Birds and Frogs

Freeman Dyson

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme of my talk tonight is this. Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

This talk is called the Einstein lecture, and I am grateful to the American Mathematical Society for inviting me to do honor to Albert Einstein. Einstein was not a mathematician, but a physicist who had mixed feelings about mathematics. On the one hand, he had enormous respect for the power of mathematics to describe the workings of nature, and he had an instinct for mathematical beauty which led him onto the right track to find nature's laws. On the other hand, he had no interest in pure mathematics, and he had no technical skill as a mathematician. In his later years he hired younger colleagues with the title of assistants to do mathematical calculations for him. His way of thinking was physical rather than mathematical. He was supreme among physicists as a bird who saw further than others. I will not talk about Einstein since I have nothing new to say.

Francis Bacon and René Descartes

At the beginning of the seventeenth century, two great philosophers, Francis Bacon in England and René Descartes in France, proclaimed the birth of modern science. Descartes was a bird, and Bacon was a frog. Each of them described his vision of the future. Their visions were very different. Bacon said, "All depends on keeping the eye steadily fixed on the facts of nature." Descartes said, "I think, therefore I am." According to Bacon, scientists should travel over the earth collecting facts, until the accumulated facts reveal how Nature works. The scientists will then induce from the facts the laws that Nature obeys. According to Descartes, scientists should stay at home and deduce the laws of Nature by pure thought. In order to deduce the laws correctly, the scientists will need only the rules of logic and knowledge of the existence of God. For four hundred years since Bacon and Descartes led the way, science has raced ahead by following both paths simultaneously. Neither Baconian empiricism nor Cartesian dogmatism has the power to elucidate Nature's secrets by itself, but both together have been amazingly successful. For four hundred years English scientists have tended to be Baconian and French scientists Cartesian. Faraday and Darwin and Rutherford were Baconians; Pascal and Laplace and Poincaré were Cartesians. Science was greatly enriched by the cross-fertilization of the two contrasting cultures. Both cultures were always at work in both countries. Newton was at heart a Cartesian, using

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pure thought as Descartes intended, and using it to demolish the Cartesian dogma of vortices. Marie Curie was at heart a Baconian, boiling tons of crude uranium ore to demolish the dogma of the indestructibility of atoms.

In the history of twentieth century mathematics, there were two decisive events, one belonging to the Baconian tradition and the other to the Cartesian tradition. The first was the International Congress of Mathematicians in Paris in 1900, at which Hilbert gave the keynote address, charting the course of mathematics for the coming century by propounding his famous list of twenty-three outstanding unsolved problems. Hilbert himself was a bird, flying high over the whole territory of mathematics, but he addressed his problems to the frogs who would solve them one at a time. The second decisive event was the formation of the Bourbaki group of mathematical birds in France in the 1930s, dedicated to publishing a series of textbooks that would establish a unifying framework for all of mathematics. The Hilbert problems were enormously successful in guiding mathematical research into fruitful directions. Some of them were solved and some remain unsolved, but almost all of them stimulated the growth of new ideas and new fields of mathematics. The Bourbaki project was equally influential. It changed the style of mathematics for the next fifty years, imposing a logical coherence that did not exist before, and moving the emphasis from concrete examples to abstract generalities. In the Bourbaki scheme of things, mathematics is the abstract structure included in the Bourbaki textbooks. What is not in the textbooks is not mathematics. Concrete examples, since they do not appear in the textbooks, are not mathematics. The Bourbaki program was the extreme expression of the Cartesian style. It narrowed the scope of mathematics by excluding the beautiful flowers that Baconian travelers might collect by the wayside.

Francis Bacon

René Descartes

Jokes of Nature

For me, as a Baconian, the main thing missing in the Bourbaki program is the element of surprise. The Bourbaki program tried to make mathematics logical. When I look at the history of mathematics, I see a succession of illogical jumps, improbable coincidences, jokes of nature. One of the most profound jokes of nature is the square root of minus one that the physicist Erwin Schrödinger put into his wave equation when he invented wave mechanics in 1926. Schrödinger was a bird who started from the idea of unifying mechanics with optics. A hundred years earlier, Hamilton had unified classical mechanics with ray optics, using the same mathematics to describe optical rays and classical particle trajectories. Schrödinger's idea was to extend this unification to wave optics and wave mechanics. Wave optics already existed, but wave mechanics did not. Schrödinger had to invent wave mechanics to complete the unification. Starting from wave optics as a model, he wrote down a differential equation for a mechanical particle, but the equation made no sense. The equation looked like the equation of conduction of heat in a continuous medium. Heat conduction has no visible relevance to particle mechanics. Schrödinger's idea seemed to be going nowhere. But then came the surprise. Schrödinger put the square root of minus one into the equation, and suddenly it made sense. Suddenly it became a wave equation instead of a heat conduction equation. And Schrödinger found to his delight that the equation has solutions corresponding to the quantized orbits in the Bohr model of the atom.

It turns out that the Schrödinger equation describes correctly everything we know about the behavior of atoms. It is the basis of all of chemistry and most of physics. And that square root of minus one means that nature works with complex numbers and not with real numbers. This discovery came as a complete surprise, to Schrödinger as well as to everybody else. According to Schrödinger, his fourteen-year-old girl friend Itha Junger said to him at the time, "Hey, you never even thought when you began that so much sensible stuff would come out of it." All through the nineteenth century, mathematicians from Abel to Riemann and Weierstrass had been creating a magnificent theory of functions of complex variables. They had discovered that the theory of functions became far deeper and more powerful when it was extended from real to complex numbers. But they always thought of complex numbers as an artificial construction, invented by human mathematicians as a useful and elegant abstraction from real life. It never entered their heads that this artificial number system that they had invented was in fact the ground on which atoms move. They never imagined that nature had got there first.

