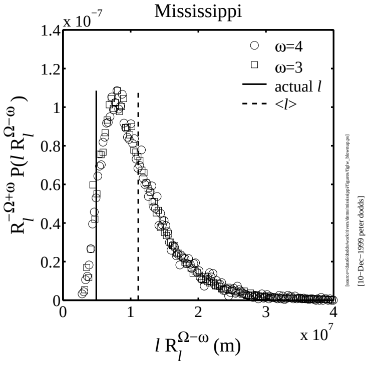

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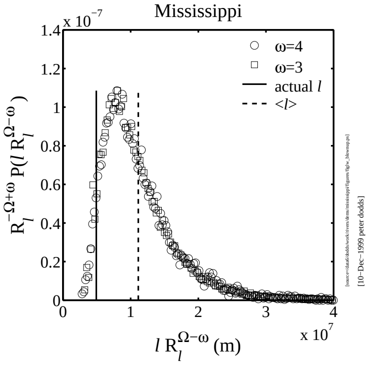

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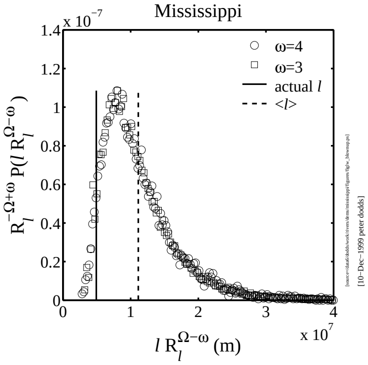

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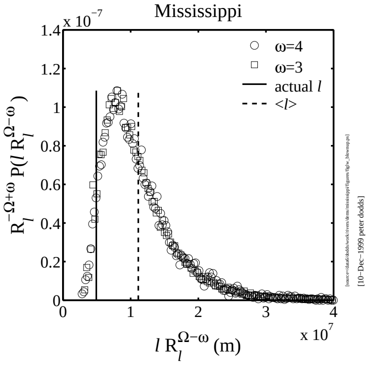

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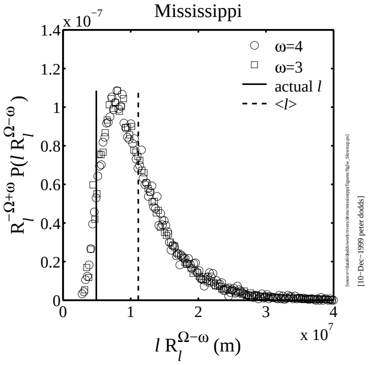

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- Actual length/Mean length = 44 %
- Okay.

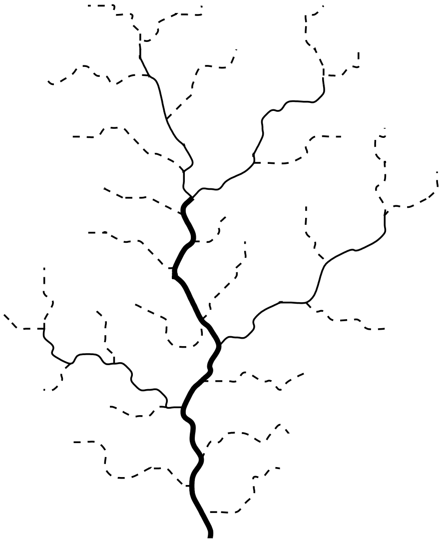
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73

	a	$\bar{a}_\Omega$	σ_a	$a/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86

Combining stream segments distributions:



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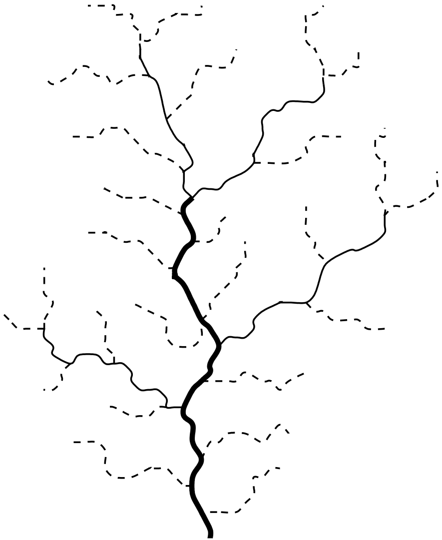

- Stream segments sum to give main stream lengths
- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_ω

Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$

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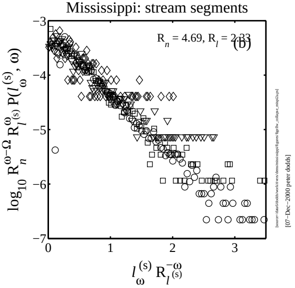


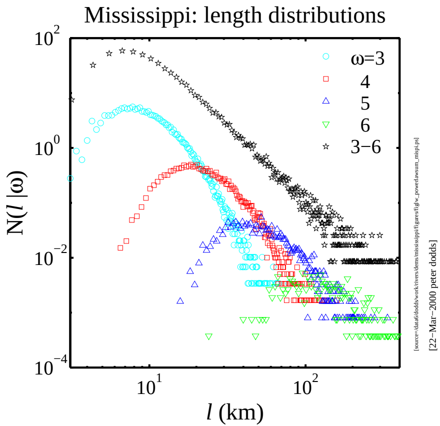
$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

Generalizing Horton's laws

- Next level up: Main stream length distributions must combine to give overall distribution for stream length



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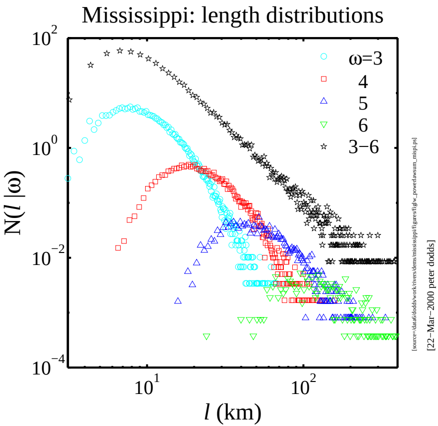
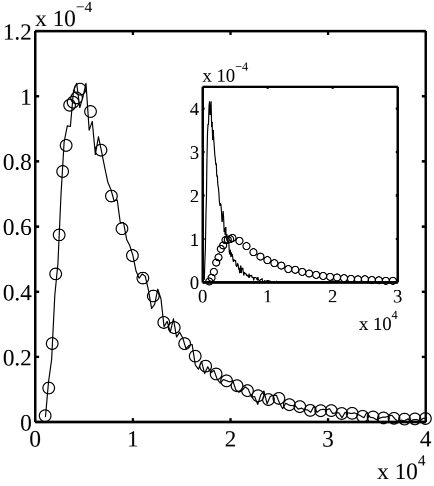


- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting...

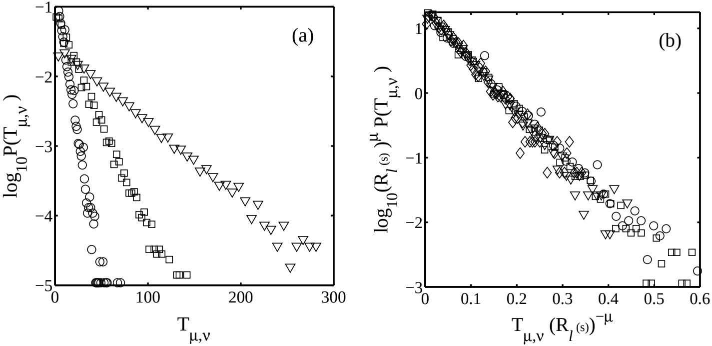
Generalizing Horton's laws

Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$.



Generalizing Tokunaga's law

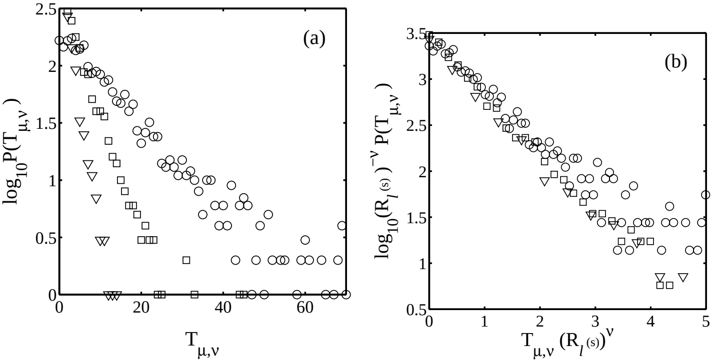
Scheidegger:



- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

Generalizing Tokunaga's law

Mississippi:



- Same data collapse for Mississippi...

