On Periodic Points

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An important part of the qualitative theory of dynamical systems, and of diffeomorphisms, is the detailed study of periodic orbits. Much work is devoted to demonstrating in specific cases such as the three-body problem that infinitely many periodic orbits exist [12].

Consider an arbitrary map $f\colon X \to X$, where X is a topological space. Denote by $N_\nu(f)$ the number of *isolated* periodic points of f, of period ν (i.e., the number of isolated fixed points of f^ν).

We are interested in studying the rate of growth of $N_\nu(f)$ (as ν varies through the positive integers). Let us say that $N_\nu(f)$ *grows at most exponentially* if there is a constant $c = c(f) < +\infty$, such that

$$N_\nu(f) \leqq c^\nu \qquad \text{for all } \nu \geqq 1 .$$

Let M be a compact differential manifold (without boundary). Let $F^k(M)$ be the space of C^k mappings of M into itself, given the C^k topology.

Using algebraic approximation techniques of Nash [8], we prove:

THEOREM. *There is a dense subset $\mathcal{E}_k \subset F^k(M)$ such that if $f \in \mathcal{E}_k$, then $N_\nu(f)$ grows at most exponentially.*

A map $f\colon M \to M$ is numerically stable if there is a neighborhood $f \in U \subset \mathcal{E}_k$ such that

$$N_\nu(f) \leqq N_\nu(g) \qquad \text{for all } g \in U,\ \nu \geqq 1 .$$

The structurally stable diffeomorphisms of Smale [11] are clearly numerically stable.

An immediate corollary of the theorem is that, if $f\colon M \to M$ is numerically stable, then $N_\nu(f)$ grows at most exponentially. (It is a happy fact that structurally stable maps, so intractible from a direct geometric point of view, are by their very definition amenable to approximation procedures.) This should be compared with the following corollary of the work of Smale [12], which gives a very sizeable class of diffeomorphisms (D_0 in Smale's terminology) an exponential lower estimate as well. In particular, Smale's class D_0 includes the structurally stable diffeomorphisms possessing a homoclinic point, so we may state.

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PROPOSITION (Smale). *Let φ: $M \to M$ be a structurally stable diffeomorphism (C^1) possessing a homoclinic point of order N. Then, there is an $l \geqq 1$ such that:*

$$N_\nu(\varphi^l) \geqq N^\nu \qquad \textit{for all } \nu \geqq 0 .$$

PROOF. Let us first state that a structurally stable diffeomorphism is in the subspace $D_0 \subset F(M)$ as required in the hypothesis of Smale's theorem [12, Theorem B]. This follows from [12, 9.1], and (2.6). Consequently, by the conclusion of Smale's theorem, T^l (for some $l > 0$) is equivalent to a shift automorphism of order N on some subset of M. Let us count periodic points of a shift automorphism on the set of doubly infinite N-imals:

$$(\cdots, \cdots, \alpha_{-\gamma}, \cdots, \alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots) .$$

Clearly such an N-imal is periodic of period ν under the shift automorphism if and only if it is, naively, periodic of period ν. There are precisely N^ν such N-imals. The proposition follows.

Combining lower and upper estimates we may say that, if f: $M \to M$ is a structurally stable diffeomorphism possessing a homoclinic point of order $N \geqq 1$, then there is a constant $c = c(f) < +\infty$, and an integer $l \geqq 1$ such that

$$1 < N^\nu \leqq N_\nu(f^l) \leqq c^\nu \qquad \text{for all } \nu \geqq 1 .$$

Structurally stable diffeomorphisms possessing homoclinic points have been constructed on S^p ($p > 1$) (Smale, not yet published) and else where. A most interesting example is the following (see [1], [11]). Let L be a linear transformation on the plane R^2 in SL(2; **Z**) (i.e., which induces an automorphism of the integral lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$) with an eigenvalue of absolute value greater than 1. Then L induces a diffeomorphisms φ_L of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. This diffeomorphism φ_L is structurally stable, possesses homoclinic points of order $N > 1$, and has a dense set of periodic points.

Let us look more closely at these diffeomorphisms. Our condition on the eigenvalues of L implies that they are both real. Assume for simplicity, that they are positive.

We may identify $\mathbf{R}^2$ in a natural way with the tangent space to any point $x \in \mathbf{T}^2$. If x is a periodic point of period ν of φ_L, then under the above identification, the differential of $(\varphi_L)^\nu$ is given simply by the linear transformation, L^ν. From this we deduce

(i) Every periodic point of φ_L is elementary (since the eigenvalues of L^ν are all absolute value different from 1).

(ii) The (Lefschetz) indices of the fixed points of $(\varphi_L)^\nu$ are all equal. (The index of an elementary fixed point x is computable from the differential of the map at x, and the differentials of $(\varphi_L)^\nu$ at the fixed points are all equal.) After (i), (ii) the Lefschetz fixed point formula applies to establish

$$(*) \qquad \pm N_\nu(\varphi_L) = 2 - \operatorname{Tr}(L^\nu) .$$

But since the eigenvalues of L are positive, $\operatorname{Tr}(L^\nu)$ is a positive integer greater than 1, so we may sharpen (*) to yield:

$$(**) \qquad N_\nu(\varphi_L) = \operatorname{Tr}(L^\nu) - 2 .$$

If we form the function

$$\zeta(s) = \exp\left(\sum_{\nu=1}^{\infty} \frac{N_\nu(\varphi_L)\cdot s^\nu}{\nu}\right)$$

where s is a complex variable, we obtain

$$\zeta(s) = (1-s)^2/\det(1 - s\cdot L)$$

which coincides with the zeta function introduced by Lang [6].

Define the zeta function of any map f as

$$\zeta_f(s) = \exp\left(\sum_{\nu=1}^{\infty} \frac{N_\nu(f)\cdot s^\nu}{\nu}\right) .$$

Our main theorem then states that ζ_f has a non-zero radius of convergence about $s = 0$ if $f \in \mathcal{E}_k$.

Let us remark that if V is an algebraic variety defined over a finite field F_q, and $X = \bar{V}$ is the set of points of V over the algebraic closure; $f\colon X \to X$ the map induced from the q^{th} power map on coordinates (with respect to some projective imbedding of V defined over F_q), the above zeta function of f is precisely the classical zeta function of V.

There are a few directions in which one might hope to improve our result:

1. Let $f\colon M \to M$ be an arbitrary differentiable map whose periodic points are all transversal. (M compact without boundary.) Does the exponential upper bound hold for $N_\nu(f)$?

