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Mathematics and Complex Systems

Richard Foote

Contemporary researchers strive to understand complex physical phenomena that involve many constituents, may be influenced by numerous forces, and may exhibit unexpected or emergent behavior. Often such "complex systems" are macroscopic manifestations of other systems that exhibit their own complex behavior and obey more elemental laws. This article proposes that areas of mathematics, even ones based on simple axiomatic foundations, have discernible layers, entirely unexpected "macroscopic" outcomes, and both mathematical and physical ramifications profoundly beyond their historical beginnings. In a larger sense, the study of mathematics itself, which is increasingly surpassing the capacity of researchers to verify "by hand," may be the ultimate complex system.

Since antiquity, humankind has pondered and debated the "unreasonable effectiveness of mathematics" in its apparent ability to explain physical phenomena, an enigma elucidated by Wigner [(*1*), p. 1]: "Mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections." In recent years the scientific community has coined the rubric "complex system" to describe phenomena, structures, aggregates, organisms, or problems that share some common themes: (i) They are inherently complicated or intricate, in that they have factors such as the number of parameters affecting the system or the rules governing interactions of components of the system; (ii) they are rarely completely deterministic, and state parameters or measurement data may only be known in terms of probabilities; (iii) mathematical models of the system are usually complex and involve nonlinear, ill-posed, or chaotic behavior; and (iv) the systems are predisposed to unexpected outcomes (so-called "emergent behavior"). Familiar examples include weather systems, biological and chemical systems, social networks, transportation and engineering infrastructure systems, and the Internet (*2*–*4*).

In these examples, the complex systems are physical phenomena that researchers attempt to model mathematically. But areas of mathematics itself may be viewed as complex systems exhibiting many of the characteristics of the physical structures, including discernible "layers" closely analogous to microscopic or macroscopic strata in physics, biology, and other sciences. As we struggle to better understand and systematize precise notions embodying physical complex systems, and ultimately to fathom the potential for mathematics to model these, a closer look at "abstract" complex mathematical systems is in order. We may then hope to glean from the theoretical setting a more rigorous grasp of some mathematical underpinnings that characterize complex systems in the natural world, and thereby make a chink in the Wigner conundrum.

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Background from Group Theory

Some developments in finite group theory are useful illustrations of the idea that aspects of mathematics represent inherently complex systems [see (*5*) for other examples]. Finite group theory emerged from a confluence of many ideas in 18th- and 19th-century abstract algebra, most notably the general insolvability of polynomial equations by radicals as pioneered by Galois around 1830 (see below). Galois showed that each polynomial is associated to a mathematical structure called a group, and the roots of the polynomial can be written in terms of radicals involving its coefficients—generalizing the familiar formula for roots of quadratic equations—if and only if the associated group can be "factored" (in a way made precise shortly) into simple group factors that all have prime order. This succinct formulation of the cornerstone of what is now known as Galois theory is possible with historical hindsight: The axiomatic formulation of the concept of a group first appeared in 1882, and shortly thereafter it was shown that every finite group—these are the ones in Galois theory—has a "unique factorization" into simple group factors (the simple group factors need not have prime order in general, however).

Before going further, a digression into some precise concepts from abstract algebra is essential for understanding the scope of the "complex system" initiated by Galois and his cohort. A group is any set G that has an operation on pairs of elements of G satisfying some simple axioms. The operation between elements a and b of G is denoted $a * b$; note that in specific groups the operation may be addition, matrix multiplication, function composition, etc., depending on the exact nature of the elements in G. The axioms for a group are:

1. The operation is associative: $a * (b * c) = (a * b) * c$, for every a, b, c in G.

2. There is an identity, denoted by the symbol 1, such that $1 * a = a * 1 = a$, for every a in G.

3. Every element a in G has an inverse, denoted by a^{-1}, such that $a^{-1} * a = a * a^{-1} = 1$.

A group is called finite if the underlying set G contains only a finite number of elements; the order of G is the number of elements in G. Groups abound in mathematics: The most familiar examples are the set of all integers, the set of all rational numbers, or the set of all real numbers, where in each instance the operation is addition, the identity is the number zero, and the inverse of an element is its negative. Likewise, the nonzero rational or real numbers (but not the integers) form a group under the operation of multiplication, where identity is the number 1 and the inverses are reciprocals. Note that in general the group operation need not be commutative; that is, we need not have $a * b = b * a$, for every a, b in G.

