Branching Networks

Complex Networks, Course 295A, Spring, 2008

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Outline

Introduction

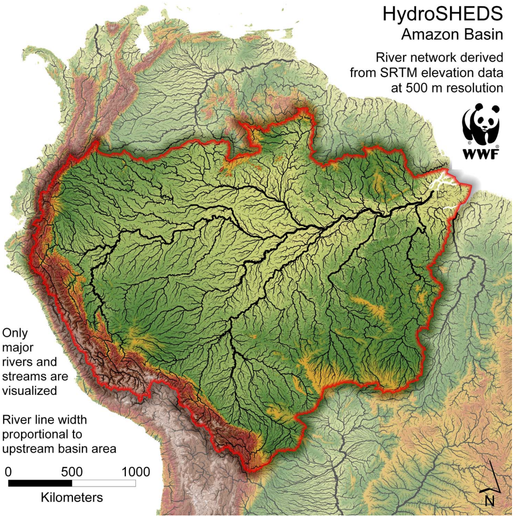
Branching networks are useful things:

- Fundamental to material supply and collection
- Supply: From one source to many sinks in 2- or 3-d.
- Collection: From many sources to one sink in 2- or 3-d.
- Typically observe hierarchical, recursive self-similar structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

Branching networks are everywhere...



http://hydrosheds.cr.usgs.gov/ (⊞)

Branching networks are everywhere...

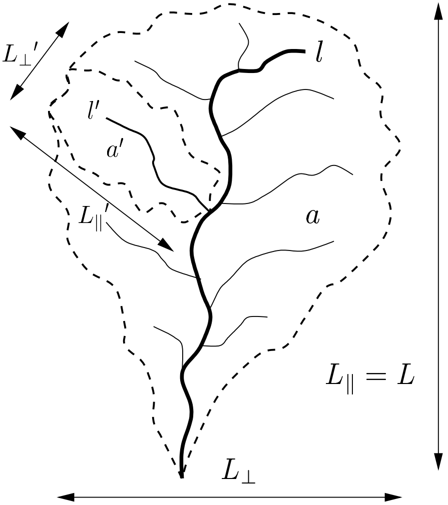
http://en.wikipedia.org/wiki/Image:Applebox.JPG (⊞)

Geomorphological networks

Definitions

- Drainage basin for a point p is the complete region of land from which overland flow drains through p.
- Definition most sensible for a point in a stream.
- Recursive structure: Basins contain basins and so on.
- In principle, a drainage basin is defined at every point on a landscape.
- On flat hillslopes, drainage basins are effectively linear.
- We treat subsurface and surface flow as following the gradient of the surface.
- Okay for large-scale networks...

Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:



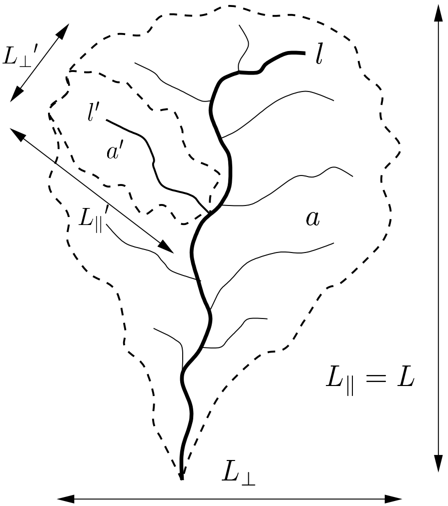
- a = drainage basin area
- ℓ = length of longest (main) stream (which may be fractal)
- $L = L_{\parallel}$ = longitudinal length of basin
- $L = L_{\perp}$ = width of basin

Allometry

Isometry: dimensions scale linearly with each other.

Allometry: dimensions scale nonlinearly.

Basin allometry



Allometric relationships:

- $$\ell \propto a^h$$
- $$\ell \propto L^d$$
- Combine above:

$$a \propto L^{d/h} \equiv L^D$$

‘Laws’

- Hack’s law (1957) [6]:

$$\ell \propto a^h$$

reportedly $0.5 < h < 0.7$

- Scaling of main stream length with basin size:

$$\ell \propto L_{\parallel}^{d}$$

reportedly $1.0 < d < 1.1$

- Basin allometry:

$$L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$$

$D < 2 \rightarrow$ basins elongate.

There are a few more 'laws': [2]

Relation:	Name or description:
$T_k = T_1 (R_T)^k$	Tokunaga's law
$\ell \sim L^d$	self-affinity of single channels
$n_\omega / n_{\omega+1} = R_n$	Horton's law of stream numbers
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	Horton's law of main stream lengths
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	Horton's law of basin areas
$\bar{s}_{\omega+1} / \bar{s}_\omega = R_s$	Horton's law of stream segment lengths
$L_\perp \sim L^H$	scaling of basin widths
$P(a) \sim a^{-\tau}$	probability of basin areas
$P(\ell) \sim \ell^{-\gamma}$	probability of stream lengths
$\ell \sim a^h$	Hack's law
$a \sim L^D$	scaling of basin areas
$\Lambda \sim a^\beta$	Langbein's law
$\lambda \sim L^\varphi$	variation of Langbein's law

Reported parameter values: [2]

Parameter:	Real networks:
R_n	3.0–5.0
R_a	3.0–6.0
$R_\ell = R_T$	1.5–3.0
T_1	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50–0.70
τ	1.43 ± 0.05
γ	1.8 ± 0.1
H	0.75–0.80
β	0.50–0.70
φ	1.05 ± 0.05

Kind of a mess...

Order of business:

1. Find out how these relationships are connected.
2. Determine most fundamental description.
3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...