2. Let $f \in \mathcal{E}_k$. Is $\zeta_f(s)$ an algebraic function of s?

3. Let V be a differentiable vector field on a compact manifold M. Denote by $N_\nu(V)$ the number of *isolated* periodic orbits of V, period *less than or equal to* ν. Does $N_\nu(V)$ grow at most exponentially for some reasonable, dense class of vector fields V? (In particular, for Nash vector fields? Notice that if this theorem is proven for any dense set of vector fields, it is also true for structurally stable vector fields. Compare Corollary 2.3 below.)

Unfortunately, these conjectures would probably not be easily established if true, while simple counter-examples could demolish them.

The key to the proof of our result is the fundamental theory of real algebraic approximations introduced by Nash [8]. In § 1, we make explicit the category of objects and morphisms introduced [8]. The Nash category is extremely useful in that it occupies a hybrid position: it is "dense" in differential topology (1.8), and it is amenable to algebraic geometric techniques.

Our main theorem is proved by showing (2.1) that, if φ is any Nash morphism, then $N_\nu(\varphi)$ grows at most exponentially (2.1).

The proof goes, in outline, as follows:

(a) One bounds $N_\nu(\varphi)$ in terms of the degree of φ^ν (which is defined to be the biprojective degree of the algebraic correspondence generated by φ^ν with respect to some fixed projective representation), (§ 3). Part of this chore is:

(b) To obtain an upper bound for the number of *isolated* real points of a real projective variety of degree d and dimension r. (Sufficient for our purposes is the upper estimate: d^{3^r}), (§ 4).

(c) Finally we must bound $\deg(\varphi^\nu)$. This is done by obtaining a weak multiplicative inequality for the degree of a composite Nash morphism in terms of the degree of its factors, (§ 3).

0. Definition of intersection

In this section we work over a fixed field k which we assume, for simplicity, to be perfect. For intersection theory we refer to [10] or [13] except that we prefer to define intersections with reference to the field k rather than to a universal domain. It makes no difference.

Consider, in a non-singular ambient variety X of dimension n, *cycles*, i.e., finite sums of the form $Z = \sum r_i V_i$ where V_i are irreducible closed subsets of X and r_i are integers. A *divisor* is a cycle Z in which each V_i has codimension 1. For a cycle Z which is non-negative (all $r_i \geqq 0$) we denote by $|Z|$ its *support*. Thus

$$|Z| = \bigcup_{r_i>0} V_i \,. \tag{0.1}$$

Let Z be a cycle and D a non-negative divisor. We shall define an intersection cycle $Z \cap D$ by the conditions

(i) $Z \cap D$ is linear in Z.

(ii) If $Z = 1\cdot V$, then

$$Z \cap D = \begin{cases} (Z\cdot D) & \text{if } V \not\subset |D|\text{, i.e., } V \text{ and } D \text{ intersect properly (intersection as in [10, p. 92]).} \\ Z & \text{if } V \subset |D|\,. \end{cases} \tag{0.2}$$

More generally, if $D = (D_1, \cdots, D_s)$ is a finite sequence of non-negative divisors, we obtain an intersection $Z \cap D$, again linear in Z, by

$$(0.2') \qquad Z \cap D = (\cdots((Z \cap D_1) \cap D_2) \cdots) \qquad (D = (D_1, \cdots, D_s)) .$$

We emphasize that this notion is completely asymmetric and depends on the ordering of D.

For a non-negative Z, $Z \cap D$ is non-negative and we have

$$(0.3) \qquad |Z \cap D| = |Z| \cap |D_1| \cap \cdots \cap |D_s| \qquad (Z \geqq 0) .$$

The crucial property of this intersection is:

PROPOSITION 0.4. *Let φ: $X' \to X$ be an étale morphism, X non-singular. Let Z be any cycle, and D a sequence of non-negative divisors on X. Then*

$$\varphi^*(Z \cap D) = \varphi^*(Z) \cap \varphi^*(D) .$$

Here $\varphi^*(Z)$ is the obvious transform of cycles and $\varphi^*(D) = (\varphi^*(D_1), \cdots, \varphi^*(D_s))$.

PROOF. By induction we may assume $s = 1$, i.e., D is a divisor. By linearity, we may assume $Z = 1 \cdot V$, V irreducible. Let $V' = \varphi^*(V)$, $D' = \varphi^*(D)$, and let $V'_1, \cdots, V'_m$ be the irreducible components of V', so that $V' = V'_1 + \cdots + V'_m$. Clearly $V'_i \subset |D'|$, all i, if $V \subset |D|$. Hence the proposition is true if $V \subset |D|$. On the other hand, each V'_i has dimension equal to that of V and the general point of V'_i maps to that of V. Hence $V'_i \not\subset |D'|$ if $V \not\subset |D|$. Thus in this case all intersections are proper, and we are immediately reduced to the following assertion of intersection theory:

LEMMA 0.5. *Let φ: $X' \to X$ be étale, X non-singular, and let V and W be subvarieties of X intersecting properly at a component C. Let V', W' be the inverse images of V and W respectively and let C' be a component of the inverse image of C. Let $\{V'_i\}$ $(i = 1, \cdots, r)$ and $\{W'_j\}$ $(j = 1, \cdots, s)$ be the components of V' and W' containing C'. Then*

$$i(C;\, V \cdot W) = \textstyle\sum_{i,j} i(C';\, V'_i \cdot W'_j) .$$

PROOF. $i(C;\, V \cdot W)$ is the multiplicity $\mu(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal induced by the diagonal of $X \times X$ in the local ring A of C^D in $V \times W \subset X \times X$ (notation as in [10, p. 77]. Let A' be the local ring of C'^D in $V' \times W'$, and $\mathfrak{a}'$ the ideal induced by the diagonal of $X' \times X'$. Since $X' \times X'$ is étale over $X \times X$, $\mathfrak{a}' = A'\mathfrak{a}$. Since A' is étale over A, $\hat{A}'$ is finite over $\hat{A}$, and so we have ([9, p. 31, Proposition 2]),

$$\mu(\mathfrak{a}) = \mu(\hat{A}\mathfrak{a}) = \mu(\hat{A}'\mathfrak{a}') = \mu(\mathfrak{a}') .$$

Now the components of $V' \times W'$ containing C'^D are $V'_i \times W'_j$. Call A'_{ij} the ring of C'^D in $V'_i \times W'_j$, and call $\mathfrak{a}'_{ij}$ the ideal induced by the diagonal. We have $i(C';\, V'_i \cdot W'_j) = \mu(\mathfrak{a}'_{ij})$. Hence it remains to show $\mu(\mathfrak{a}') = \sum_{i,j} \mu(\mathfrak{a}'_{ij})$. But this follows from [7, Theorem 23.5].