Another joke of nature is the precise linearity of quantum mechanics, the fact that the possible states of any physical object form a linear space.

Before quantum mechanics was invented, classical physics was always nonlinear, and linear models were only approximately valid. After quantum mechanics, nature itself suddenly became linear. This had profound consequences for mathematics. During the nineteenth century Sophus Lie developed his elaborate theory of continuous groups, intended to clarify the behavior of classical dynamical systems. Lie groups were then of little interest either to mathematicians or to physicists. The nonlinear theory of Lie groups was too complicated for the mathematicians and too obscure for the physicists. Lie died a disappointed man. And then, fifty years later, it turned out that nature was precisely linear, and the theory of linear representations of Lie algebras was the natural language of particle physics. Lie groups and Lie algebras were reborn as one of the central themes of twentieth century mathematics.

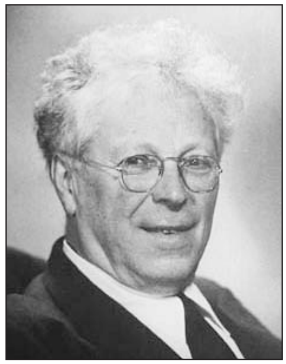
A third joke of nature is the existence of quasi-crystals. In the nineteenth century the study of crystals led to a complete enumeration of possible discrete symmetry groups in Euclidean space. Theorems were proved, establishing the fact that in three-dimensional space discrete symmetry groups could contain only rotations of order three, four, or six. Then in 1984 quasi-crystals were discovered, real solid objects growing out of liquid metal alloys, showing the symmetry of the icosahedral group, which includes five-fold rotations. Meanwhile, the mathematician Roger Penrose discovered the Penrose tilings of the plane. These are arrangements of parallelograms that cover a plane with pentagonal long-range order. The alloy quasi-crystals are three-dimensional analogs of the two-dimensional Penrose tilings. After these discoveries, mathematicians had to enlarge the theory of crystallographic groups to include quasi-crystals. That is a major program of research which is still in progress.

A fourth joke of nature is a similarity in behavior between quasi-crystals and the zeros of the Riemann Zeta function. The zeros of the zeta-function are exciting to mathematicians because they are found to lie on a straight line and nobody understands why. The statement that with trivial exceptions they all lie on a straight line is the famous Riemann Hypothesis. To prove the Riemann Hypothesis has been the dream of young mathematicians for more than a hundred years. I am now making the outrageous suggestion that we might use quasi-crystals to prove the Riemann Hypothesis. Those of you who are mathematicians may consider the suggestion frivolous. Those who are not mathematicians may consider it uninteresting. Nevertheless I am putting it forward for your serious consideration. When the physicist Leo Szilard was young, he became dissatisfied with the ten commandments of Moses and wrote a new set of ten commandments to replace them. Szilard's second commandment says: "Let your acts be directed towards a worthy goal, but do not ask if they can reach it: they are to be models and examples, not means to an end." Szilard practiced what he preached. He was the first physicist to imagine nuclear weapons and the first to campaign actively against their use. His second commandment certainly applies here. The proof of the Riemann Hypothesis is a worthy goal, and it is not for us to ask whether we can reach it. I will give you some hints describing how it might be achieved. Here I will be giving voice to the mathematician that I was fifty years ago before I became a physicist. I will talk first about the Riemann Hypothesis and then about quasi-crystals.

There were until recently two supreme unsolved problems in the world of pure mathematics, the proof of Fermat's Last Theorem and the proof of the Riemann Hypothesis. Twelve years ago, my Princeton colleague Andrew Wiles polished off Fermat's Last Theorem, and only the Riemann Hypothesis remains. Wiles' proof of the Fermat Theorem was not just a technical stunt. It required the discovery and exploration of a new field of mathematical ideas, far wider and more consequential than the Fermat Theorem itself. It is likely that any proof of the Riemann Hypothesis will likewise lead to a deeper understanding of many diverse areas of mathematics and perhaps of physics too. Riemann's zeta-function, and other zeta-functions similar to it, appear ubiquitously in number theory, in the theory of dynamical systems, in geometry, in function theory, and in physics. The zeta-function stands at a junction where paths lead in many directions. A proof of the hypothesis will illuminate all the connections. Like every serious student of pure mathematics, when I was young I had dreams of proving the Riemann Hypothesis. I had some vague ideas that I thought might lead to a proof. In recent years, after the discovery of quasi-crystals, my ideas became a little less vague. I offer them here for the consideration of any young mathematician who has ambitions to win a Fields Medal.

Quasi-crystals can exist in spaces of one, two, or three dimensions. From the point of view of physics, the three-dimensional quasi-crystals are the most interesting, since they inhabit our three-dimensional world and can be studied experimentally. From the point of view of a mathematician, one-dimensional quasi-crystals are much more interesting than two-dimensional or three-dimensional quasi-crystals because they exist in far greater variety. The mathematical definition of a quasi-crystal is as follows. A quasi-crystal is a distribution of discrete point masses whose Fourier transform is a distribution of discrete point frequencies. Or to say it more briefly, a quasi-crystal is a pure point distribution that has a pure point spectrum. This definition includes as a special case the ordinary crystals, which are periodic distributions with periodic spectra.