Generalizing Tokunaga's law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

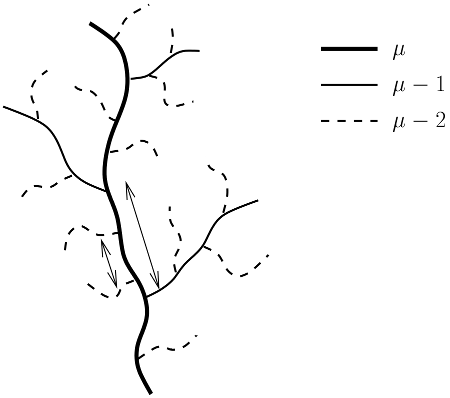
$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



Generalizing Tokunaga's law

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- $\Rightarrow$ random spatial distribution of stream segments

Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

where

  - p_ν = probability of absorbing an order ν side stream

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  where
  - $p_\nu =$ probability of absorbing an order ν side stream
  - $\tilde{p}_\mu =$ probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

Generalizing Tokunaga's law

- Now deal with thing:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

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- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

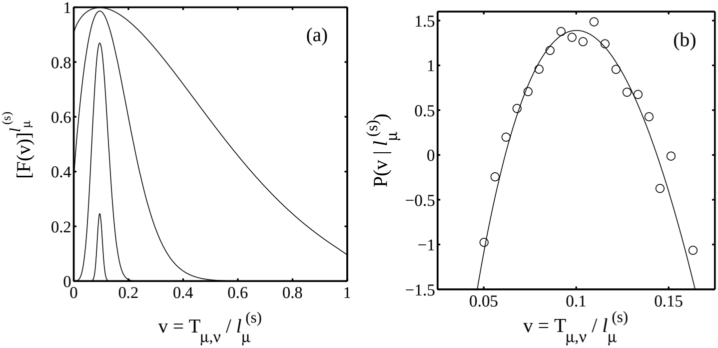
where

$$F(v) = \left(\frac{1 - v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:



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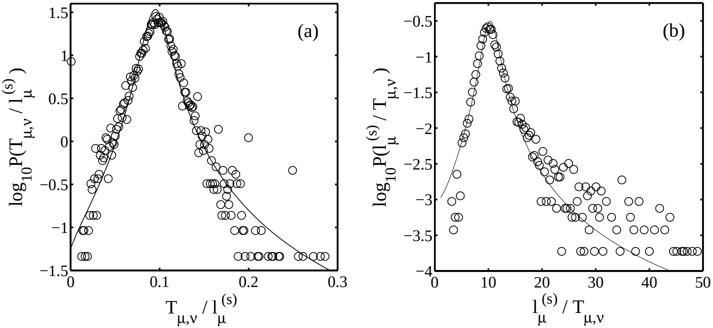

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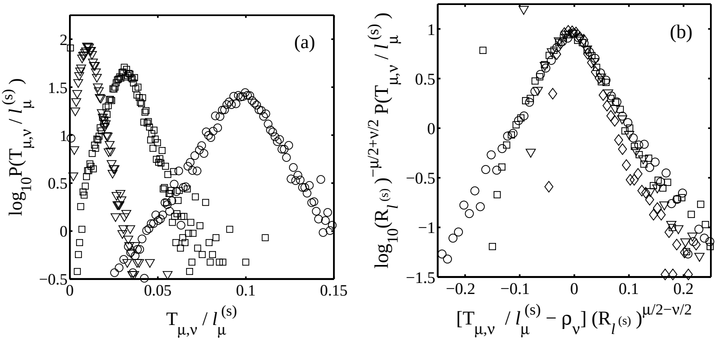

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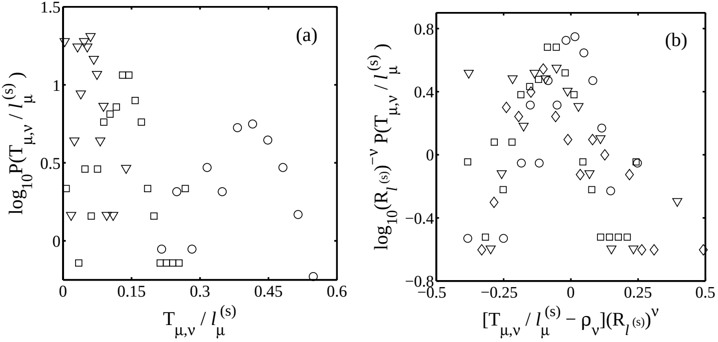
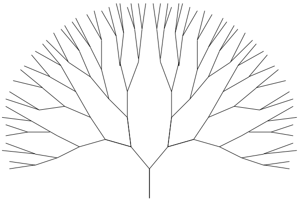