Stream Ordering:

Method for describing network architecture:

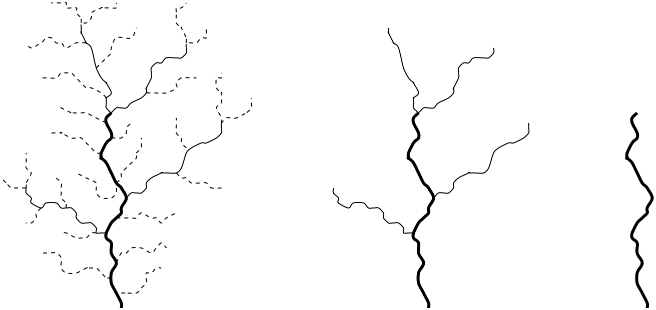
- Introduced by Horton (1945) [7]
- Modified by Strahler (1957) [16]
- Term: Horton-Strahler Stream Ordering [11]
- Can be seen as iterative trimming of a network.

Stream Ordering:

Some definitions:

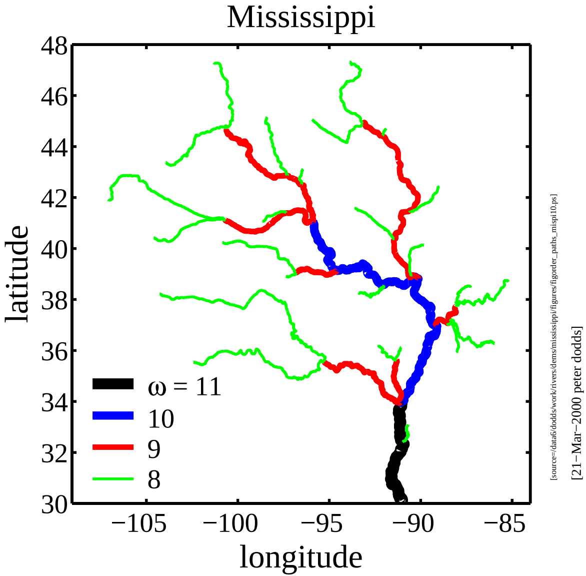
- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, \ldots$ for stream order.

Stream Ordering:



1. Label all source streams as order $\omega = 1$ and remove.
2. Label all new source streams as order $\omega = 2$ and remove.
3. Repeat until one stream is left (order = Ω)
4. Basin is said to be of the order of the last stream removed.
5. Example above is a basin of order $\Omega = 3$.

Stream Ordering—A large example:



Stream Ordering:

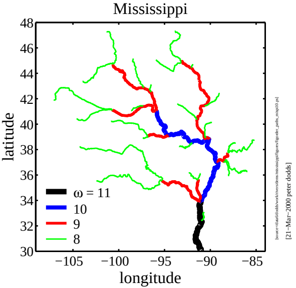
Another way to define ordering:

- As before, label all source streams as order $\omega = 1$.
- Follow all labelled streams downstream
- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 ($\omega + 1$).

- If streams of different orders ω_1 and ω_2 meet, then the resultant stream has order equal to the largest of the two.
- Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.



Stream Ordering:

One problem:

- Resolution of data messes with ordering
- Micro-description changes (e.g., order of a basin may increase)
- ... but relationships based on ordering appear to be robust to resolution changes.

Stream Ordering:

Utility:

- Stream ordering helpfully discretizes a network.
- Goal: understand network architecture

Stream Ordering:

Resultant definitions:

- A basin of order Ω has n_ω streams (or sub-basins) of order ω.
  - $n_\omega > n_{\omega+1}$
- An order ω basin has area a_ω.
- An order ω basin has a main stream length ℓ_ω.
- An order ω basin has a stream segment length s_ω
  1. an order ω stream segment is only that part of the stream which is actually of order ω
  2. an order ω stream segment runs from the basin outlet up to the junction of two order $\omega - 1$ streams

Horton's laws

Self-similarity of river networks

- First quantified by Horton (1945) [7], expanded by Schumm (1956) [14]

Three laws:

- Horton's law of stream numbers:

$$\boxed{n_\omega / n_{\omega+1} = R_n > 1}$$

- Horton's law of stream lengths:

$$\boxed{\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell > 1}$$

- Horton's law of basin areas:

$$\boxed{\bar{a}_{\omega+1} / \bar{a}_\omega = R_a > 1}$$

Horton's laws

Horton's Ratios:

- So... Horton's laws are defined by three ratios:

$$R_n, \ R_\ell, \ \text{and } R_a.$$

- Horton's laws describe exponential decay or growth:

$$\begin{aligned} n_\omega &= n_{\omega-1}/R_n \\ &= n_{\omega-2}/R_n^2 \\ &\vdots \\ &= n_1/R_n^{\omega-1} \\ &= n_1 e^{-(\omega-1)\ln R_n} \end{aligned}$$

Horton's laws

Similar story for area and length:

- $$\bar{a}_\omega = \bar{a}_1 e^{(\omega-1)\ln R_a}$$
- $$\bar{\ell}_\omega = \bar{\ell}_1 e^{(\omega-1)\ln R_\ell}$$
- As stream order increases, number drops and area and length increase.

Horton's laws

A few more things:

- Horton's laws are laws of averages.
- Averaging for number is across basins.
- Averaging for stream lengths and areas is within basins.
- Horton's ratios go a long way to defining a branching network...
- But we need one other piece of information...

Horton's laws

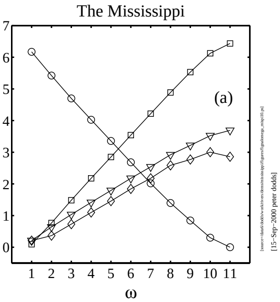
A bonus law:

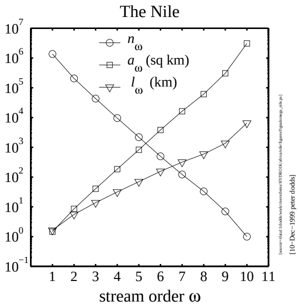
- Horton's law of stream segment lengths:

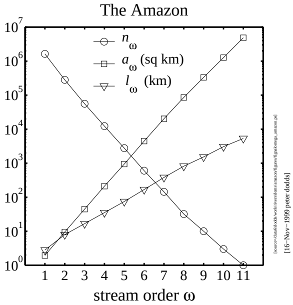
$$\boxed{\bar{s}_{\omega+1}/\bar{s}_{\omega} = R_s > 1}$$

- Can show that $R_s = R_\ell$.