Excluding the ordinary crystals, quasi-crystals in three dimensions come in very limited variety, all of them associated with the icosahedral group. The two-dimensional quasi-crystals are more numerous, roughly one distinct type associated with each regular polygon in a plane. The two-dimensional quasi-crystal with pentagonal symmetry is the famous Penrose tiling of the plane. Finally, the one-dimensional quasi-crystals have a far richer structure since they are not tied to any rotational symmetries. So far as I know, no complete enumeration of one-dimensional quasi-crystals exists. It is known that a unique quasi-crystal exists corresponding to every Pisot-Vijayaraghavan number or PV number. A PV number is a real algebraic integer, a root of a polynomial equation with integer coefficients, such that all the other roots have absolute value less than one, [1]. The set of all PV numbers is infinite and has a remarkable topological structure. The set of all one-dimensional quasi-crystals has a structure at least as rich as the set of all PV numbers and probably much richer. We do not know for sure, but it is likely that a huge universe of one-dimensional quasi-crystals not associated with PV numbers is waiting to be discovered.



Abram Besicovitch

Hermann Weyl

Photograph of A. Besicovitch from AMS archives. Photo of H. Weyl from the archives of Peter Roquette, used with permisssion.

Here comes the connection of the one-dimensional quasi-crystals with the Riemann hypothesis. If the Riemann hypothesis is true, then the zeros of the zeta-function form a one-dimensional quasi-crystal according to the definition. They constitute a distribution of point masses on a straight line, and their Fourier transform is likewise a distribution of point masses, one at each of the logarithms of ordinary prime numbers and prime-power numbers. My friend Andrew Odlyzko has published a beautiful computer calculation of the Fourier transform of the zeta-function zeros, [6]. The calculation shows precisely the expected structure of the Fourier transform, with a sharp discontinuity at every logarithm of a prime or prime-power number and nowhere else.

My suggestion is the following. Let us pretend that we do not know that the Riemann Hypothesis is true. Let us tackle the problem from the other end. Let us try to obtain a complete enumeration and classification of one-dimensional quasi-crystals. That is to say, we enumerate and classify all point distributions that have a discrete point spectrum. Collecting and classifying new species of objects is a quintessentially Baconian activity. It is an appropriate activity for mathematical frogs. We shall then find the well-known quasi-crystals associated with PV numbers, and also a whole universe of other quasi-crystals, known and unknown. Among the multitude of other quasi-crystals we search for one corresponding to the Riemann zeta-function and one corresponding to each of the other zeta-functions that resemble the Riemann zeta-function. Suppose that we find one of the quasi-crystals in our enumeration with properties that identify it with the zeros of the Riemann zeta-function. Then we have proved the Riemann Hypothesis and we can wait for the telephone call announcing the award of the Fields Medal.

These are of course idle dreams. The problem of classifying one-dimensional quasi-crystals is horrendously difficult, probably at least as difficult as the problems that Andrew Wiles took seven years to explore. But if we take a Baconian point of view, the history of mathematics is a history of horrendously difficult problems being solved by young people too ignorant to know that they were impossible. The classification of quasi-crystals is a worthy goal, and might even turn out to be achievable. Problems of that degree of difficulty will not be solved by old men like me. I leave this problem as an exercise for the young frogs in the audience.

Abram Besicovitch and Hermann Weyl

Let me now introduce you to some notable frogs and birds that I knew personally. I came to Cambridge University as a student in 1941 and had the tremendous luck to be given the Russian mathematician Abram Samoilovich Besicovitch as my supervisor. Since this was in the middle of World War Two, there were very few students in Cambridge, and almost no graduate students. Although I was only seventeen years old and Besicovitch was already a famous professor, he gave me a great deal of his time and attention, and we became life-long friends. He set the style in which I began to work and think about mathematics. He gave wonderful lectures on measure-theory and integration, smiling amiably when we laughed at his glorious abuse of the English language. I remember only one occasion when he was annoyed by our laughter. He remained silent for a while and then said, "Gentlemen. Fifty million English speak English you speak. Hundred and fifty million Russians speak English I speak."

Besicovitch was a frog, and he became famous when he was young by solving a problem in elementary plane geometry known as the Kakeya problem. The Kakeya problem was the following. A line segment of length one is allowed to move freely in a plane while rotating through an angle of 360 degrees. What is the smallest area of the plane that it can cover during its rotation? The problem was posed by the Japanese mathematician Kakeya in 1917 and remained a famous unsolved problem for ten years. George Birkhoff, the leading American mathematician at that time, publicly proclaimed that the Kakeya problem and the four-color problem were the outstanding unsolved problems of the day. It was widely believed that the minimum area was $\pi/8$, which is the area of a three-cusped hypocycloid. The three-cusped hypocycloid is a beautiful three-pointed curve. It is the curve traced out by a point on the circumference of a circle with radius one-quarter, when the circle rolls around the inside of a fixed circle with radius three-quarters. The line segment of length one can turn while always remaining tangent to the hypocycloid with its two ends also on the hypocycloid. This picture of the line turning while touching the inside of the hypocycloid at three points was so elegant that most people believed it must give the minimum area. Then Besicovitch surprised everyone by proving that the area covered by the line as it turns can be less than ϵ for any positive ϵ.

Besicovitch had actually solved the problem in 1920 before it became famous, not even knowing that Kakeya had proposed it. In 1920 he published the solution in Russian in the *Journal of the Perm Physics and Mathematics Society*, a journal that was not widely read. The university of Perm, a city 1,100 kilometers east of Moscow, was briefly a refuge for many distinguished mathematicians after the Russian revolution. They published two volumes of their journal before it died amid the chaos of revolution and civil war. Outside Russia the journal was not only unknown but unobtainable. Besicovitch left Russia in 1925 and arrived at Copenhagen, where he learned about the famous Kakeya problem that he had solved five years earlier. He published the solution again, this time in English in the *Mathematische Zeitschrift*. The Kakeya problem as Kakeya proposed it was a typical frog problem, a concrete problem without much connection with the rest of mathematics. Besicovitch gave it an elegant and deep solution, which revealed a connection with general theorems about the structure of sets of points in a plane.