Outline

Models

Random subnetworks on a Bethe lattice [15]



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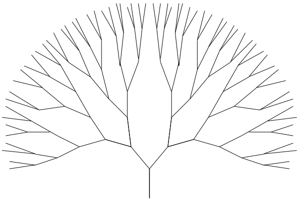

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Models

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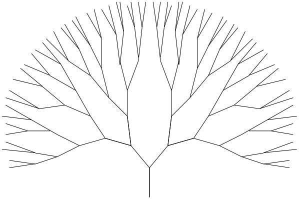

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Branching Networks

Introduction

River Networks
Definitions
Allometry
Laws
Stream Ordering
Horton's Laws
Tokunaga's Law
Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models

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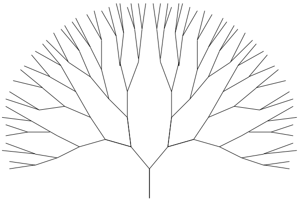

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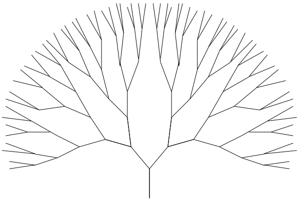

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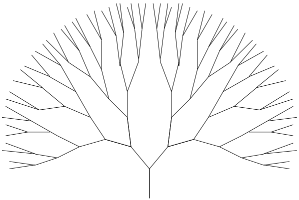

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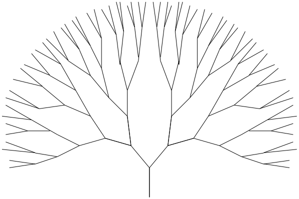

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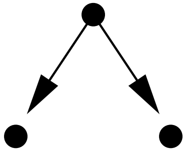
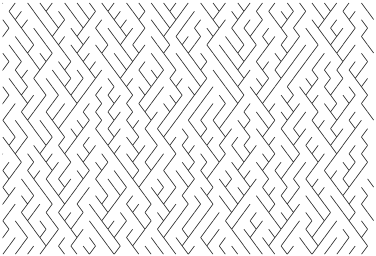
- So let's move on...

Scheidegger's model

Directed random networks [12, 13]





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$$P(\searrow) = P(\swarrow) = 1/2$$

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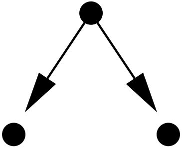

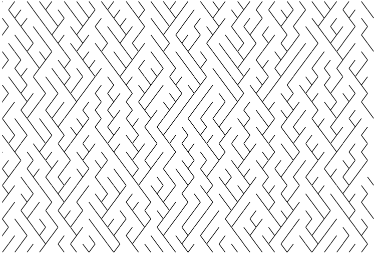

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-

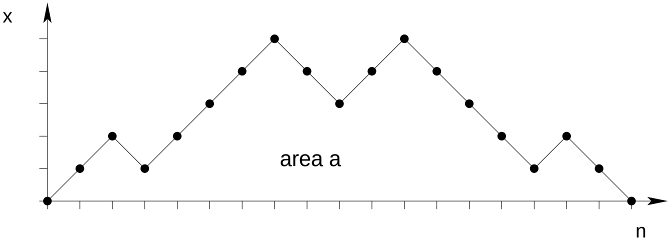
$$P(\searrow) = P(\swarrow) = 1/2$$

- Functional form of all scaling laws exhibited but exponents differ from real world [18, 19, 17]

A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks



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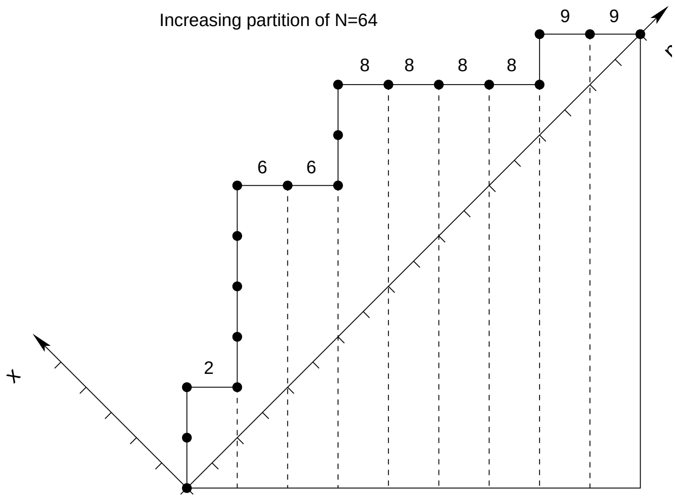

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- R_n and R_ℓ have not been derived analytically.

Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al.[11]

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- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [9]

Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

$d \Rightarrow \ell \propto L_{\parallel}^{d}$ (stream self-affinity).

References I

H. de Vries, T. Becker, and B. Eckhardt.
Power law distribution of discharge in ideal networks.
Water Resources Research, 30(12):3541–3543, December 1994.

P. S. Dodds and D. H. Rothman.
Unified view of scaling laws for river networks.
Physical Review E, 59(5):4865–4877, 1999. pdf (⊞)

P. S. Dodds and D. H. Rothman.
Geometry of river networks. II. Distributions of component size and number.
Physical Review E, 63(1):016116, 2001. pdf (⊞)

P. S. Dodds and D. H. Rothman.
Geometry of river networks. III. Characterization of component connectivity.
Physical Review E, 63(1):016117, 2001. pdf (⊞)

References II

N. Goldenfeld.
Lectures on Phase Transitions and the Renormalization Group, volume 85 of *Frontiers in Physics*.
Addison-Wesley, Reading, Massachusetts, 1992.

J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957.

R. E. Horton.
Erosional development of streams and their drainage basins; hydrophysical approach to quatitative morphology.
Bulletin of the Geological Society of America, 56(3):275–370, 1945.

References III

J. W. Kirchner.
Statistical inevitability of Horton's laws and the apparent randomness of stream channel networks.
Geology, 21:591–594, July 1993.

A. Maritan, F. Colaiori, A. Flammini, M. Cieplak, and J. R. Banavar.
Universality classes of optimal channel networks.
Science, 272:984–986, 1996. pdf (⊞)

S. D. Peckham.
New results for self-similar trees with applications to river networks.
Water Resources Research, 31(4):1023–1029, April 1995.

References IV

I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.

A. E. Scheidegger.
A stochastic model for drainage patterns into an intramontane trench.
Bull. Int. Assoc. Sci. Hydrol., 12(1):15–20, 1967.
.

A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.
.

References V

S. A. Schumm.
Evolution of drainage systems and slopes in badlands at Perth Amboy, New Jersey.
Bulletin of the Geological Society of America, 67:597–646, May 1956.

R. L. Shreve.
Infinite topologically random channel networks.
Journal of Geology, 75:178–186, 1967.

A. N. Strahler.
Hypsometric (area altitude) analysis of erosional topography.
Bulletin of the Geological Society of America, 63:1117–1142, 1952.

.

References VI

H. Takayasu.
Steady-state distribution of generalized aggregation system with injection.
Physcial Review Letters, 63(23):2563–2565, December 1989.

H. Takayasu, I. Nishikawa, and H. Tasaki.
Power-law mass distribution of aggregation systems with injection.
Physical Review A, 37(8):3110–3117, April 1988.

M. Takayasu and H. Takayasu.
Apparent independency of an aggregation system with injection.
Physical Review A, 39(8):4345–4347, April 1989.

References VII

D. G. Tarboton, R. L. Bras, and I. Rodríguez-Iturbe.
Comment on "On the fractal dimension of stream networks" by Paolo La Barbera and Renzo Rosso.
Water Resources Research, 26(9):2243–4, September 1990.

E. Tokunaga.
The composition of drainage network in Toyohira River Basin and the valuation of Horton's first law.
Geophysical Bulletin of Hokkaido University, 15:1–19, 1966.

E. Tokunaga.
Consideration on the composition of drainage networks and their evolution.
Geographical Reports of Tokyo Metropolitan University, 13:1–27, 1978.
.

References VIII

E. Tokunaga.
Ordering of divide segments and law of divide segment numbers.
Transactions of the Japanese Geomorphological Union, 5(2):71–77, 1984.

G. K. Zipf.
Human Behaviour and the Principle of Least-Effort.
Addison-Wesley, Cambridge, MA, 1949.