# Fractal dimensions

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# Fractal dimensions

The term ‘fractal’ (from the Latin *fractus*, meaning ‘broken’), introduced by Benoit Mandelbrot about 25 years ago, is used to characterize spatial and/or temporal phenomena that are continuous but not differentiable. Geometrically, a fractal is a rough or fragmented geometric shape that can be subdivided into parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractal geometry is the study of geometric shapes that seem chaotic or irregular when compared with those of standard geometry (line, square, sphere, etc.) but exhibit extreme orderliness because they possess a property of invariance under suitable contractions or dilations [13]. Fractal objects are highly nontrivial representations of the two fundamental symmetries of nature, dilation ($\mathbf{r} \rightarrow a\mathbf{r}$) and translation ($\mathbf{r} \rightarrow \mathbf{r} + b$). There are many mathematical structures that are fractals (e.g. Sierpinski gasket, Koch snowflake, Peano curve, Mandelbrot set, and Lorenz attractor) and many examples of fractal-like objects in nature, such as clouds, mountains, turbulence, coastlines, shells, cauliflowers, and leaves, that do not correspond to simple, Euclidean geometric shapes.

Fractal properties include self-similarity or affinity, scale symmetry, scale independence or invariance, heterogeneity, complexity, and infinite length or detail. ‘Self-similar’ here has two meanings. One can understand ‘similar’ as a loose everyday synonym of ‘analogous’. But there is also the strict textbook sense of ‘contracting similarity’. It expresses that each part is a linear geometric reduction of the whole, with the same reduction ratios in all directions. Random fractals are self-similar only in a statistical sense; to describe them it is more appropriate to use the term ‘scale invariance’ than self-similarity. There are many different self-similar processes. However, most studies have considered those that have a stationary increment. More recent developments have extended, in particular, to include self-affine, in that the reductions are still linear but the reduction ratios in different directions are different. Fractal structures do not have a single length scale, while fractal processes (e.g. **time series**) cannot be characterized by a single time scale. Fractal theory offers methods for describing the inherent irregularity of natural objects. In fractal analysis, the Euclidean concept of ‘length’ is viewed as a process. This process is characterized by a constant parameter $D$ known as the fractal (or fractional) dimension.

There are many fractal dimensions introduced in mathematical and physical literature (e.g. [13], [16], [19] and [22]). Fractal dimensions can be positive, negative, complex, fuzzy, multifractal, etc. The two most commonly used are the Hausdorf dimension and capacity. Hausdorf in 1919 introduced the definition of the Hausdorf dimension based on a method of covering. Let $D > 0$ and $\varepsilon > 0$ be real numbers. Cover a set $E$ by countable spheres whose diameters are all smaller than $\varepsilon$. Denoting the radii of the spheres by $r_1, r_2, \ldots, r_k$, the $D$-dimensional Hausdorf measure is defined by the following equation:

$$M_D(E) \equiv \lim_{\varepsilon \to 0} \inf_{r_k < \varepsilon} \sum_k r_k^D \qquad (1)$$

For any given set $E$, this quantity is proved to vary from infinity to zero at a special value of $D$, denoted by $D_{\mathrm{H}}$, which is the Hausdorf dimension.

Since a rigorous calculation of the Hausdorf dimension is generally very difficult another fractal dimension, the capacity dimension, is more practically useful. Kolmogorov introduced this dimension in 1959 and like the Hausdorf dimension it is based on the idea of covering. Let the considered shape be a bounded set in $d$-dimensional Euclidean space. Cover the set by $d$-dimensional spheres of identical radius $1/\varepsilon$. The capacity dimension is given by

$$D_C \equiv \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log 1/\varepsilon} \qquad (2)$$

where $N(\varepsilon)$ denotes the minimal number of spheres. For example, consider a Koch snowflake curve formed by repeatedly replacing ___ with $\wedge$, where each of the four new lines is one-third the length of the old line. Blowing up the snowflake curve by a factor of three results in a snowflake curve four times as large (one of the old snowflake curves can be placed on each of the four segments $\wedge$, $4 = 3^{\log_3 4}$). So, the curve length has a dimension of $\log_3 4 (= \log 4 / \log 3 = 1.2618)$. Since the dimension 1.261 is larger than the dimension 1 of the lines making up the curve, the Koch curve is more complicated than a line, but seems less complicated than the plane-filling Peano curve. Another example is the Sierpinski gasket. This is perhaps the most studied two-dimensional fractal structure because it can be regarded as a prototype of fractal lattices with an infinite hierarchy of loops (see Figure 1). When constructing this fractal, three of the four equilateral triangles generated within the triangles obtained in the previous step are kept. Since the linear size of the triangles is halved in every iteration, the fractal dimension of the resulting object is $\log 3/\log 2 = 1.5849$.

Capacity dimension is a special case of the Hausdorf dimension with the restriction that the radii of spheres are the same. This dimension often coincides with the Hausdorf dimension but sometimes differs. Mathematically, we can prove the following inequality for any shape:

$$D_C \geq D_H \tag{3}$$

Therefore, a fractal can be defined as a set for which the Hausdorf dimension strictly exceeds the topological dimension [13]. Topological dimension is the 'normal' idea of dimension; a point has topological dimension 0, a line has topological dimension 1, a surface has topological dimension 2, etc.