The Besicovitch style is seen at its finest in his three classic papers with the title, “On the fundamental geometric properties of linearly measurable plane sets of points”, published in *Mathematische Annalen* in the years 1928, 1938, and 1939. In these papers he proved that every linearly measurable set in the plane is divisible into a regular and an irregular component, that the regular component has a tangent almost everywhere, and the irregular component has a projection of measure zero onto almost all directions. Roughly speaking, the regular component looks like a collection of continuous curves, while the irregular component looks nothing like a continuous curve. The existence and the properties of the irregular component are connected with the Besicovitch solution of the Kakeya problem. One of the problems that he gave me to work on was the division of measurable sets into regular and irregular components in spaces of higher dimensions. I got nowhere with the problem, but became permanently imprinted with the Besicovitch style. The Besicovitch style is architectural. He builds out of simple elements a delicate and complicated architectural structure, usually with a hierarchical plan, and then, when the building is finished, the completed structure leads by simple arguments to an unexpected conclusion. Every Besicovitch proof is a work of art, as carefully constructed as a Bach fugue.

A few years after my apprenticeship with Besicovitch, I came to Princeton and got to know Hermann Weyl. Weyl was a prototypical bird, just as Besicovitch was a prototypical frog. I was lucky to overlap with Weyl for one year at the Princeton Institute for Advanced Study before he retired from the Institute and moved back to his old home in Zürich. He liked me because during that year I published papers in the *Annals of Mathematics* about number theory and in the *Physical Review* about the quantum theory of radiation. He was one of the few people alive who was at home in both subjects. He welcomed me to the Institute, in the hope that I would be a bird like himself. He was disappointed. I remained obstinately a frog. Although I poked around in a variety of mud-holes, I always looked at them one at a time and did not look for connections between them. For me, number theory and quantum theory were separate worlds with separate beauties. I did not look at them as Weyl did, hoping to find clues to a grand design.

Weyl’s great contribution to the quantum theory of radiation was his invention of gauge fields. The idea of gauge fields had a curious history. Weyl invented them in 1918 as classical fields in his unified theory of general relativity and electromagnetism, [7]. He called them “gauge fields” because they were concerned with the non-integrability of measurements of length. His unified theory was promptly and publicly rejected by Einstein. After this thunderbolt from on high, Weyl did not abandon his theory but moved on to other things. The theory had no experimental consequences that could be tested. Then in 1929, after quantum mechanics had been invented by others, Weyl realized that his gauge fields fitted far better into the quantum world than they did into the classical world, [8]. All that he needed to do, to change a classical gauge into a quantum gauge, was to change real numbers into complex numbers. In quantum mechanics, every quantum of electric charge carries with it a complex wave function with a phase, and the gauge field is concerned with the non-integrability of measurements of phase. The gauge field could then be precisely identified with the electromagnetic potential, and the law of conservation of charge became a consequence of the local phase invariance of the theory.

Chen Ning (Frank) Yang

Yuri Manin

Photo of F. Yang courtesy of SUNY Stony Brook. Photo of Y. Manin courtesy of Northwestern University.

Weyl died four years after he returned from Princeton to Zürich, and I wrote his obituary for the journal *Nature*, [3]. "Among all the mathematicians who began their working lives in the twentieth century," I wrote, "Hermann Weyl was the one who made major contributions in the greatest number of different fields. He alone could stand comparison with the last great universal mathematicians of the nineteenth century, Hilbert and Poincaré. So long as he was alive, he embodied a living contact between the main lines of advance in pure mathematics and in theoretical physics. Now he is dead, the contact is broken, and our hopes of comprehending the physical universe by a direct use of creative mathematical imagination are for the time being ended." I mourned his passing, but I had no desire to pursue his dream. I was happy to see pure mathematics and physics marching ahead in opposite directions.

The obituary ended with a sketch of Weyl as a human being: "Characteristic of Weyl was an aesthetic sense which dominated his thinking on all subjects. He once said to me, half joking, 'My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful'. This remark sums up his personality perfectly. It shows his profound faith in an ultimate harmony of Nature, in which the laws should inevitably express themselves in a mathematically beautiful form. It shows also his recognition of human frailty, and his humor, which always stopped him short of being pompous. His friends in Princeton will remember him as he was when I last saw him, at the Spring Dance of the Institute for Advanced Study last April: a big jovial man, enjoying himself splendidly, his cheerful frame and his light step giving no hint of his sixty-nine years."

The fifty years after Weyl's death were a golden age of experimental physics and observational astronomy, a golden age for Baconian travelers picking up facts, for frogs exploring small patches of the swamp in which we live. During these fifty years, the frogs accumulated a detailed knowledge of a large variety of cosmic structures and a large variety of particles and interactions. As the exploration of new territories continued, the universe became more complicated. Instead of a grand design displaying the simplicity and beauty of Weyl's mathematics, the explorers found weird objects such as quarks and gamma-ray bursts, weird concepts such as supersymmetry and multiple universes. Meanwhile, mathematics was also becoming more complicated, as exploration continued into the phenomena of chaos and many other new areas opened by electronic computers. The mathematicians discovered the central mystery of computability, the conjecture represented by the statement P is not equal to NP. The conjecture asserts that there exist mathematical problems which can be quickly solved in individual cases but cannot be solved by a quick algorithm applicable to all cases. The most famous example of such a problem is the traveling salesman problem, which is to find the shortest route for a salesman visiting a set of cities, knowing the distance between each pair. All the experts believe that the conjecture is true, and that the traveling salesman problem is an example of a problem that is P but not NP. But nobody has even a glimmer of an idea how to prove it. This is a mystery that could not even have been formulated within the nineteenth-century mathematical universe of Hermann Weyl.