Finite groups arise naturally in many contexts; the most familiar are as groups of symmetries of geometric or physical objects. A regular n-sided planar figure or n-gon has $2n$ symmetries, and these symmetries form a group under the operation *, where $a * b$ denotes "first apply symmetry b and follow that by applying symmetry a to the n-gon" (this is just function composition, as one computes the composition $f[g(x)]$ of functions f and g in calculus). Such symmetry groups are not, in general, commutative—for example, the group of six symmetries of an equilateral triangle.

A nonempty subset H of a finite group G is called a subgroup of G if it is closed under the group operation: Whenever a, b belong to the subset H, their group product $a * b$ also belongs to H (in this situation, H is a group in its own right). A subgroup H of a group G is called normal if it satisfies a "weak commutativity" rule: $aH = Ha$, for every a in G, where aH is the set of all products $a * h$, over all h in H (likewise, Ha is the set of all $h * a$).

The fundamental importance of normal subgroups, first realized by Galois, is that the whole group G may be "factored" or "collapsed" into a new group by "collapsing" the set of all the elements of H to a single point, and likewise calling each set aH a single point in this new group; the operation in this new group inherits from the original group via the rule $(aH) * (bH) = (a * b)H$, where a and b are arbitrary elements of G. The resulting new group is called the quotient group of G by H and is denoted as G/H. This quotient group has order equal to the order of G divided by the order of H, and there is a natural "projection map" from the original group to the quotient group that is compatible with both their operations.

We may view the group G as "covering" the group G/H by n sheets, where H has n elements. Figure 1 illustrates this when G is the group of real numbers (operation addition), $G/H = U$ is the unit circle in the complex plane (which is a group under multiplication of complex numbers), and the projection map is $\varphi: \theta \rightarrow \exp(i\theta)$ (where H is the subgroup of G consisting of integer multiples of 2π, and n is infinite).

It is reasonable to view a group G possessing a normal subgroup H as "factored" into two smaller groups, H and G/H, and many global properties of the original group might be inferred from properties of these "factors." Indeed, this

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process may be continued in a finite group by factoring both H and G/H in turn by their normal subgroups, and so on inductively until one obtains a sequence of factors that cannot be broken down further. These irreducible factors, the "primes" of group theory, are the simple groups: A group is called simple if it has at least two elements and its only normal subgroups are the obvious ones—namely G itself and the subgroup of order 1 consisting of only the identity element.

The above process illustrates that every finite group has a "factorization" into simple groups. It is also quite elementary (the Jordan-Hölder Theorem) to prove that for any given G, although different routes may be taken to achieve factorizations into its simple components, the number of factors and the intrinsic type of the factors are independent of the route (*5*). In other words, every finite group possesses an essentially unique factorization into simple factors, in much the same way as an integer factors uniquely into primes. The simple groups are thus the primes or "atoms" of finite group theory.

Galois exploited this "atomic" decomposition in an astounding way. It is quite elementary to show that finite groups whose orders are prime numbers are always simple (there is essentially a unique group of prime order p for each prime number p). Galois proved that to each polynomial of degree n there is associated a finite group in such a way that the roots of the polynomial may be expressed in terms of radical expressions involving its coefficients (as is the case for quadratics) if and only if the simple factors for its group all have prime order. He went on to show that there are polynomials of degree 5 (or any higher degree) whose groups contain simple factors that are not of prime order; this is the famous "insolvability of the quintic" theorem of Galois and Abel. This theorem was the first inkling of the complexity of this branch of mathematics: The existence of a simple group containing 60 elements—which is the smallest simple group of nonprime order—is the obstruction to there being an algebraic formula for the roots of a quintic polynomial.