Horton's laws in the real world:







Horton's laws-at-large

Blood networks:

- Horton's laws hold for sections of cardiovascular networks
- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton's law.

Horton's laws

Observations:

- Horton's ratios vary:

R_n	3.0–5.0
R_a	3.0–6.0
R_ℓ	1.5–3.0

- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...
- ... but they don't explain how networks are structured.

Tokunaga's law

Delving deeper into network architecture:

- Tokunaga (1968) identified a clearer picture of network structure [21, 22, 23]
- As per Horton-Strahler, use stream ordering.
- Focus: describe how streams of different orders connect to each other.
- Tokunaga's law is also a law of averages.

Network Architecture

Definition:

- $T_{\mu,\nu}$ = the average number of side streams of order ν that enter as tributaries to streams of order μ
- μ, ν = 1, 2, 3, ...
- $\mu \geq \nu + 1$
- Recall each stream segment of order μ is 'generated' by two streams of order $\mu - 1$
- These generating streams are not considered side streams.

Network Architecture

Tokunaga's law

- Property 1: Scale independence—depends only on difference between orders:

$$T_{\mu,\nu} = T_{\mu-\nu}$$

- Property 2: Number of side streams grows exponentially with difference in orders:

$$T_{\mu,\nu} = T_1 (R_T)^{\mu-\nu-1}$$

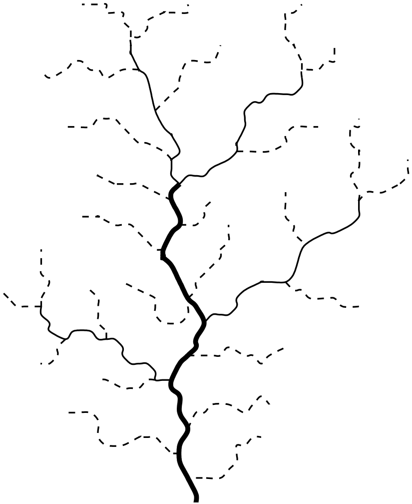
- We usually write Tokunaga's law as:

$$\boxed{T_k = T_1 (R_T)^{k-1}} \quad \text{where } R_T \simeq 2$$

Tokunaga's law—an example:

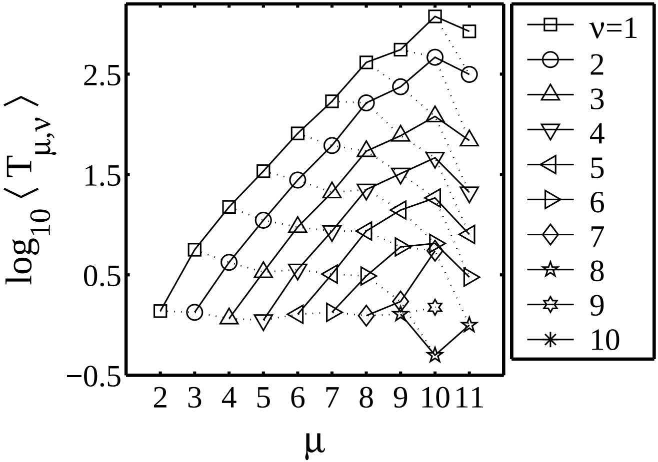
$T_1 \simeq 2$

$R_T \simeq 4$



The Mississippi

A Tokunaga graph:



Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's law has three parameters and Tokunaga has two parameters.
- R_n, R_ℓ, and R_s versus T_1 and R_T.
- To make a connection, clearest approach is to start with Tokunaga's law...
- Known result: Tokunaga $\rightarrow$ Horton [21, 22, 23, 10, 2]

Let us make them happy

We need one more ingredient:

Space-fillingness

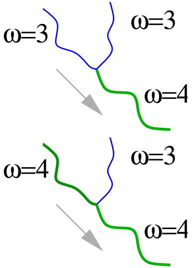
- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}} = \frac{\sum_{\omega=1}^{\Omega} n_\omega s_\omega}{a_\Omega}$$

More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:



1. Running into another stream of order ω and generating a stream of order $\omega + 1$...
   - $2n_{\omega+1}$ streams of order ω do this
2. Running into and being absorbed by a stream of higher order $\omega' > \omega$...
   - $n'_\omega T_{\omega'-\omega}$ streams of order ω do this

More with the happy-making thing

Putting things together:

- $$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$

- Substitute in $T_{\omega'-\omega} = T_1 (R_T)^{\omega'-\omega-1}$:

$$n_\omega = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1 (R_T)^{\omega'-\omega-1} n_{\omega'}$$

- Shift index to $k = \omega' - \omega$:

$$n_\omega = 2n_{\omega+1} + \sum_{k=1}^{\Omega-\omega} T_1 (R_T)^{k-1} n_{\omega+k}$$

More with the happy-making thing

Create Horton ratios:

- Divide through by $n_{\omega+1}$:

$$\frac{n_\omega}{n_{\omega+1}} = \frac{2\cancel{n_{\omega+1}}}{\cancel{n_{\omega+1}}} + \sum_{k=1}^{\Omega-\omega} T_1 (R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}}$$

- Left hand side looks good but we have $n_{\omega+k}/n_{\omega+1}$'s hanging around on the right.
- Recall, we want to show $R_n = n_\omega/n_{\omega+1}$ is a constant, independent of ω...

More with the happy-making thing

Finding Horton ratios:

- Letting $\Omega \rightarrow \infty$, we have

$$\frac{n_\omega}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} \frac{n_{\omega+k}}{n_{\omega+1}} \qquad (1)$$

- The ratio $n_{\omega+k}/n_{\omega+1}$ can only be a function of k due to self-similarity (which is implicit in Tokunaga's law).
- The ratio $n_\omega/n_{\omega+1}$ is independent of ω and depends only on T_1 and R_T.
- Can now call $n_\omega/n_{\omega+1} = R_n$.
- Immediately have $n_{\omega+k}/n_{\omega+1} = R_n^{-(k-1)}$.
- Plug into Eq. (1)...