Frank Yang and Yuri Manin

The last fifty years have been a hard time for birds. Even in hard times, there is work for birds to do, and birds have appeared with the courage to tackle it. Soon after Weyl left Princeton, Frank Yang arrived from Chicago and moved into Weyl's old house. Yang took Weyl's place as the leading bird among my generation of physicists. While Weyl was still alive, Yang and his student Robert Mills discovered the Yang-Mills theory of non-Abelian gauge fields, a marvelously elegant extension of Weyl's idea of a gauge field, [11]. Weyl's gauge field was a classical quantity, satisfying the commutative law of multiplication. The Yang-Mills theory had a triplet of gauge fields which did not commute. They satisfied the commutation rules of the three components of a quantum mechanical spin, which are generators of the simplest non-Abelian Lie algebra A_2. The theory was later generalized so that the gauge fields could be generators of any finite-dimensional Lie algebra. With this generalization, the Yang-Mills gauge field theory provided the framework for a model of all the known particles and interactions, a model that is now known as the Standard Model of particle physics. Yang put the finishing touch to it by showing that Einstein's theory of gravitation fits into the same framework, with the Christoffel three-index symbol taking the role of gauge field, [10].

In an appendix to his 1918 paper, added in 1955 for the volume of selected papers published to celebrate his seventieth birthday, Weyl expressed his final thoughts about gauge field theories (my translation), [12]: "The strongest argument for my theory seemed to be this, that gauge invariance was related to conservation of electric charge in the same way as coordinate invariance was related to conservation of energy and momentum." Thirty years later Yang was in Zürich for the celebration of Weyl's hundredth birthday. In his speech, [12], Yang quoted this remark as evidence of Weyl's devotion to the idea of gauge invariance as a unifying principle for physics. Yang then went on, "Symmetry, Lie groups, and gauge invariance are now recognized, through theoretical and experimental developments, to play essential roles in determining the basic forces of the physical universe. I have called this the principle that symmetry dictates interaction." This idea, that symmetry dictates interaction, is Yang's generalization of Weyl's remark. Weyl observed that gauge invariance is intimately connected with physical conservation laws. Weyl could not go further than this, because he knew only the gauge invariance of commuting Abelian fields. Yang made the connection much stronger by introducing non-Abelian gauge fields. With non-Abelian gauge fields generating nontrivial Lie algebras, the possible forms of interaction between fields become unique, so that symmetry dictates interaction. This idea is Yang's greatest contribution to physics. It is the contribution of a bird, flying high over the rain forest of little problems in which most of us spend our lives.

Another bird for whom I have a deep respect is the Russian mathematician Yuri Manin, who recently published a delightful book of essays with the title *Mathematics as Metaphor* [5]. The book was published in Moscow in Russian, and by the American Mathematical Society in English. I wrote a preface for the English version, and I give you here a short quote from my preface. "*Mathematics as Metaphor* is a good slogan for birds. It means that the deepest concepts in mathematics are those which link one world of ideas with another. In the seventeenth century Descartes linked the disparate worlds of algebra and geometry with his concept of coordinates, and Newton linked the worlds of geometry and dynamics with his concept of fluxions, nowadays called calculus. In the nineteenth century Boole linked the worlds of logic and algebra with his concept of symbolic logic, and Riemann linked the worlds of geometry and analysis with his concept of Riemann surfaces. Coordinates, fluxions, symbolic logic, and Riemann surfaces are all metaphors, extending the meanings of words from familiar to unfamiliar contexts. Manin sees the future of mathematics as an exploration of metaphors that are already visible but not yet understood. The deepest such metaphor is the similarity in structure between number theory and physics. In both fields he sees tantalizing glimpses of parallel concepts, symmetries linking the continuous with the discrete. He looks forward to a unification which he calls the quantization of mathematics.

"Manin disagrees with the Baconian story, that Hilbert set the agenda for the mathematics of the twentieth century when he presented his famous list of twenty-three unsolved problems to the International Congress of Mathematicians in Paris in 1900. According to Manin, Hilbert's problems were a distraction from the central themes of mathematics. Manin sees the important advances in mathematics coming from programs, not from problems. Problems are usually solved by applying old ideas in new ways. Programs of research are the nurseries where new ideas are born. He sees the Bourbaki program, rewriting the whole of mathematics in a more abstract language, as the source of many of the new ideas of the twentieth century. He sees the Langlands program, unifying number theory with geometry, as a promising source of new ideas for the twenty-first. People who solve famous unsolved problems may win big prizes, but people who start new programs are the real pioneers."

The Russian version of *Mathematics as Metaphor* contains ten chapters that were omitted from the English version. The American Mathematical Society decided that these chapters would not be of interest to English language readers. The omissions are doubly unfortunate. First, readers of the English version see only a truncated view of Manin, who is perhaps unique among mathematicians in his broad range of interests extending far beyond mathematics. Second, we see a truncated view of Russian culture, which is less compartmentalized than English language culture, and brings mathematicians into closer contact with historians and artists and poets.

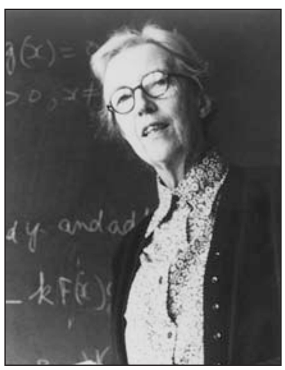
John von Neumann

Another important figure in twentieth century mathematics was John von Neumann. Von Neumann was a frog, applying his prodigious technical skill to solve problems in many branches of mathematics and physics. He began with the foundations of mathematics. He found the first satisfactory set of axioms for set-theory, avoiding the logical paradoxes that Cantor had encountered in his attempts to deal with infinite sets and infinite numbers. Von Neumann's axioms were used by his bird friend Kurt Gödel a few years later to prove the existence of undecidable propositions in mathematics. Gödel's theorems gave birds a new vision of mathematics. After Gödel, mathematics was no longer a single structure tied together with a unique concept of truth, but an archipelago of structures with diverse sets of axioms and diverse notions of truth. Gödel showed that mathematics is inexhaustible. No matter which set of axioms is chosen as the foundation, birds can always find questions that those axioms cannot answer.