Suppose now that $X = \mathbf{P}^m \times \cdots \times \mathbf{P}^m$ (r factors) is a product of projective spaces. Then the rational equivalence ring $A(X)$ is $\mathbf{Z}[x_1, \cdots, x_r]/J$ where J is the ideal generated by the set $\{x_1^{m+1}, \cdots, x_r^{m+1}\}$, and where x_i is represented by the divisor $\mathrm{pr}_i^*(H)$, H a hyperplane of $\mathbf{P}^m$ and pr_i the i^{th} projection. If Z is any cycle, denote by $a(Z)$ its class in $A(X)$. For a non-negative Z all the coefficients of $a(Z)$ are non-negative integers, as is easily seen. We define the *degree* of Z as

(0.6) $\deg Z =$ sum of the coefficients of $a(Z)$.

This yields the following variant of the Bezout theorem:

COROLLARY 0.7. *Let Z be a non-negative cycle and $D = (D_1, \cdots, D_s)$ a sequence of positive divisors in $X = \mathbf{P}^m \times \cdots \times \mathbf{P}^m$. Set* $\deg D = \prod \deg D_i$. *Then*

$$\deg (Z \cap D) \leqq (\deg Z)(\deg D) .$$

1. Definition of Nash manifold

In what follows, we consider schemes and morphisms of finite type over spec $\mathbf{R}$. Given such a scheme X, denote by $\bar{X}$ its space of real points. $\bar{X}$ may be regarded as a real analytic space [13] in a natural way. Thus we obtain a functor from schemes of finite type over spec $\mathbf{R}$ to real analytic spaces. This functor commutes with fibered products and respects closed (resp. open) immersions.

THEOREM 1.1. *Let φ: $Y \to X$ be an étale morphism (of schemes over* $\mathbf{R}$) *and let $P \in \bar{Y}$. Then a neighborhood of P in $\bar{Y}$ is isomorphic to its image in $\bar{X}$.*

PROOF. Let Q be the image of P. Imbed X in affine space in a neighborhood of Q. By [4, Theorem 7.6], Y may be generated locally by one element y over X. Lifting the (monic) equation of y to the ambient affine space, we get an étale extension of this affine space which induces φ. Taking into account the compatibility of $\bar{X}$ with fibered products, this reduces us to the case $X =$ affine space.

Let $(x_1, \cdots, x_n)$ be coordinates and apply [4, Theorem 7.6] again to write Y locally as zeros of a polynomial $f(x_1, \cdots, x_n; y)$, monic in y. Since Y is étale over X at P, $\partial f/\partial y \neq 0$ at P. But f is a real polynomial, so we can apply the implicit function theorem to the zeros $\bar{Y}$ of f in $\mathbf{R}^{n+1}$. We find a unique solution real analytic near P for y in terms of (x), and the theorem follows.

COROLLARY 1.2. *If P be a real simple point of X of dimension n, then $\bar{X}$ will be analytically isomorphic to an open set of $\mathbf{R}^n$ in a neighborhood of P.*

For, by the jacobian criterion for simple points, X is étale over affine n-space at P.

We turn now to the notion of Nash manifold, introduced by Nash in [8].* On the (real) space $\mathbf{R}^n$, consider the sheaf $\mathcal{A}$ of local rings of Nash functions obtained in either of the following equivalent ways. Taking $U \subset \mathbf{R}^n$ open and connected, we will define $\mathcal{A}(U)$:

(1.3) (i) Form the inductive set I, whose members consist of

(a) a connected real variety V with an étale map to affine n-space $\mathbf{A}^n$ and

(b) a continuous lifting $v: U \to \bar{V}$, and whose ordering relation is $(V, v) \leqq (V', v')$ if there exists a map (necessarily unique) $V' \to V$ such that the diagrams

$$\begin{tikzcd} V' \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & A^n & \end{tikzcd} \qquad \begin{tikzcd} \bar{V}' \arrow[rr] & & \bar{V} \\ & U \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commute. Set $\mathcal{A}(U) = \lim_I \Gamma(V, O_V)$ (O_V the structure sheaf of V).

(ii) Let $\mathcal{A}(U) =$ ring of real analytic functions f on U which are algebraically dependent on the coordinate functions of $\mathbf{R}^n$, i.e., which satisfy a non-trivial polynomial equation with coefficients in the ring $\mathbf{R}[x_1, \cdots, x_n]$ (x_i the coordinate functions).

To see that these definitions are equivalent and give a sheaf, argue as follows: We associate with an element of $\Gamma(V, O_V)$ $((V, v) \in I$ as in (i)$)$ the function it induces on U to get a map of the first ring to the second. Given $P \in U$ the local ring in sense (i) is the *henselization* $\tilde{o}$ of the local ring o of affine space $\mathbf{A}^n$ at P [7], [2]. By a result of Nagata [7, Theorem 44.1], this ring is the algebraic closure of o in its completion $\hat{o}$. The local ring in sense (ii) is obviously the algebraic closure of o in the ring $\bar{o}$ of analytic functions at P. Since we have $o \subset \tilde{o} \subset \bar{o} \subset \hat{o}$, it follows that the local rings are equal. Thus equality of (i), (ii) will follow if we show that the presheaves of the two definitions are sheaves. In definition (ii) it is clear that no non-zero section of $\mathcal{A}(U)$ can be zero on any smaller non-empty open set. Hence it suffices in fact to show that definition (i) gives a sheaf. Again it is clear that a section of $\mathcal{A}(U)$ can not be zero on a smaller non-empty open set. What has to be shown is that, if $\{U_i\}$ is an open cover of U and if $f_i \in \mathcal{A}(U_i)$ agree on the connected components of the intersections $U_i \cap U_j$, then there is an $f \in \mathcal{A}(U)$ inducing f_i on U_i.