More with the happy-making thing

Finding Horton ratios:

- Now have:

$$R_n = 2 + \sum_{k=1}^{\infty} T_1 (R_T)^{k-1} R_n^{-(k-1)}$$

$$= 2 + T_1 \sum_{k=1}^{\infty} (R_T/R_n)^{k-1}$$

$$= 2 + T_1 \frac{1}{1 - R_T/R_n}$$

- Rearrange to find:

$$(R_n - 2)(1 - R_T/R_n) = T_1$$

More with the happy-making thing

Finding R_n in terms of T_1 and R_T:

- We are here: $(R_n - 2)(1 - R_T/R_n) = T_1$
- $\times R_n$ to find quadratic in R_n:

$$(R_n - 2)(R_n - R_T) = T_1 R_n$$

- $$R_n^2 - (2 + R_T + T_1)R_n + 2R_T = 0$$
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}$$

Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.
- Assume side streams are roughly separated by distance $1/\rho_{\mathrm{dd}}$.
- For an order ω stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{\mathrm{dd}}^{-1}\left(1 + \sum_{k=1}^{\omega-1} T_k\right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{\mathrm{dd}}^{-1}\left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1}\right) \propto R_T^\omega$$

Horton and Tokunaga are happy

Altogether then:

- $$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$
- Recall $R_\ell = R_s$ so

$$\boxed{R_\ell = R_T}$$

- And from before:

$$\boxed{R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}}$$

Horton and Tokunaga are happy

Some observations:

- R_n and R_ℓ depend on T_1 and R_T.
- Seems that R_a must as well...
- Suggests Horton's laws must contain some redundancy
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between statistical distributions. [3, 4]

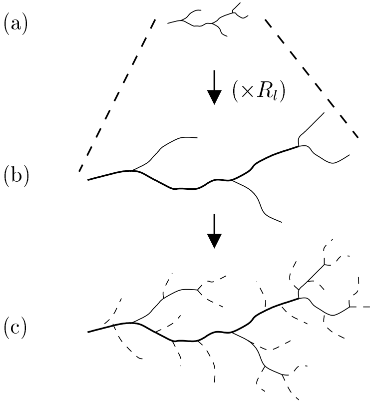
Horton and Tokunaga are happy

The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.
- $$R_T = R_\ell,$$
- $$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$
- Suggests we should be able to argue that Horton's laws imply Tokunaga's laws (if drainage density is uniform)...

Horton and Tokunaga are friends

From Horton to Tokunaga [2]



- Assume Horton's laws hold for number and length
- Start with an order ω stream
- Scale up by a factor of R_ℓ, orders increment
- Maintain drainage density by adding new order 1 streams

Horton and Tokunaga are friends

...and in detail:

- Must retain same drainage density.
- Add an extra $(R_\ell - 1)$ first order streams for each original tributary.
- Since number of first order streams is now given by T_{k+1} we have:

$$T_{k+1} = (R_\ell - 1)\left(\sum_{i=1}^{k} T_i + 1\right).$$

- For large ω, Tokunaga's law is the solution—let's check...

Horton and Tokunaga are friends

Just checking:

- Substitute Tokunaga's law $T_i = T_1 R_T^{i-1} = T_1 R_\ell^{\,i-1}$ into

$$T_{k+1} = (R_\ell - 1)\left(\sum_{i=1}^{k} T_i + 1\right)$$

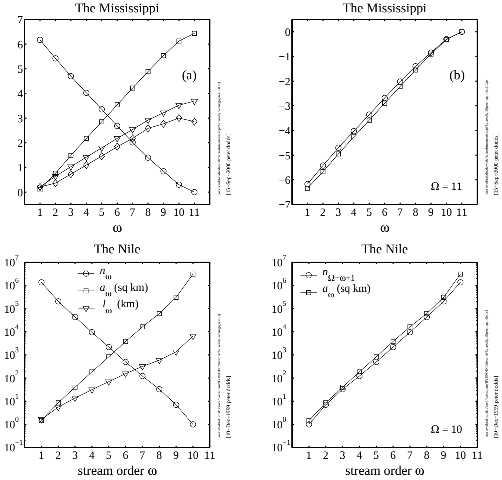
-

$$T_{k+1} = (R_\ell - 1)\left(\sum_{i=1}^{k} T_1 R_\ell^{\,i-1} + 1\right)$$

$$= (R_\ell - 1)\, T_1 \left(\frac{R_\ell^{\,k} - 1}{R_\ell - 1} + 1\right)$$

$$\simeq (R_\ell - 1)\, T_1 \frac{R_\ell^{\,k}}{R_\ell - 1} = T_1 R_\ell^k \quad \text{... yep.}$$

Horton's laws of area and number:



- In right plots, stream number graph has been flipped vertically.
- Highly suggestive that $R_n \equiv R_a$...

Measuring Horton ratios is tricky:

- How robust are our estimates of ratios?
- Rule of thumb: discard data for two smallest and two largest orders.

Mississippi:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	5.27	5.26	2.48	2.30	1.00
[2, 5]	4.86	4.96	2.42	2.31	1.02
[2, 7]	4.77	4.88	2.40	2.31	1.02
[3, 4]	4.72	4.91	2.41	2.34	1.04
[3, 6]	4.70	4.83	2.40	2.35	1.03
[3, 8]	4.60	4.79	2.38	2.34	1.04
[4, 6]	4.69	4.81	2.40	2.36	1.02
[4, 8]	4.57	4.77	2.38	2.34	1.05
[5, 7]	4.68	4.83	2.36	2.29	1.03
[6, 7]	4.63	4.76	2.30	2.16	1.03
[7, 8]	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024

Amazon:

ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream lengths in a order Ω basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\mathrm{dd}}$$

$$\propto \sum_{\omega=1}^{\Omega} \underbrace{R_n^{\Omega-\omega} \cdot \overbrace{1}^{n_\Omega}}_{n_\omega} \underbrace{\bar{s}_1 \cdot R_s^{\omega-1}}_{\bar{s}_\omega}$$

$$= \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n}\right)^\omega$$

Reducing Horton's laws:

Continued ...