This and many other properties of groups fall into the realm of elementary abstract algebra normally covered in a substantial college-level introduction [see (*6*)]. One may argue that this elementary, axiomatic field of mathematics, although perhaps complicated and mathematically deep, does not constitute a true complex system in the current scientific parlance. Indeed, one might contend that finite group theory, being completely deterministic, cannot beget emergent behavior, but perhaps can only produce "unexpected" results (or, to a Platonist, unexpected theorems may be uncovered). The import of this case study, however, is to reexamine some of the fundamental premises of what attributes truly are the hallmarks of complex systems. After all, it is futile to try to predict truly unpredictable or random systems; rather, the aim of complex systems is to explain and understand the local ("atomic") workings and rules of a system, and then to make global connections and inferences based on this knowledge. Finite group theory may help to serve as a paradigm for this thesis. Pursuing our historical explication, we discern at least three "layers" or "jumps in complexity," even within this seemingly elementary discrete realm.

The Classification of Finite Simple Groups

In the 1880s the quest began to classify finite simple groups, in the sense of listing them all in "families" that enjoy common structural properties. For example, given any natural number $n > 1$ and any finite field F (such as the integers modulo p, for any prime p), the set of all $n \times n$ matrices of determinant 1 with entries from F becomes a simple group after we factor out the scalar matrices (except in a handful of small cases). This collection of (linear) simple groups constitutes one family, indexed by the two parameters n and F [see (*5*) for other examples and precise definitions]. By the early 1960s, 18 infinite families of simple groups were known. In addition, there were five simple groups not belonging to any of these families, the five Mathieu groups; these were named "sporadic simple groups." At that time, however, there was little progress toward showing that this list was complete.

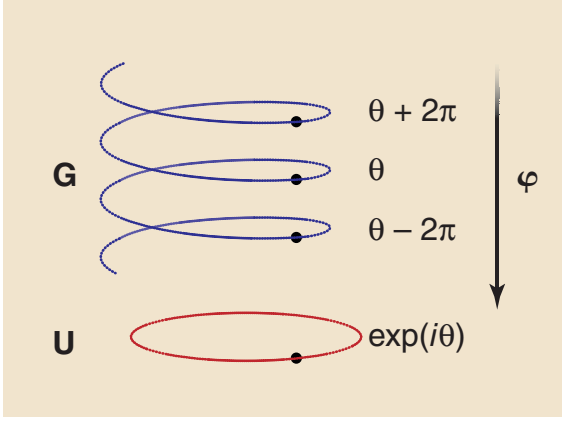


Fig. 1. Projection map φ from the real numbers G onto the circle group U.

A breakthrough to the next level of complexity occurred in 1962 when Feit and Thompson proved the celebrated Odd Order Theorem (*7*): The only simple groups containing an odd number of elements are those of prime order. This proof, which occupies an entire journal issue, consists of 255 pages of intricate, delicate, and ingenious mathematics; considerable efforts to simplify it since then have met with only modest success. This exemplifies a not uncommon phenomenon in mathematics: A theorem with a strikingly elementary formulation may—initially at least—necessitate a very complicated proof. The Feit-Thompson Theorem, although a fully anticipated result, nonetheless spawned the first "quantum jump" in technical virtuosity that practitioners would need in order to surmount problems in this arena. Numerous proofs more than 100 pages in length sprang forth, including Thompson's successor to the Odd Order Theorem, the N-group Theorem (*8*), requiring more than 400 pages of proof.

In the early 1960s, a number of experts in the field felt that the complete determination of all finite simple groups—the Classification, as it was termed—was well within reach, and the 18 families plus five sporadic Mathieu groups would constitute the complete list. A blow to this presumption fell around 1965 when Janko discovered a sixth sporadic simple group. A cascade of other sporadic simple groups emerged in subsequent years, sometimes singly and sometimes in mini-families of two or three. The Classification (*9*) was essentially completed around 1980, although a final 1200-page component appeared in print only in 2005.