Von Neumann went on from the foundations of mathematics to the foundations of quantum mechanics. To give quantum mechanics a firm mathematical foundation, he created a magnificent theory of rings of operators. Every observable quantity is represented by a linear operator, and the peculiarities of quantum behavior are faithfully represented by the algebra of operators. Just as Newton invented calculus to describe classical dynamics, von Neumann invented rings of operators to describe quantum dynamics.

Von Neumann made fundamental contributions to several other fields, especially to game theory and to the design of digital computers. For the last ten years of his life, he was deeply involved with computers. He was so strongly interested in computers that he decided not only to study their design but to build one with real hardware and software and use it for doing science. I have vivid memories of the early days of von Neumann's computer project at the Institute for Advanced Study in Princeton. At that time he had two main scientific interests, hydrogen bombs and meteorology. He used his computer during the night for doing hydrogen bomb calculations and during the day for meteorology. Most of the people hanging around the computer building in daytime were meteorologists. Their leader was Jule Charney. Charney was a real meteorologist, properly humble in dealing with the inscrutable mysteries of the weather, and skeptical of the ability of the computer to solve the mysteries. John von Neumann was less humble and less skeptical. I heard von Neumann give a lecture about the aims of his project. He spoke, as he always did, with great confidence. He said, "The computer will enable us to divide the atmosphere at any moment into stable regions and unstable regions. Stable regions we can predict. Unstable regions we can control." Von Neumann believed that any unstable region could be pushed by a judiciously applied small perturbation so that it would move in any desired direction. The small perturbation would be applied by a fleet of airplanes carrying smoke generators, to absorb sunlight and raise or lower temperatures at places where the perturbation would be most effective. In particular, we could stop an incipient hurricane by identifying the position of an instability early enough, and then cooling that patch of air before it started to rise and form a vortex. Von Neumann, speaking in 1950, said it would take only ten years to build computers powerful enough to diagnose accurately the stable and unstable regions of the atmosphere. Then, once we had accurate diagnosis, it would take only a short time for us to have control. He expected that practical control of the weather would be a routine operation within the decade of the 1960s.

John von Neumann



Mary Cartwright

Photograph of Mary Cartright courtesy of The Mistress and Fellows, Girton College, Cambridge.

Von Neumann, of course, was wrong. He was wrong because he did not know about chaos. We now know that when the motion of the atmosphere is locally unstable, it is very often chaotic. The word "chaotic" means that motions that start close together diverge exponentially from each other as time goes on. When the motion is chaotic, it is unpredictable, and a small perturbation does not move it into a stable motion that can be predicted. A small perturbation will usually move it into another chaotic motion that is equally unpredictable. So von Neumann's strategy for controlling the weather fails. He was, after all, a great mathematician but a mediocre meteorologist.

Edward Lorenz discovered in 1963 that the solutions of the equations of meteorology are often chaotic. That was six years after von Neumann died. Lorenz was a meteorologist and is generally regarded as the discoverer of chaos. He discovered the phenomena of chaos in the meteorological context and gave them their modern names. But in fact I had heard the mathematician Mary Cartwright, who died in 1998 at the age of 97, describe the same phenomena in a lecture in Cambridge in 1943, twenty years before Lorenz discovered them. She called the phenomena by different names, but they were the same phenomena. She discovered them in the solutions of the van der Pol equation which describe the oscillations of a nonlinear amplifier, [2]. The van der Pol equation was important in World

War II because nonlinear amplifiers fed power to the transmitters in early radar systems. The transmitters behaved erratically, and the Air Force blamed the manufacturers for making defective amplifiers. Mary Cartwright was asked to look into the problem. She showed that the manufacturers were not to blame. She showed that the van der Pol equation was to blame. The solutions of the van der Pol equation have precisely the chaotic behavior that the Air Force was complaining about. I heard all about chaos from Mary Cartwright seven years before I heard von Neumann talk about weather control, but I was not far-sighted enough to make the connection. It never entered my head that the erratic behavior of the van der Pol equation might have something to do with meteorology. If I had been a bird rather than a frog, I would probably have seen the connection, and I might have saved von Neumann a lot of trouble. If he had known about chaos in 1950, he would probably have thought about it deeply, and he would have had something important to say about it in 1954.

Von Neumann got into trouble at the end of his life because he was really a frog but everyone expected him to fly like a bird. In 1954 there was an International Congress of Mathematicians in Amsterdam. These congresses happen only once in four years and it is a great honor to be invited to speak at the opening session. The organizers of the Amsterdam congress invited von Neumann to give the keynote speech, expecting him to repeat the act that Hilbert had performed in Paris in 1900. Just as Hilbert had provided a list of unsolved problems to guide the development of mathematics for the first half of the twentieth century, von Neumann was invited to do the same for the second half of the century. The title of von Neumann's talk was announced in the program of the congress. It was "Unsolved Problems in Mathematics: Address by Invitation of the Organizing Committee". After the congress was over, the complete proceedings were published, with the texts of all the lectures except this one. In the proceedings there is a blank page with von Neumann's name and the title of his talk. Underneath, it says, "No manuscript of this lecture was available."