- $$a_\Omega \propto \frac{R_n^\Omega}{R_s} \bar{s}_1 \sum_{\omega=1}^{\Omega} \left(\frac{R_s}{R_n} \right)^\omega$$
$$= \frac{R_n^\Omega}{R_s} \bar{s}_1 \frac{R_s}{R_n} \frac{1-(R_s/R_n)^\Omega}{1-(R_s/R_n)}$$
$$\sim R_n^{\Omega-1} \bar{s}_1 \frac{1}{1-(R_s/R_n)} \text{ as } \Omega \nearrow$$
- So, a_Ω is growing like R_n^Ω and therefore:

$$\boxed{R_n \equiv R_a}$$

Reducing Horton's laws:

Not quite:

- ... But this only a rough argument as Horton's laws do not imply a strict hierarchy
- Need to account for sidebranching.
- Problem set 1 question....

Equipartitioning:

Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

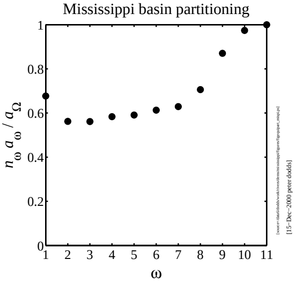
- Reason:

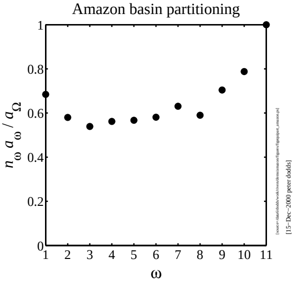
$$n_\omega \propto (R_n)^{-\omega}$$

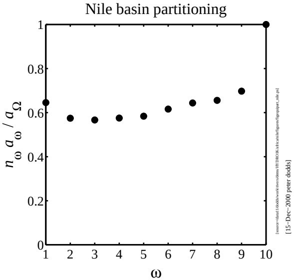
$$\bar{a}_\omega \propto (R_a)^{\omega} \propto n_\omega^{-1}$$

Equipartitioning:

Some examples:







Scaling laws

The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
- We have connected Tokunaga's and Horton's laws
- Only two Horton laws are independent ($R_n = R_a$)
- Only two parameters are independent: $(T_1, R_T) \Leftrightarrow (R_n, R_s)$

Scaling laws

A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a? $P(a) \propto a^{-\tau}$ for large a
- Q: What is probability that the longest stream from p has length ℓ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
- Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$

Scaling laws

Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [24]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
- Our task is always to illuminate the mechanism...

Scaling laws

Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story [20, 1, 2]
- Let's work on $P(\ell)$...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
- (We know they deviate from strict laws for low ω and high ω but not too much.)
- Next: place stick between teeth. Bite stick. Proceed.

Scaling laws

Finding γ:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_>(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\mathrm{d}\ell$$

-

$$P_>(\ell_*) = 1 - P(\ell < \ell_*)$$

- Also known as the exceedance probability.

Scaling laws

Finding γ:

- The connection between $P(x)$ and $P_>(x)$ when $P(x)$ has a power law tail is simple:
- Given $P(\ell) \sim \ell^{-\gamma}$ large ℓ then for large enough ℓ_*

$$P_>(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\,\mathrm{d}\ell$$

$$\sim \int_{\ell=\ell_*}^{\ell_{\max}} \ell^{-\gamma}\mathrm{d}\ell$$

$$= \left.\frac{\ell^{-\gamma+1}}{-\gamma+1}\right|_{\ell=\ell_*}^{\ell_{\max}}$$

$$\propto \ell_*^{-\gamma+1} \quad \text{for } \ell_{\max} \gg \ell_*$$

Scaling laws

Finding γ:

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- Approximate $P_>(\ell_*)$ as

$$P_>(\ell_*) = \frac{N_>(\ell_*; \Delta)}{N_>(0; \Delta)}.$$

  where $N_>(\ell_*; \Delta)$ is the number of sites with main stream length $> \ell_*$.
- Use Horton's law of stream segments: $s_\omega / s_{\omega-1} = R_s$...

Scaling laws

Finding γ:

- Set $\ell_* = \ell_\omega$ for some $1 \ll \omega \ll \Omega$.
- $$P_>(\ell_\omega) = \frac{N_>(\ell_\omega; \Delta)}{N_>(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}$$
- Δ's cancel
- Denominator is $a_\Omega \rho_{\mathrm{dd}}$, a constant.
- So... using Horton's laws...

$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$

Scaling laws

Finding γ:

- We are here:

$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$

- Cleaning up irrelevant constants:

$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega'}$$

- Change summation order by substituting $\omega'' = \Omega - \omega'$.
- Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)

Scaling laws

Finding γ:

- $$P_{>}(\ell_\omega) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$
- Since $R_n < R_s$ and $1 \ll \omega \ll \Omega$,

$$P_{>}(\ell_\omega) \propto \left(\frac{R_n}{R_s}\right)^{\Omega-\omega} \propto \left(\frac{R_n}{R_s}\right)^{-\omega}$$

again using $\sum_{i=0}^{n} a^n = (a^{i+1} - 1)/(a - 1)$

Scaling laws

Finding γ:

- Nearly there:

$$P_{>}(\ell_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}$$

- Need to express right hand side in terms of ℓ_ω.
- Recall that $\ell_\omega \simeq \bar{\ell}_1 R_\ell^{\omega-1}$.
- $$\ell_\omega \propto R_\ell^\omega = R_s^\omega = e^{\omega \ln R_s}$$

Scaling laws

Finding γ:

- Therefore:

$$P_{>}(\ell_\omega) \propto e^{-\omega \ln(R_n/R_s)} = \left(e^{\omega \ln R_s}\right)^{-\ln(R_n/R_s)/\ln(R_s)}$$