Consider first any $f \in \mathcal{A}(U)$. We will represent f by a function on a canonically determined variety étale over $\mathbf{A}^n$ as follows: Let $(V, v) \in I$, $f_0 \in \Gamma(V, O_V)$ represent f, and let $\Gamma \subset V \times \mathbf{A}^1$ be the graph of f_0. Let $W \subset \mathbf{A}^{n+1}$ be the closure of its image under the map induced by $V \times \mathbf{A}^1 \to \mathbf{A}^n \times \mathbf{A}^1$. Now if $\bar{f}$

* Actually, Nash introduces the projective version of what follows.

is the function $U \to \mathbf{R}$ induced by f then the graph of $\bar{f}$ has dimension n in the sense of [5]. Since W has dimension n it follows that W is the Zariski closure of $\bar{f}$ in $\mathbf{A}^{n+1}$. For, a real variety of dimension n has a real locus of dimension at most n. (This is seen by induction as follows: It is true if $n = 0$. Suppose it is known for $n - 1$, and let X be an affine variety of dimension n and $P \in \bar{X}$. Let Y be the intersection of X with an ε-sphere about P. Y is an algebraic variety of dimension $n - 1$, and so we may apply the definition of dimension [5] and the induction hypothesis.) Hence W is determined by f. Let W' be the normalization of W. The map $V \to W$ lifts to a map $V \to W'$ and W' is étale over $\mathbf{A}^n$ (under the projection $\mathbf{A}^n \times \mathbf{A}^1 \to \mathbf{A}^n$) at each point of the image of V. Replace W' by the subvariety of those points at which W' is étale over $\mathbf{A}^n$, and let w: $U \to W'$ be the map induced by v and $V \to W'$. For reasons of dimension, w is the only lifting of the graph of $\bar{f}$ to $\bar{W}'$. The function obtained as $W' \to \mathbf{A}^n \times \mathbf{A}^1 \to \mathbf{A}^1$ clearly represents f, and so f is represented canonically on $(W', w) \in I$.

Returning to the problem of showing (i) gives a sheaf, let $f_i \in \mathcal{A}(U_i)$ be given agreeing on the overlaps. Each f_i is represented canonically on some variety W_i with a lifting w_i: $U_i \to \bar{W}_i$. Using connectedness of U and the fact that $f_i = f_j$ on $U_i \cap U_j$, one sees that the W_i are all canonically isomorphic, say to W, and that the w_i glue together to give a lifting w: $U \to \bar{W}$. Moreover, the functions f_i are all represented by the same element $f_0 \in \Gamma(W, O_W)$ and f_0 represents the required $f \in \mathcal{A}(U)$.

DEFINITION 1.4.* *A Nash manifold of dimension n is a local ringed Hausdorff space which is locally isomorphic to an open subset of* $\mathbf{R}^n$ *with the sheaf* $\mathcal{A}$. *By morphism we mean as local ringed space.*

Let V be a real variety of dimension n, and $U \subset \bar{V}$ an open set consisting of simple points of V. Then V has locally an étale map to affine space $\mathbf{A}^n$ at any point of U, and one obtains from these maps and from definition (i) of $\mathcal{A}$ a structure, clearly uniquely determined, of Nash manifold for U. Given a Nash manifold N, an isomorphism of N with such a U will be called a *realization of N on V*. The set of all real simple points of V is the maximal such set U and the structure of Nash manifold on this set will be denoted by $\tilde{V}$. $\tilde{V}$ is not a functor of V.

By a *submanifold* of a Nash manifold N, we mean a subspace [3] of N which is a Nash manifold.

THEOREM 1.5. *Let V be a real variety. A submanifold N of $\tilde{V}$ is uniquely determined by its support. Given a compact subset S of the space of real*

* One could define the more general structure of Nash space.

simple points of V, S is the support of a Nash submanifold of $\tilde{V}$ of dimension n if and only if

(i) *S is the support of a compact real analytic submanifold of $\bar{V}$, and*

(ii) *the Zariski closure of S in V has dimension n.**

PROOF. Since V can be covered by finitely many affines, we may assume V is in affine space $\mathbf{A}^m$. Suppose N is a Nash submanifold with support $S \subset \mathbf{R}^m$. N can be covered by a finite number of open sets $\{U_i\}$ each isomorphic to an open subset of $\tilde{\mathbf{A}}^n$ ($=\mathbf{R}^n$ with the sheaf $\mathfrak{A}$). For each i there is a variety W_i étale over $\mathbf{A}^n$, a lifting of U_i to W_i, and functions $f_1, \cdots, f_m$ on W_i which induce the coordinate functions of the imbedding of N in $\mathbf{A}^m$ on U_i. Since the f_j are also analytic functions on U_i by (1.1), this shows (i) holds for S. Moreover, viewing U_i as contained in $\mathbf{A}^m$, it is a subset of the closure of the image of W_i under the map given by the functions $\{f_j\}$. Thus the Zariski closure, say W, of U_i has dimension at most n. Since we already know S is the support of a real analytic manifold, it is clear that in fact $S \subset W$ and so the Zariski closure of S has dimension at most n. On the other hand, S has dimension n in the sense of [5] and can therefore not be contained in the real locus of a variety of dimension $<n$ as was seen above.

To show that N is determined by S is a local problem and so we may take $N = U_i$, i.e., N realized on a variety W with a map to V. Let $P \in N$. Since by assumption N is a submanifold of $\tilde{V}$, the local ring of $\tilde{V}$ at P maps onto that of N at P. These local rings are the henselizations of the local rings of V and W at P respectively. Since the henselization of a local ring may be obtained as a limit of rings étale and of finite type over the given one [2], it follows that there is a variety W' étale over W with a real point P' lying over P, a variety V' étale over V, and an *immersion* $W' \hookrightarrow V'$ commuting with $W \to V$. Since V' is étale ever V, $\tilde{V}'$ is locally isomorphic to $\tilde{V}$, and so we may put primes on everything; i.e., we may suppose N is *realized* on a subvariety W of V. W is then the Zariski closure of S. If N_1 were another structure on S, we could also assume N_1 realized on a subvariety of V. This subvariety would also be the Zariski closure of S, i.e., W. Thus N and N_1 would be realized on the same variety W and hence their local rings would be the same, so $N = N_1$.