The Classification theorem states that every finite simple group belongs to one of the following families:

1. The groups of prime order,
2. The alternating groups A_n for $n > 4$,
3. The 16 families of groups of Lie type over finite fields (each has infinitely many members), or
4. The 26 sporadic simple groups.

This culmination—deemed by Gorenstein (*10*) the Enormous Theorem—required 10,000 to 15,000 journal pages spread over some 500 articles, written mostly between 1950 and 1980 by more than 100 mathematicians. This is the genome project of finite group theory! It astounds not only by virtue of its technical complexity, length, and intricacy, but by the emergence of a wholly unexpected and enigmatic gaggle of sporadic exceptions. The rudimentary "elementary particle" structural axioms for group theory have produced an ultimate "atomic" array of simple groups of astonishing beauty, symmetry, and diversity.

The Classification provides an essential foundation for unraveling the structure of finite groups. However, just as molecules and compounds are not completely determined by their list of constituent elements, a given finite group is not uniquely determined only by its list of simple factors; how the factors are "bonded" is crucial. For example, in Fig. 1 the group G of real numbers factors as $G/H = U$, the circle group, where H is the group of integer multiples of 2π; however, this factorization is not "symmetric": There is no subgroup of K of G where G/K is H; moreover, there are many different groups having the same two factors H and U. Constructing larger groups from smaller constituents falls under the rubric of the "extension problem," which is ubiquitous throughout the study of mathematical structures. Even for finite groups, a categorical solution to the extension problem is unattainable, and the universe of finite groups

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may be viewed as an infinitely variegated category whose "states" or "molecules" are the groups themselves. In this context, the Classification is a prototype for the study of other complex systems, inasmuch as it provides both structural "germs" and manageable subsystems from which meaningful results may be extracted.

Comparison of the Classification to genome research is not facetious, as the intent, scope, and ramifications of these two massive endeavors are not dissimilar. It would divert us too far afield to list new results that have accrued from the Classification, some within group theory itself but many in other areas of mathematics (*5*). However, one promontory epitomizing the next "layer" of mathematics may be viewed as an outgrowth of the Classification.

The Monster and Moonshine

The largest of the 26 sporadic simple groups is the Fischer-Griess "Monster," containing about 10^{54} elements; it is the nexus of a new level of complexity. Around 1978, McKay noticed striking coincidences between the dimensions of the smallest linear spaces on which the Monster could act and Fourier coefficients of the classical modular function $j(\tau)$, well known from complex analysis and Riemann surface theory (*5*). The smallest nontrivial action (i.e., linear representation) of the Monster occurs in dimension 196,883, which is 1 less than the coefficient of the linear term in the q-series expansion of $j(\tau)$, where $q = \exp(i\tau)$. McKay's observations—which were expanded, honed, and systematized by Thompson, Conway, Norton, and others (*11*)—provided an almost exact one-to-one correspondence between classes of elements in the Monster and certain modular functions associated to Riemann surfaces of genus zero (i.e., compact one-dimensional complex manifolds with no holes). The connections were astonishing and mysterious: remarkable "coincidences" between a structure that emerged from finite group-theoretic research and modular functions over the complex numbers, an area familiar to mathematicians and physicists for more than 100 years. Conway coined the fanciful moniker "Monstrous Moonshine" for them, partly to reflect the slightly "illicit" (and tantalizing) nature of what, at the time, was based almost entirely on dimly lit speculation.