- $$\propto \ell_\omega^{\,-\ln(R_n/R_s)/\ln R_s}$$

- $$= \ell_\omega^{-(\ln R_n - \ln R_s)/\ln R_s}$$

- $$= \ell_\omega^{-\ln R_n/\ln R_s + 1}$$

- $$= \ell_\omega^{-\gamma+1}$$

Scaling laws

Finding γ:

- And so we have:

$$\gamma = \ln R_n / \ln R_s$$

- Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law

Scaling laws

Hack's law: [6]

- $$\ell \propto a^h$$
- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect h to Horton ratios:

$$\ell_\omega \propto R_s^\omega \text{ and } a_\omega \propto R_n^\omega$$

- Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}\right)^{\ln R_s / \ln R_n}$$

$$\propto (R_n^\omega)^{\ln R_s / \ln R_n} = a_\omega^{\ln R_s / \ln R_n} \Rightarrow \boxed{h = \ln R_s / \ln R_n}$$

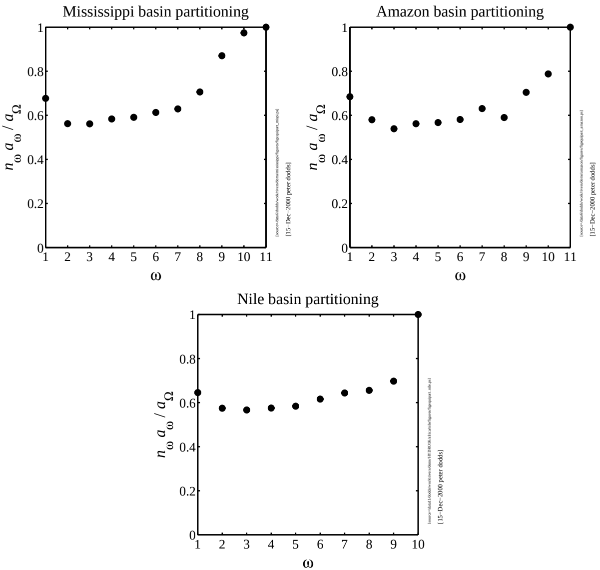
Connecting exponents

Only 3 parameters are independent:
e.g., take d, R_n, and R_s

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1(R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega/n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1}/\bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1}/\bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log R_s / \log R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$

Equipartitioning reexamined:

Recall this story:



Equipartitioning

- What about
$$P(a) \sim a^{-\tau} \qquad ?$$
- Since $\tau > 1$, suggests no equipartitioning:
$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$
- $P(a)$ overcounts basins within basins...
- while stream ordering separates basins...

Fluctuations

Moving beyond the mean:

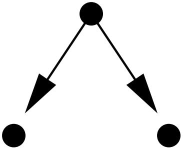
- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

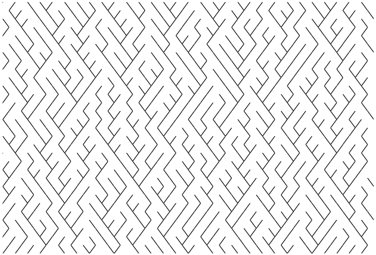
$$\bar{s}_{\omega}/\bar{s}_{\omega-1} = R_s$$

- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure
- See into the heart of randomness...

A toy model—Scheidegger's model

Directed random networks [12, 13]





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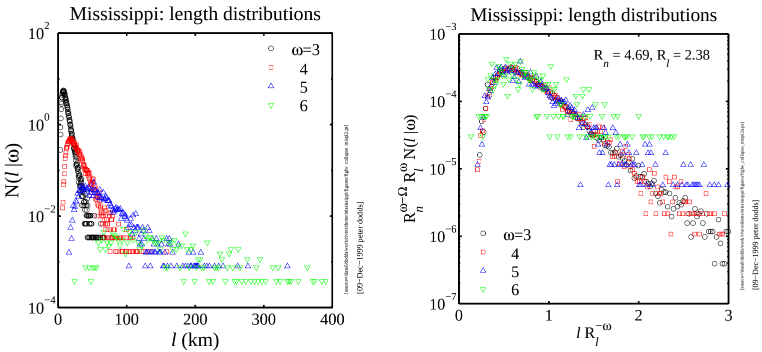
$$P(\searrow) = P(\swarrow) = 1/2$$

- Flow is directed downwards
- Useful and interesting test case—more later...

Generalizing Horton's laws

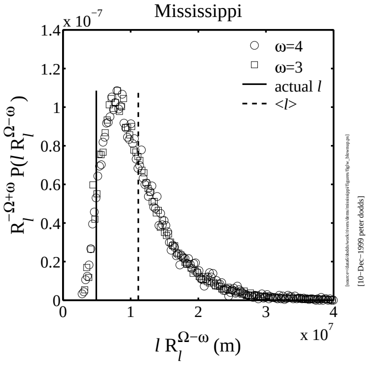
- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$
- $\bar{a}_\omega \propto (R_a)^\omega \Rightarrow N(a|\omega) = (R_n^2)^{-\omega} F_a(a/R_n^\omega)$



- Scaling collapse works well for intermediate orders
- All moments grow exponentially with order

Generalizing Horton's laws

- How well does overall basin fit internal pattern?



- Actual length = 4920 km (at 1 km res)
- Predicted Mean length = 11100 km
- Predicted Std dev = 5600 km
- Actual length/Mean length = 44 %
- Okay.