Now suppose S is given satisfying (i), (ii), and let W be the Zariski closure of S. Let α denote the given sheaf of analytic functions on S, and ρ the sheaf on S obtained by taking, for U connected and open in S, $\rho(U)$ to be the ring of rational functions on W regular on U. Because of the dimensions involved,

* Most of this is in Nash [8].

no such function can be zero on U, and so $\rho(U)$ may be regarded as a subring of $\alpha(U)$ in the natural way. Let $\nu(U)$ be the algebraic closure of $\rho(U)$ in $\alpha(U)$. This yields a sheaf ν on S and we claim ν gives S the structure of Nash manifold.

We may suppose V affine. Let $P \in S$, and choose n coordinate fnnctions $x_1, \cdots, x_n$ which are uniformizing parameters for the analytic structure at P. These functions determine an analytic isomorphism of some neighborhood U of P with an open set U' of $\mathbf{R}^m$. Since W has dimension n, any function in $\rho(U)$ is algebraic over the coordinates $x_1, \cdots, x_n$ and so it follows from (1.3) (ii) that the sheaf ν on U is isomorphic with the sheaf $\mathcal{A}$ on U', as required.

Let V be a real variety and $P \in \bar{V}$ a point. By a *branch* of V at P, we mean a minimal prime ideal of the henselization $\tilde{A}$ of the local ring A of V at P. (Thus the locus $x^2 - y^2 = 0$ in the plane has two branches at the origin, while $x^2 + y^2 = 0$ has one branch.) Writing $\tilde{A}$ as limit of étale extensions of A, one sees that one can choose a real variety V' étale over V and a point $P' \in \bar{V}'$ lying over P such that the branches of V at P are in one-one correspondence with minimal primes of the local ring A' of V' at P'. This fact, together with Theorem 1.1, allows us to define in an obvious way the notion of germ of the space of real points, say $\bar{B}$, of a branch B of V at P. If $\bar{V}_0$ denotes the germ of $\bar{V}$ at P, then B is closed in V_P.

COROLLARY 1.6. *Let V be a real variety and N a submanifold of $\tilde{V}$. Let W be the Zariski closure of N in V. At each point $P \in N$, the germ of the support of N is the germ of the space of real points of some simple branch of W.*

COROLLARY 1.7. *With the above notation, let W' denote the normalization of W. The inclusion $N \to \tilde{V}$ factors uniquely through a map $N \to \tilde{W}'$, and in this way N is realized on W'.*

We will call a Nash manifold *projective* if it has a closed imbedding as submanifold of $\tilde{\mathbf{P}}^m$ ($\mathbf{P}^m$ = projective space/$\mathbf{R}$). To assume N projective is the same as to assume N compact and imbeddable in $\tilde{\mathbf{A}}^m$. For, if $N \subset \tilde{\mathbf{A}}^m$ is compact, it is closed in the projective space $\tilde{\mathbf{P}}^m$ obtained by adding a hyperplane at infinity. Conversely if N is in $\tilde{\mathbf{P}}^m$, then N is also contained in the real part of the affine subset V of $\mathbf{P}^m$ obtained by removing the locus $x_0^2 + x_1^2 + \cdots + x_m^2 = 0$ ((x) projective coordinates), and V is closed in some affine space.

The principal results of Nash [8] may be stated as:

THEOREM 1.8 (Nash). *Let M be a compact C^k-manifold ($k \geqq 1$). Then M is isomorphic to the underlying C^k structure of a projective Nash mani-*

fold $N(M)$. As Nash manifold, $N(M)$ is determined up to (non-unique) isomorphism. Given a C^k-differentiable (iso) morphism φ: $N \to N'$ between projective Nash manifolds, and given $\varepsilon > 0$, there is a Nash (iso) morphism ψ: $N \to N'$ such that φ, ψ differ by less than ε in the C^k norm.

2. The main theorems

In § 4 we will prove

THEOREM 2.1. *Let N be a projective Nash manifold, and let φ: $N \to N$ be a morphism. Then $N_\nu(\varphi)$ grows at most exponentially.*

Let M be a compact differentiable manifold (without boundary) and let η: $N \to M$ be a projective Nash structure on M. Set $\mathcal{E}_k \subset F^k(M)$ to be the subset of C^k-differentiable maps which are Nash morphisms with respect to the Nash structure η. Then by (1.8), $\mathcal{E}_k \subset F^k(M)$ is dense in $F^k(M)$.

Consequently:

THEOREM 2.2. *There is a dense subset $\mathcal{E}_k \subset F^k(M)$ such that if $f \in \mathcal{E}_k$, then $N_\nu(f)$ grows at most exponentially.*

COROLLARY 2.3. *Let f: $M \to M$ be a numerically stable C^k-map. Then $N_\nu(f)$ grows at most exponentially.*

PROOF. Let $f \in U \subset F^k(M)$ be a neighborhood for which

$$N_\nu(f) \leqq N_\nu(g) \qquad \text{for all } g \in U,\ \nu \geqq 1\ .$$

Since $\mathcal{E}_k$ is dense, there is a map $g \in U \cap \mathcal{E}_k$. The corollary follows. Consequently the exponential upper bound is valid for structurally stable diffeomorphisms as well.

In the next section we give a shorter proof of the following weaker theorem. Recall that a periodic point P of φ is transversal if the graph of φ^ν intersects the diagonal $\Delta \subset N \times N$ transversally at (P, P).

THEOREM 2.4. *Let N be a Nash manifold and let φ: $N \to N$ be a morphism. Suppose that every isolated periodic point of φ is transversal. Then $N_\nu(\varphi)$ grows at most exponentially.*

This theorem is already enough to enable us to obtain the following result, the proof of which occupies the remainder of this section.

THEOREM 2.5. *Let M be a compact differentiable manifold (without boundary), and let φ: $M \to M$ be a structurally stable C^1 differentiable isomorphism. Then $N_\nu(\varphi)$ grows at most exponentially.*

PROOF. By the theorems of Nash (1.8), M may be realized as the underlying differentiable structure of a Nash submanifold N of projective space, and φ may be approximated in a C^1 manner by a morphism $\tilde{\varphi}$: $N \to N$. If we choose the approximation close enough, then $\tilde{\varphi}$ will be topologically conjugate to φ and will induce a structurally stable diffeomorphism on M. Since $N_\nu(\varphi)$ is an invariant of topological conjugacy, we are reduced to the case that $\tilde{\varphi}$ induces φ. Then the theorem follows if we combine (2.4) with

PROPOSITION 2.6. *If φ: $M \to M$ is a structurally stable diffeomorphism, then every periodic point of φ is transversal.*

REMARK. Let P be a periodic point of order ν. To say P is transversal is equivalent to saying that the jacobian of φ^ν at P has no non-zero eigenvector of eigenvalue 1. P is called elementary if the jacobian of φ^ν has no eigenvalue of absolute value 1. One can show that every periodic point is elementary but we do not need it.