Much work has been done in an attempt to explain the Moonshine connections (*12*), culminating in a full verification of the Moonshine conjectures by Borcherds (*13*), for which he was awarded the Fields Medal in 1998. Building on work of others, Borcherds developed new mathematical structures, now known as Borcherds-Kac-Moody algebras, generalizing the familiar notions of simple Lie algebras, and used these as the cornerstone of his proof. Perhaps even more startling, indeed at yet another "level" of complexity, is Borcherds's use of ideas from perturbative string theory (or conformal field theory) in this work. In hindsight, Gagnnon [(*12*), p. 27] observed, "Almost every facet of Moonshine finds a natural formulation in conformal field theory, where it often was discovered first." Yet although Borcherds established the rigorous mathematics verifying the conjectures, it is perhaps safer to characterize his contribution as shedding more light on, rather than fully illuminating, the root causes for these connections.

There is some speculation that there is a 26-dimensional model of space-time for which the Monster is its group of symmetries. If true, this would be an ultimate layer of complexity on which simple groups have left their imprint. A recent conference (*14*) highlighted the groundbreaking importance of this research: "The [Borcherds-Kac-Moody] algebra $V^\natural$ seems to be the natural object to define the Monster and it is conjectured that other (maybe even all) sporadic simple groups arise as automorphism groups of some vertex operator algebras as well. On the other hand, vertex operator algebras are essentially the chiral algebras of conformal quantum field theories and the latter—providing a concept to describe symmetry in two-dimensional critical systems—are one of the objects of greatest interest in modern physics. There exists a special class of quantum field theories called minimal models which are believed to constitute in a certain sense the building blocks of all other conformal field theories and the classification of all such minimal models is actually one of the most important problems in theoretical physics and has attracted growing research interest during the last years." Indeed, in the spirit of Wolfram (*15*), one might speculate even further that some of the fundamental laws of nature or cosmology are, in some fashion yet to be uncovered, compatible with the group axioms, which may then shed some new light on the "unreasonable effectiveness of mathematics."

Conclusion

Scientific research, as with physical systems, evolves over time through generally incremental changes punctuated by breakthroughs, culminations, and unforeseen outcomes. As the field of finite group theory epitomizes, new results in a given area often initiate surges of effort and focus the expenditure of resources; as such, the pursuit of knowledge has facets in common with biological colonies, financial markets, etc. The human element is an essential agent in the evolution of mathematics as a complex system, and the "layers" of complexity mirror the "knowledge states" in this adaptive process. Computers are now taking an increasingly important (and controversial) role in both verifying and discovering new mathematics [see, e.g., (*5*, *16*)]; and so we stand on the threshold of a new dynamic where mathematics, the very foundation of science, may produce and build on results that are humanly unverifiable by even the combined effort of the community, and the veracity of certain results may only be known with some degree of probability.

To reiterate, this case study is intended not just as a primer of advances in finite group theory, but as a test case from which we may glean some general principles for both the characterization and study of complex systems. Evidently, complex systems may evolve from structures according to very elementary rules or transition laws; the seemingly "deterministic" nature of such foundations may belie their ultimate intricacy and unpredictability. A combination of technical depth and breadth of relevance should be essential facets of any complex system. Moreover, each should have "layers" of depth that are reasonably discernible to experts, even if there may be some disagreement about the precise nature of this term or where the "boundaries" of the layers lie. There should be some cross-fertilization of ideas, outcomes, and motivations spanning the layers (even if practitioners work primarily in only one layer).

A complex system must have a substantial impact on systems other than itself; from its study, some larger principles, insights, techniques, or connections should accrue. Emergent behavior is not sufficient to characterize a complex system; rather, a legitimate hallmark is unexpected behavior that leads to deeper understanding of the system or relationships to other phenomena not heretofore considered relevant to the system.

Finally, the study of complex systems should be the exclusive purview of no one but the responsibility of everyone: Each scientist, mathematician, or researcher unfurls the mysteries of nature and humankind in small, deliberate steps. Science embodies the ability to verify, reproduce, and convince others of the veracity of one's discoveries, so the work of scientists is inherently incremental and precise. On the other hand, it is incumbent on us all to work toward enhancing the understanding of "big picture" issues within our own disciplines and beyond; each of our disciplines must itself exhibit the inherent facets of a complex system, or our research is surely nugatory.

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