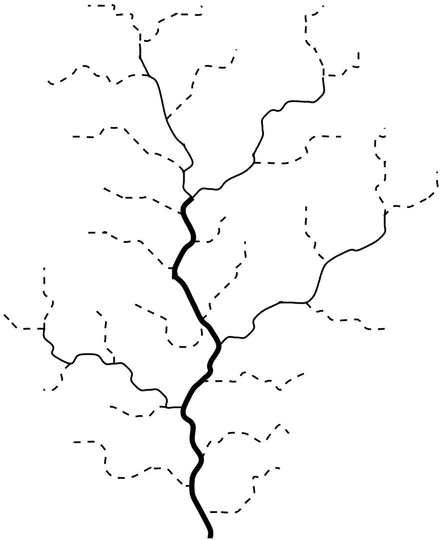
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73

	a	$\bar{a}_\Omega$	σ_a	$a/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86

Combining stream segments distributions:



- Stream segments sum to give main stream lengths
-

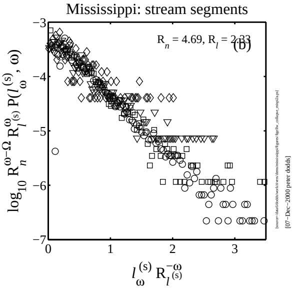
$$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$

- $P(\ell_\omega)$ is a convolution of distributions for the s_ω

Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



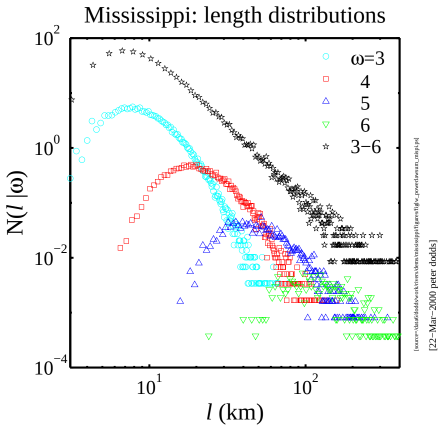
$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

Generalizing Horton's laws

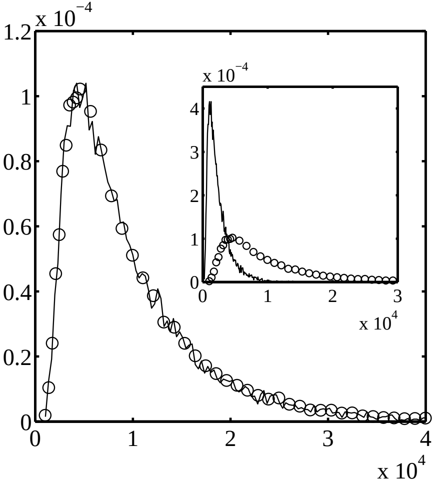
- Next level up: Main stream length distributions must combine to give overall distribution for stream length



- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting...

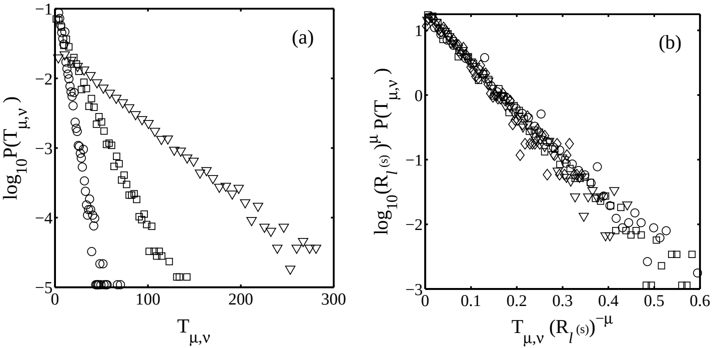
Generalizing Horton's laws

Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$.



Generalizing Tokunaga's law

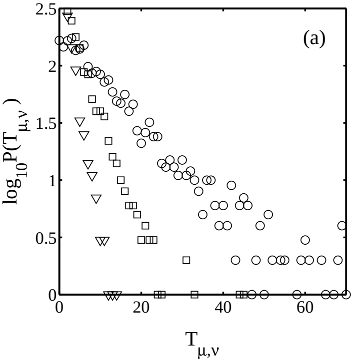
Scheidegger:

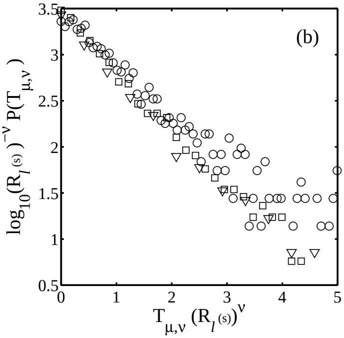


- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

Generalizing Tokunaga's law

Mississippi:





- Same data collapse for Mississippi...

Generalizing Tokunaga's law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu}/(R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

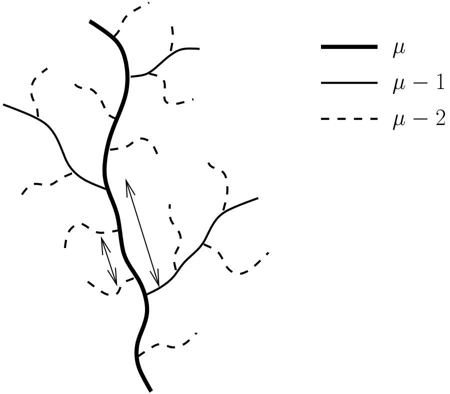
$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_{\mu} \simeq 1/(R_s)^{\mu-1}\xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
- $\Rightarrow$ random spatial distribution of stream segments

Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

Generalizing Tokunaga's law

- Now deal with thing:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = N x^{-1/2} \left[F(y/x)\right]^x$$