By compactness of M, if φ is transversal, then $N_\nu(\varphi)$ is finite for each $\nu > 0$. By [12, 9.1], elementary diffeomorphisms are dense. Hence if φ is structurally stable, then φ is topologically conjugate to an elementary diffeomorphism.

Consequently we have

LEMMA 2.7. *If φ is structurally stable, then $N_\nu = N_\nu(\varphi)$ is finite for all $\nu \geqq 0$.*

PROOF. N_ν is an invariant of topological conjugacy class.

Now let f: $M \to M$ be an arbitrary diffeomorphism; $p \in M$ a periodic point of f of order ν. Let L denote the jacobian of f^ν at p, and identify some coordinate neighborhood $U \subset M$ about p with an open subset of the tangent space at p.

LEMMA 2.8. *Let $\varepsilon > 0$ be given. Then there is an open set $U_0 \subset U$ about p, and a diffeomorphism g: $M \to M$ such that*

(i) $\delta(f, g) \leqq \varepsilon$

(ii) $g^\nu = L$ *on* U_0.

(Here δ is some C^1-metric on $F^1(M)$).

PROOF. Let us take ν to be the minimal period of p (no loss of generality), and assume U chosen small enough that

$$U, f U, \cdots, f^{\nu-1} U \subset M$$

are disjoint. Write

$$f^\nu(x) = L(x) + r(x) \qquad \text{for } x \in U$$

where

$$\lim_{||x|| \to 0} r(x)/||x|| = 0 .$$

Set $\sigma_0(x)$ to be a C^∞-function such that

(a) $\sigma_0(x) = 1$ if $\| x \| > 2$
(b) $\sigma_0(x) = 0$ if $\| x \| < 1$.
Let $\sigma_\lambda(x) = \sigma_0(\lambda \cdot x)$, and set

$$f'(x) = L(x) + \sigma_\lambda(x) \cdot r(x) \qquad x \in U .$$

Given any $\varepsilon_0 > 0$, an elementary calculation shows that there is a λ_0 such that if $\lambda \geqq \lambda_0$, then

$$\delta_U(f', f^\nu) < \varepsilon_0 .$$

Define $g\colon M \to M$ to be the following differentiable map:

$$g(x) = f(x) \qquad \text{if } x \notin f^{\nu-1}U$$
$$g(x) = f' \circ f^{1-\nu}(x) \qquad \text{if } x \in f^{\nu-1}U .$$

Notice that if λ_0 is chosen so that

$$\{x \mid \| x \| \leqq 2/\lambda_0\} \subset U ,$$

then g is differentiable. Choose $\varepsilon_0 > 0$ sufficiently small, and we get

$$\delta(g, f) < \varepsilon ;$$

and we may also assure ourselves that g is a diffeomorphism by making our original choice of ε small enough, since f is a diffeomorphism.

Finally, if

$$U_0 = \{x \mid \| x \| < 1/\lambda\} \subset U ,$$

then $g^\nu(x) = gf^{\nu-1}(x) = f'(x) = L(x)$ for $x \in U_0$, since $\sigma_\lambda(x) = 0$.

PROOF OF PROPOSITION 2.6. Assume φ is structurally stable and non-transversal. Then there is a periodic point $P \in M$ of order ν with $U \subset M$ and L as above, such that

$$L(x_0) = x_0 \qquad \text{for some } x_0 \neq 0,\ x_0 \in U .$$

Since φ is structurally stable there is an $\varepsilon > 0$ such that, if $\delta(\varphi, \psi) < \varepsilon$, then ψ is topologically conjugate to φ. Use this $\varepsilon > 0$ in Lemma 2.8, to obtain a ψ such that $\psi^\nu(x) = L(x)$ for $x \in U_0$. Thus there is a t_0 such that if $t \leqq t_0$, $t \cdot x_0 \in U_0$, and

$$\psi^\nu(t \cdot x_0) = L(tx_0) = t \cdot x_0 .$$

Consequently ψ has an infinite number of periodic points of order ν. Since ψ is topologically conjugate to φ, this contradicts Lemma 2.7. Proposition 2.6 is therefore proved.

3. Proof of Theorem 2.4

In this section we will use the definitions and notation of § 0. Let $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ be morphisms of n-dimensional Nash manifolds where X, Y, Z are submanifolds of $\widetilde{\mathbf{P}}^m$ with Zariski closures A, B, C respectively. We assume A, B, C are normal varieties so that by Corollary 1.7 the Nash manifolds are realized on the varieties. Let us denote by Γ_φ the graph of φ, and by $\Lambda_\varphi \subset \mathbf{P}^m \times \mathbf{P}^m$ its Zariski closure.

Let $D = (D_1, \cdots, D_s)$ be a sequence of positive divisors of $\mathbf{P}^m \times \mathbf{P}^m \times \mathbf{P}^m$ such that $|D_1| \cap \cdots \cap |D_s| = \Lambda_\varphi \times C$.

LEMMA 3.1. *The Zariski closure $\Lambda_{\psi\varphi}$ of the graph of $\psi\varphi$ occurs with positive coefficient in the cycle* $\mathrm{pr}_{13*}((A \times \Lambda_\psi) \cap D)$.

Here pr_{13*} is the projection of $\mathbf{P}^m \times \mathbf{P}^m \times \mathbf{P}^m$ on the product of the first and third factors, and the star denotes the usual direct image operation on cycles.

PROOF. Let Λ be the Zariski closure in $(\mathbf{P}^m)^3$ of $(X \times \Gamma_\psi) \cap (\Gamma_\varphi \times Z)$. Clearly Λ has dimension n (1.5) and projects to $\Lambda_{\psi\varphi}$, also of dimension n. Hence by definition of pr_{13*}, it suffices to show Λ occurs with positive coefficient in $(A \times \Lambda_\psi) \cap D$.