What happened? I know what happened, because I was there in the audience, at 3:00 p.m. on Thursday, September 2, 1954, in the Concertgebouw concert hall. The hall was packed with mathematicians, all expecting to hear a brilliant lecture worthy of such a historic occasion. The lecture was a huge disappointment. Von Neumann had probably agreed several years earlier to give a lecture about unsolved problems and had then forgotten about it. Being busy with many other things, he had neglected to prepare the lecture. Then, at the last moment, when he remembered that he had to travel to Amsterdam and say something about mathematics, he pulled an old lecture from the 1930s out of a drawer and dusted it off. The lecture was about rings of operators, a subject that was new and fashionable in the 1930s. Nothing about unsolved problems. Nothing about the future. Nothing about computers, the subject that we knew was dearest to von Neumann's heart. He might at least have had something new and exciting to say about computers. The audience in the concert hall became restless. Somebody said in a voice loud enough to be heard all over the hall, "Aufgewärmte Suppe", which is German for "warmed-up soup". In 1954 the great majority of mathematicians knew enough German to understand the joke. Von Neumann, deeply embarrassed, brought his lecture to a quick end and left the hall without waiting for questions.

Weak Chaos

If von Neumann had known about chaos when he spoke in Amsterdam, one of the unsolved problems that he might have talked about was weak chaos. The problem of weak chaos is still unsolved fifty years later. The problem is to understand why chaotic motions often remain bounded and do not cause any violent instability. A good example of weak chaos is the orbital motions of the planets and satellites in the solar system. It was discovered only recently that these motions are chaotic. This was a surprising discovery, upsetting the traditional picture of the solar system as the prime example of orderly stable motion. The mathematician Laplace two hundred years ago thought he had proved that the solar system is stable. It now turns out that Laplace was wrong. Accurate numerical integrations of the orbits show clearly that neighboring orbits diverge exponentially. It seems that chaos is almost universal in the world of classical dynamics.

Chaotic behavior was never suspected in the solar system before accurate long-term integrations were done, because the chaos is weak. Weak chaos means that neighboring trajectories diverge exponentially but never diverge far. The divergence begins with exponential growth but afterwards remains bounded. Because the chaos of the planetary motions is weak, the solar system can survive for four billion years. Although the motions are chaotic, the planets never wander far from their customary places, and the system as a whole does not fly apart. In spite of the prevalence of chaos, the Laplacian view of the solar system as a perfect piece of clockwork is not far from the truth.

We see the same phenomena of weak chaos in the domain of meteorology. Although the weather in New Jersey is painfully chaotic, the chaos has firm limits. Summers and winters are unpredictably mild or severe, but we can reliably predict that the temperature will never rise to 45 degrees Celsius or fall to minus 30, extremes that are often exceeded in India or in Minnesota. There is no conservation law of physics that forbids temperatures from rising as high in New Jersey as in India, or from falling as low in New Jersey as in Minnesota. The weakness of chaos has been essential to the long-term survival of life on this planet. Weak chaos gives us a challenging variety of weather while protecting us from fluctuations so severe as to endanger our existence. Chaos remains mercifully weak for reasons that we do not understand. That is another unsolved problem for young frogs in the audience to take home. I challenge you to understand the reasons why the chaos observed in a great diversity of dynamical systems is generally weak.

The subject of chaos is characterized by an abundance of quantitative data, an unending supply of beautiful pictures, and a shortage of rigorous theorems. Rigorous theorems are the best way to give a subject intellectual depth and precision. Until you can prove rigorous theorems, you do not fully understand the meaning of your concepts. In the field of chaos I know only one rigorous theorem, proved by Tien-Yien Li and Jim Yorke in 1975 and published in a short paper with the title, "Period Three Implies Chaos", [4]. The Li-Yorke paper is one of the immortal gems in the literature of mathematics. Their theorem concerns nonlinear maps of an interval onto itself. The successive positions of a point when the mapping is repeated can be considered as the orbit of a classical particle. An orbit has period N if the point returns to its original position after N mappings. An orbit is defined to be chaotic, in this context, if it diverges from all periodic orbits. The theorem says that if a single orbit with period three exists, then chaotic orbits also exist. The proof is simple and short. To my mind, this theorem and its proof throw more light than a thousand beautiful pictures on the basic nature of chaos. The theorem explains why chaos is prevalent in the world. It does not explain why chaos is so often weak. That remains a task for the future. I believe that weak chaos will not be understood in a fundamental way until we can prove rigorous theorems about it.

String Theorists

I would like to say a few words about string theory. Few words, because I know very little about string theory. I never took the trouble to learn the subject or to work on it myself. But when I am at home at the Institute for Advanced Study in Princeton, I am surrounded by string theorists, and I sometimes listen to their conversations. Occasionally I understand a little of what they are saying. Three things are clear. First, what they are doing is first-rate mathematics. The leading pure mathematicians, people like Michael Atiyah and Isadore Singer, love it. It has opened up a whole new branch of mathematics, with new ideas and new problems. Most remarkably, it gave the mathematicians new methods to solve old problems that were previously unsolvable. Second, the string theorists think of themselves as physicists rather than mathematicians. They believe that their theory describes something real in the physical world. And third, there is not yet any proof that the theory is relevant to physics. The theory is not yet testable by experiment. The theory remains in a world of its own, detached from the rest of physics. String theorists make strenuous efforts to deduce consequences of the theory that might be testable in the real world, so far without success.

My colleagues Ed Witten and Juan Maldacena and others who created string theory are birds, flying high and seeing grand visions of distant ranges of mountains. The thousands of humbler practitioners of string theory in universities around the world are frogs, exploring fine details of the mathematical structures that birds first saw on the horizon. My anxieties about string theory are sociological rather than scientific. It is a glorious thing to be one of the first thousand string theorists, discovering new connections and pioneering new methods. It is not so glorious to be one of the second thousand or one of the tenth thousand. There are now about ten thousand string theorists scattered around the world. This is a dangerous situation for the tenth thousand and perhaps also for the second thousand. It may happen unpredictably that the fashion changes and string theory becomes unfashionable. Then it could happen that nine thousand string theorists lose their jobs. They have been trained in a narrow specialty, and they may be unemployable in other fields of science.