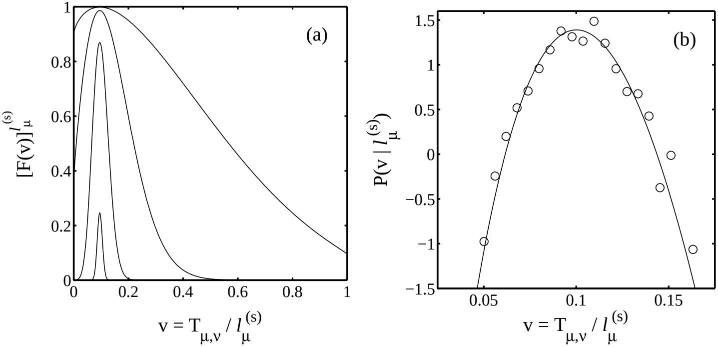
where

$$F(v) = \left(\frac{1 - v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

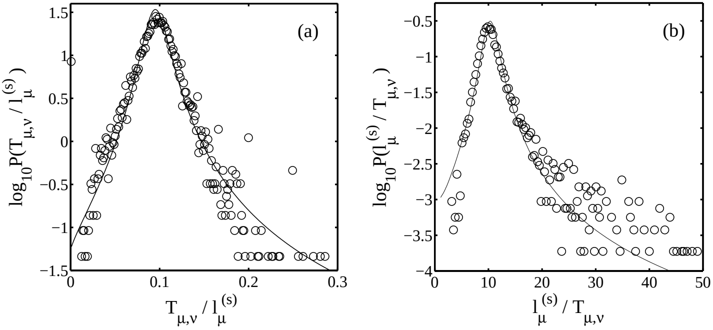
Scheidegger:



Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

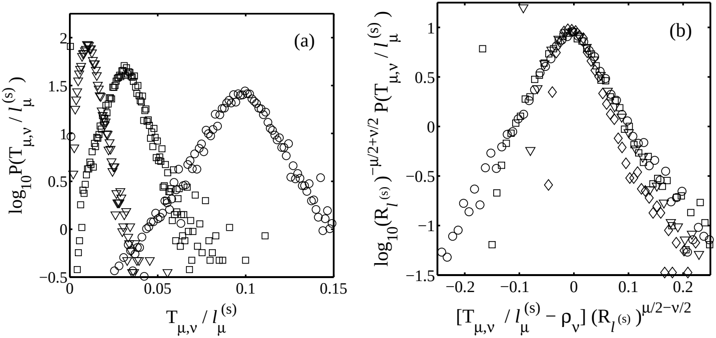
Scheidegger:



Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

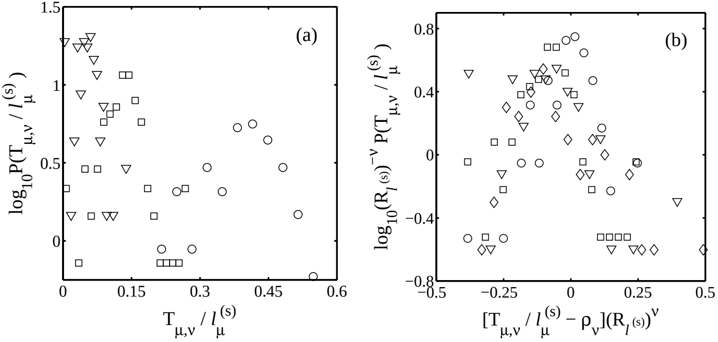
Scheidegger:



Generalizing Tokunaga's law

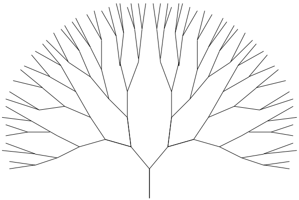
- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Mississippi:



Models

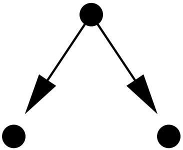
Random subnetworks on a Bethe lattice [15]

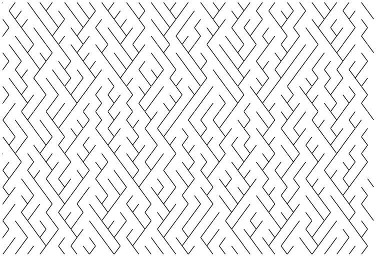


- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of "Statistical inevitability" of river network statistics [8]
- But Bethe lattices unconnected with surfaces.
- In fact, Bethe lattices $\simeq$ infinite dimensional spaces (oops).
- So let's move on...

Scheidegger's model

Directed random networks [12, 13]





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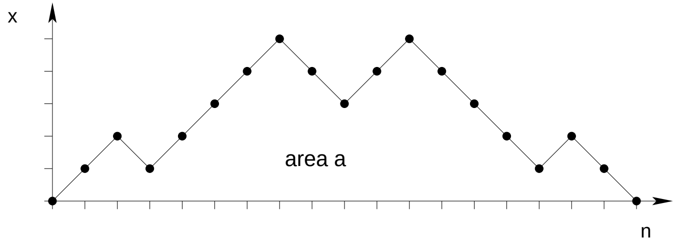
$$P(\searrow) = P(\swarrow) = 1/2$$

- Functional form of all scaling laws exhibited but exponents differ from real world [18, 19, 17]

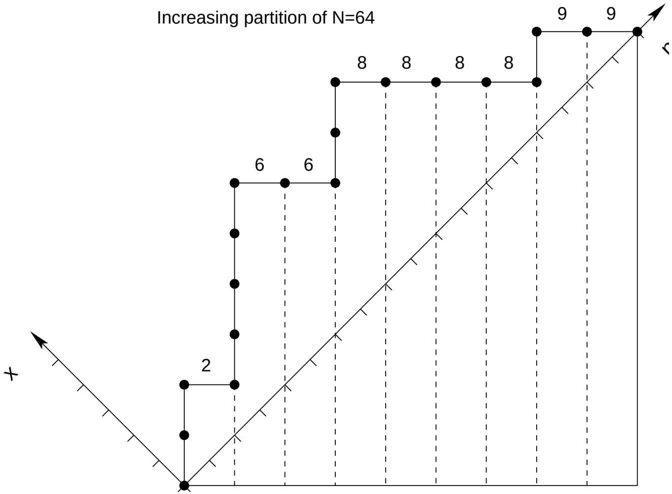
A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks



Scheidegger's model



Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions:

- $$P(n) \sim \frac{1}{2\sqrt{\pi}}\, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:

  $$\ell \propto a^{2/3}.$$

- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.

Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [11]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\ (\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [9]

Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

$d \Rightarrow \ell \propto L_{\parallel}^{d}$ (stream self-affinity).

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