Let $P \in X$, $Q = \varphi(P)$, $R = \psi(Q)$. In a neighborhood of Q we can find an étale map $B' \to B$ and a lifting of Y to B' such that ψ is induced by a morphism $\psi'\colon B' \to C$ (this follows from (1.3) (i)). The map $B' \to B$ can be induced locally as base extension on an étale map $e\colon W \to \mathbf{P}^m$. Let ε denote the map $1 \times e \times 1\colon \mathbf{P}^m \times W \times \mathbf{P}^m \to (\mathbf{P}^m)^3$. Because the dimensions are the same, $A \times \Lambda_{\psi'}$ (here $\Lambda_{\psi'}$ is the *graph* of ψ') is an irreducible component of $\varepsilon^*(A \times \Lambda_\psi)$. Therefore, by linearity of the intersection and Proposition 0.4, it suffices to show that some component of $\varepsilon^*(\Lambda)$ occurs with positive coefficients in $(A \times \Lambda_{\psi'}) \cap \varepsilon^*(D)$. One such component is the Zariski closure Λ' of $(X \times \Gamma_\psi) \cap (\Gamma_\varphi \times Z)$ in $\mathbf{P}^m \times W \times \mathbf{P}^m$. Hence it suffices to show (0.3) that Λ' is a component of $|A \times \Lambda_{\psi'}| \cap |\varepsilon^*(\Lambda_\varphi \times C)|$. Certainly $\Lambda' \subset |A \times \Lambda_{\psi'}| \cap |\varepsilon^*(\Lambda_\varphi \times C)|$.

Now $\varepsilon^*(\Lambda_\varphi \times C) = ((1 \times e)^*(\Lambda_\varphi)) \times C$. Each component U of $(1 \times e)^*(\Lambda_\varphi)$ has dimension n and projects generically surjectively onto A. Hence above a generic point P' of A lie finitely many points of U. Since $\Lambda_{\psi'}$ is the graph of a map, it follows that finitely many points of $|A \times \Lambda_{\psi'}| \cap |U \times C|$ lie above P'. Thus each component of $|A \times \Lambda_{\psi'}| \cap |\varepsilon^*(\Lambda_\varphi \times C)|$ mapping generically surjectively onto A has dimension n. Since Λ' also maps generically surjectively onto A and has dimension n, we are done.

Let φ be as above. We set

$$\deg \varphi = \deg \Lambda_\varphi \,. \tag{3.2}$$

LEMMA. 3.3. *With the above notation,*

$$\deg \psi\varphi \leqq \deg \psi \deg D \deg A \,.$$

PROOF. By Lemma 3.1, $\deg \psi\varphi \leqq \deg \mathrm{pr}_{13*} (A \times \Lambda_\psi) \cap D)$. Write the rational equivalence ring of $(\mathbf{P}^m)^3$ as $\mathbf{Z}[x_1, x_2, x_3]/J$ (notation as in § 0). For a cycle Z with

$$a(Z) = \sum_{i,j,k} a_{ijk} x_1^i x_2^j x_3^k ,$$

we have

$$a(\mathrm{pr}_{13*}(Z)) = \sum_{i,k} a_{imk} x_1^i x_3^k .$$

Hence since $(A \times \Lambda_\psi) \cap D$ is a positive cycle, pr_{13*} drops its degree and so

$$\begin{aligned} \deg \psi\varphi &\leqq \deg \big((A \times \Lambda_\psi) \cap D\big) \\ &\leqq \deg (A \times \Lambda_\psi) \deg D && \text{by (0.7)} \\ &= \deg \big((A \times \mathbf{P}^m \times \mathbf{P}^m)\cdot(\mathbf{P}^m \times \Lambda_\psi)\big) \deg D \\ &= \deg (A \times \mathbf{P}^m \times \mathbf{P}^m) \deg (\mathbf{P}^m \times \Lambda_\psi) \deg D \\ &= \deg A \deg \psi \deg D . \end{aligned}$$

COROLLARY 3.4. *Let N be a projective Nash manifold, and $\varphi: N \to N$ a morphism. There is a number k such that*

$$\deg \varphi^\nu \leqq k^\nu , \qquad \textit{all } \nu > 0 .$$

Let $V \subset \mathbf{P}^m$ be the Zariski closure of N in a projective imbedding of N. We may take V normal. If $D = (D_1, \cdots, D_s)$ is a sequence of positive divisors on $(\mathbf{P}^m)^3$ which cuts out $\Lambda_\varphi \times V$ (meaning $|D_1| \cap \cdots \cap |D_s| = |\Lambda_\varphi \times V|$) then by the above lemma

$$\deg \varphi^{\nu+1} \leqq \deg \varphi^\nu \deg V \deg D .$$

Hence we may take $k = \deg(\mathrm{id}_V) \deg V \deg D$.

Suppose still that the normal variety V is the Zariski closure of N, and let $\Delta \subset V \times V \subset \mathbf{P}^m \times \mathbf{P}^m$ be the diagonal. Let $\delta = (\delta_1, \cdots, \delta_r)$ be a sequence of positive divisors of $\mathbf{P}^m \times \mathbf{P}^m$ which cuts out Δ. Theorem 2.4 follows immediately from the corollary and from the proposition below:

PROPOSITION 3.5. *Let $\varphi: N \to N$ be a morphism all of whose fixed points are transversal. Then*

$$N_1(\varphi) \leqq \deg \varphi \deg \delta .$$

PROOF. We have $N_1(\varphi) = \#(\Gamma_\varphi \cap \Delta)$, when $\#$ denotes cardinality of the point set. Hence by the Bezout inequality (0.7) it suffices to show

(3.6) If $P \in \Gamma_\varphi \cap \Delta$, the variety P appears with positive coefficient in the cycle $\Lambda_\varphi \cap \delta$.

Let P be such a point. In a neighborhood of P, we may choose V' étale over V and a lifting of N to V' such that φ is induced by a morphism $\varphi': V' \to V$, and we may induce $V' \to V$ by an étale map $e: W \to \mathbf{P}^m$. Let P' be the lifting of P. $\Lambda_{\varphi'}$ is a component of $(e \times 1)^*(\Lambda_\varphi)$ passing through P', and so by linearity of the intersection (0.4) and (0.3), it suffices to show that the local component of $|\Lambda_{\varphi'}| \cap |(e \times 1)^*(\Delta)|$ at P' is P' itself. To show this it suffices, in view of the dimensions appearing, to show that $\Lambda_{\varphi'}$ and $(e \times 1)^*(\Delta)$ intersect properly at P', viewing them as subvarieties of $V' \times V$. This will be if their tangent spaces are transversal there. But since they are non-singular real varieties and P' is a real point, it suffices to calculate the real analytic tangent spaces. Analytically, the local situation at P' is isomorphic to that at $P \in \Gamma_\varphi \cap \Delta \subset \bar{V} \times \bar{V}$, and here the transversality is just the assumption of the proposition.