Why are so many young people attracted to string theory? The attraction is partly intellectual. String theory is daring and mathematically elegant. But the attraction is also sociological. String theory is attractive because it offers jobs. And why are so many jobs offered in string theory? Because string theory is cheap. If you are the chairperson of a physics department in a remote place without much money, you cannot afford to build a modern laboratory to do experimental physics, but you can afford to hire a couple of string theorists. So you offer a couple of jobs in string theory, and you have a modern physics department. The temptations are strong for the chairperson to offer such jobs and for the young people to accept them. This is a hazardous situation for the young people and also for the future of science. I am not saying that we should discourage young people from working in string theory if they find it exciting. I am saying that we should offer them alternatives, so that they are not pushed into string theory by economic necessity.

Finally, I give you my own guess for the future of string theory. My guess is probably wrong. I have no illusion that I can predict the future. I tell you my guess, just to give you something to think about. I consider it unlikely that string theory will turn out to be either totally successful or totally useless. By totally successful I mean that it is a complete theory of physics, explaining all the details of particles and their interactions. By totally useless I mean that it remains a beautiful piece of pure mathematics. My guess is that string theory will end somewhere between complete success and failure. I guess that it will be like the theory of Lie groups, which Sophus Lie created in the nineteenth century as a mathematical framework for classical physics. So long as physics remained classical, Lie groups remained a failure. They were a solution looking for a problem. But then, fifty years later, the quantum revolution transformed physics, and Lie algebras found their proper place. They became the key to understanding the central role of symmetries in the quantum world. I expect that fifty or a hundred years from now another revolution in physics will happen, introducing new concepts of which we now have no inkling, and the new concepts will give string theory a new meaning. After that, string theory will suddenly find its proper place in the universe, making testable statements about the real world. I warn you that this guess about the future is probably wrong. It has the virtue of being falsifiable, which according to Karl Popper is the hallmark of a scientific statement. It may be demolished tomorrow by some discovery coming out of the Large Hadron Collider in Geneva.

Manin Again

To end this talk, I come back to Yuri Manin and his book *Mathematics as Metaphor*. The book is mainly about mathematics. It may come as a surprise to Western readers that he writes with equal eloquence about other subjects such as the collective unconscious, the origin of human language, the psychology of autism, and the role of the trickster in the mythology of many cultures. To his compatriots in Russia, such many-sided interests and expertise would come as no surprise. Russian intellectuals maintain the proud tradition of the old Russian intelligentsia, with scientists and poets and artists and musicians belonging to a single community. They are still today, as we see them in the plays of Chekhov, a group of idealists bound together by their alienation from a superstitious society and a capricious government. In Russia, mathematicians and composers and film-producers talk to one another, walk together in the snow on winter nights, sit together over a bottle of wine, and share each others' thoughts.

Manin is a bird whose vision extends far beyond the territory of mathematics into the wider landscape of human culture. One of his hobbies is the theory of archetypes invented by the Swiss psychologist Carl Jung. An archetype, according to Jung, is a mental image rooted in a collective unconscious that we all share. The intense emotions that archetypes carry with them are relics of lost memories of collective joy and suffering. Manin is saying that we do not need to accept Jung's theory as true in order to find it illuminating.

More than thirty years ago, the singer Monique Morelli made a recording of songs with words by Pierre MacOrlan. One of the songs is *La Ville Morte*, the dead city, with a haunting melody tuned to Morelli's deep contralto, with an accordion singing counterpoint to the voice, and with verbal images of extraordinary intensity. Printed on the page, the words are nothing special:

> *"En pénétrant dans la ville morte,*
> *Je tenait Margot par le main…*
> *Nous marchions de la nécropole,*
> *Les pieds brisés et sans parole,*
> *Devant ces portes sans cadole,*
> *Devant ces trous indéfinis,*
> *Devant ces portes sans parole*
> *Et ces poubelles pleines de cris".*

"As we entered the dead city, I held Margot by the hand…We walked from the graveyard on our bruised feet, without a word, passing by these doors without locks, these vaguely glimpsed holes, these doors without a word, these garbage cans full of screams."

I can never listen to that song without a disproportionate intensity of feeling. I often ask myself why the simple words of the song seem to resonate with some deep level of unconscious memory, as if the souls of the departed are speaking through Morelli's music. And now unexpectedly in Manin's book I find an answer to my question. In his chapter, "The Empty City Archetype", Manin describes how the archetype of the dead city appears again and again in the creations of architecture, literature, art and film, from ancient to modern times, ever since human beings began to congregate in cities, ever since other human beings began to congregate in armies to ravage and destroy them. The character who speaks to us in MacOrlan's song is an old soldier who has long ago been part of an army of occupation. After he has walked with his wife through the dust and ashes of the dead city, he hears once more:

> *"Chansons de charme d'un clairon*
> *Qui fleurissait une heure lointaine*
> *Dans un rêve de garnison".*

"The magic calls of a bugle that came to life for an hour in an old soldier's dream".

The words of MacOrlan and the voice of Morelli seem to be bringing to life a dream from our collective unconscious, a dream of an old soldier wandering through a dead city. The concept of the collective unconscious may be as mythical as the concept of the dead city. Manin's chapter describes the subtle light that these two possibly mythical concepts throw upon each other. He describes the collective unconscious as an irrational force that powerfully pulls us toward death and destruction. The archetype of the dead city is a distillation of the agonies of hundreds of real cities that have been destroyed since cities and marauding armies were invented. Our only way of escape from the insanity of the collective unconscious is a collective consciousness of sanity, based upon hope and reason. The great task that faces our contemporary civilization is to create such a collective consciousness.

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