4. Proof of Theorem 2.1

Let V again denote the Zariski closure in $\mathbf{P}^m$ of the Nash manifold N, with V assumed normal. Let $\varphi: N \to N$ be a morphism, and $\delta = (\delta_1, \cdots, \delta_r)$ a sequence of positive divisors cutting out the diagonal Δ of $V \times V$ in $\mathbf{P}^m \times \mathbf{P}^m$.

We want to recover information about isolated fixed points P of φ from the cycle $\Lambda_\varphi \cap \delta$. To do this we adopt the following terminology. A real point P of a real variety U will be said to be an *isolated real point* of a branch B of U at P if the germ $\bar{B}$ is the point P (as germ, cf. § 1).

LEMMA 4.1. *Let P be an isolated fixed point of φ. There is a variety U occurring with positive coefficients in $\Lambda_\varphi \cap \delta$ and a branch B of U at P on which P is an isolated real point.*

PROOF. we may as before choose in a neighborhood of P an étale morphism $V' \to V$ induced by base extension from an etale map e: $W \to \mathbf{P}^m$, and a lifting of N to V' such that φ is induced by a morphism φ': $V' \to V$. Let U' be a component of $|\Lambda_{\varphi'}| \cap |(e \times 1)^*(\Delta)|$ passing through the lifting P' of P. It is easily seen that P' is an isolated real point of the variety U', hence of any branch of U' at P'. Set $U = (e \times 1)(U')$ Then $\dim U = \dim U'$, hence U' is a component of $(e \times 1)^*(U)$, and since $e \times 1$ is étale, any branch of U' is a branch of U. Taking into account the results of § 0, we are through.

For a real variety U define

(4.2) $I(U) =$ the number of isolated real points of the branches of U, and extend the symbol to cycles by linearity. To prove Theorem 2.1, it will suffice to show

THEOREM 4.3. *Let $Z \subset \mathbf{P}^m$ be a positive real cycle all of whose components are of dimension at most n. Set $d = \deg Z$. Then*

$$I(Z) \leqq d^{3^n} .^*$$

For, suppose (4.3) known and set $Z_\nu = \Lambda_{\varphi^\nu} \cap \delta$. Then by Lemma 4.1,

* This estimate can be improved slightly.

$$N_\nu(\varphi) \leqq I(Z_\nu)\ .$$

By Corollary 3.4 and Bezout, there is a k such that

$$\deg Z_\nu \leqq k^\nu \deg \delta\ , \qquad \text{for all } \nu > 0\ .$$

Hence

$$\deg Z_\nu \leqq l^\nu \qquad \text{for some } l\ .$$

Since all components of Z_ν have dimension at most $n = \dim V$, we have by Theorem 4.3,

$$N_\nu(\varphi) \leqq I(Z_\nu) \leqq (l^\nu)^{3^n} = (l^{3^n})^\nu = K^\nu\ .$$

PROOF OF THEOREM 4.3. It is true if $n = 0$. So suppose it true if $\dim Z < n$. We may assume $Z = 1\cdot V$, V a variety of dimension n and degree d. Consider a projection $\pi\colon \mathbf{P}^m \to \mathbf{P}^{n+1}$ followed by a projection $\pi'\colon \mathbf{P}^{n+1} \to \mathbf{P}^n$. Let $V' = \pi(V)$. We require that π and π' are defined over the reals, that the center of π (resp. π') not meet V (resp. V'), and that the map $V \to V'$ be birational. Then the induced maps $V \to V'$, $V' \to \mathbf{P}^n$ are finite. Since V' is a hypersurface, the discriminant for the map $V' \to \mathbf{P}^n$ is defined. Let Δ be its divisor on $\mathbf{P}^n$, and let W be the cycle $V\cdot(\pi'\pi)^*(\Delta)$ (the rational map $\pi'\pi$ is defined at each point on V, so we may treat it as a morphism for the purposes of this intersection, which is proper). Since the image V' of V in $\mathbf{P}^{n+1}$ has degree d, Δ has degree $d(d-1) \leqq d^2$. Since V has degree d,

$$\deg W = \deg \Delta \deg V \leqq d^3\ .$$

W is of pure dimension $n-1$. So by induction, it suffices to show $I(W) \geqq I(U)$. For $(d^3)^{3^{n-1}} = d^{3^n}$.

To show this, we show that for each $P \in \bar{V}$, $I_P(W) \geqq I_P(V)$, where $I_P(V)$ denotes the number of isolated real points of branches of V at P. Let Q be the image of P in $\mathbf{P}^n$, and replace the symbol $\mathbf{P}^n$ by U. If $U_1 \to U$ is an étale map and Q_1 is a point of $\bar{U}_1$ lying over Q then the base change from U to U_1 preserves all interesting local features of our situation. Let us denote by the subscript 1 the base extension. Then Δ_1 is the discriminant for the map $V' \to U_1$, and $W_1 = V_1\cdot(\pi_1'\pi_1))^*(\Delta_1)$ by Lemma 0.5. If $P_1 = Q_1 \times_Q P$, then we also have $I_{P_1}(W_1) = I_P(W)$ and $I_{P_1}(V_1) = I_P(V)$. Hence we may replace U by U_1. If U_1 is chosen so that the branches of V_1 at P_1 are in one-one correspondence with the local components of V_1 at P_1, this means we may assume that the branches of V at P correspond to local components. The cycle W, being an intersection, is additive over components of V, and so is $I_P(V)$. Thus if V_0 is a component of V at P and W $W_0 = V_0\cdot(\pi'\pi)^*(\Delta)$, it suffices to show $I_P(W_0) \geqq I_P(V_0)$. Here $I_P(V_0)$ is one or zero according as P is or is not an isolated point of $\bar{V}_0$. If zero there is nothing to prove. So suppose P isolated on $\bar{V}_0$. Then, by Corollary 0.2 and the assumption that $n > 0$, P must be singular for V, hence $\pi(P)$ is singular for V' and so certainly $Q \in |\Delta|$. Therefore $P \in |W_0|$. But P is an isolated point of $\bar{V}_0$ and $|\bar{W}_0| \subset \bar{V}_0$. So P is certainly isolated on every branch of $|\bar{W}_0|$ at P. Hence $I_P(W_0) > 0$ as required.

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