These definitions of dimension are rigorous. However, a basic difficulty arises when applying them to real problems. In both definitions dimensions are defined in the limit $\varepsilon \to 0$, but length 0 is not a physical concept as a consequence of the uncertainty principle. In addition to this 'inner-length scale' below which scale invariance breaks down, there is in each natural problem also an 'outer-length scale' above which scale invariance does not hold. Thus, experimentally, there must be limitations on the observed scaling range on both sides (e.g. [12] and [19]).

Many physically feasible methods of calculating fractal dimensions have been developed recently. We can classify them into five categories:

1. changing coarse graining level (also called box-counting methods) (i.e. the relationship $N(\varepsilon) \propto \varepsilon^{-D}$, where $N(\varepsilon)$ is a number measure corresponding to squares with the side length scale unit $\varepsilon$ and $D$ is the fractal dimension);
2. using the fractal measure relations (i.e. perimeter/area/volume methods: $L \propto S^{1/2} \propto V^{1/3} \propto X^{1/D}$);
3. using the correlation functions (i.e. autocorrelations, semivariograms, etc.);
4. using the **distribution functions** (i.e. hyperbolic distribution, stable Levy distribution, etc.); and
5. using the power spectrum (i.e. Fourier transformation, filters, **wavelets**, etc.). For more information about estimating fractal dimensions, see [4], [5], [13], [19], [21] and [22].

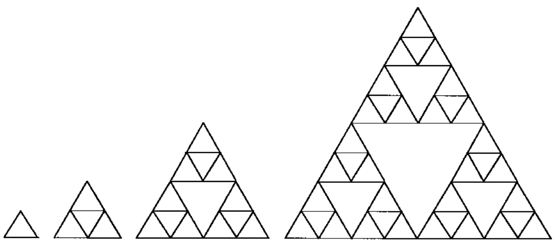

**Figure 1** The first four steps in the construction of the Sierpinski gasket that has loops on all length scales

Since the above-defined fractal dimensions are scale-independent, they are only valid when $N(\varepsilon)$ follows an exact power law; therefore, they may not be as useful for describing more complicated phenomena in nature with only one number [10, 19]. We need to extend the fractal dimension. Most important, we should be able to generalize the fractal dimension so as to depend on the scale of observation. According to (2), it seems to be most natural to define a scale-dependent fractal dimension by the slope of the graph of $N(\varepsilon)$, that is, the derivative of $\log N(\varepsilon)$ with respect to $\log \varepsilon$:

$$D(\varepsilon) = -\frac{\mathrm{d}\log N(\varepsilon)}{\mathrm{d}\log \varepsilon} \tag{4}$$

Solving (4) inversely, we can have the following expression:

$$N(E) = N(\varepsilon)\exp\left[-\int_{\varepsilon}^{E} \frac{D(r)}{r}\,\mathrm{d}r\right] \tag{5}$$

Equation (5) shows how the scale-dependent fractal dimension links observations of different scales.

Another important extension is to introduce new quantities to describe spatial fluctuations of fractal dimensions in order to cope with objects the local fractal dimensions of which change from part to part. A generalized information dimension and multifractal or $f-\alpha$ spectra are introduced for the purpose [13, 19, 20, 22]. We will not discuss them further here.

The concept and methodology of fractals have been used to study many multiscale ecological phenomena and provide us with a new insight on analyzing and quantifying ecological patterns and dynamics [6–8, 10, 11, 15, 17, 18]. The fractal approach is becoming mainstream in ecology because understanding the origins of fractal scaling can provide valuable information about the origins of systems' complexity, and also the driving mechanisms inherent to these systems with their intricate dynamical structure [1, 7]. As a property of fractals, self-similarity – repeating in time and/or space – can also substantially simplify the mathematical modeling of the phenomena themselves [2]. In ecology, there are several widely accepted fractal scaling relations, especially in size–**abundance** spectra or distributions, species–area curves, allometric scaling [23], life history invariants [3], self-thinning processes [9], and so on. These scaling relations have been helping ecologists to achieve a better understanding of ecological patterns and processes.

Due to limited space, only one example, namely the well-known species–area curve $S = CA^z$ (the number of species is a power law function of a sampled patch of area $A$ with scaling exponent $z$), is used here to show how a fractal approach can help us to solve this ecological puzzle. Milne [14] and Yamasaki et al. [24] used a fractal approach to this old problem and concluded that the number of species on the island is constrained by not only the size of the island but also the shape of the island. But the questions about why is the $z$-value of the relationship generally between 0.15 and 0.4 and why is there so much variation in $z$ among surveys remain. Here we use theories of island **biogeography** and fractal dynamics to construct a model coupling dynamics of species (colonization–extinction) and habitat heterogeneity. Consider that the total energy supporting the number of species on earth is limited. We can generalize MacArther–Wilson's model to write a simple fractal–logistic-type species dynamics as $\mathrm{d}S/\mathrm{d}t = cS^{\nu} - dS^{\nu'}$, where $S$ is the number of species, $\nu$ and $\nu'$ are scaling exponents regulating dynamics of species (i.e. colonization–extinction, competition–colonization trade-off, etc.), and $c$ and $d$ are constant coefficients. Now we assume that the total area of suitable habitat, $a$, is also limited and it should be proportional to the total sampled area, $A$, i.e. $a \propto A$. (We could consider a general fractal-like increase of habitat space on 'landscape', i.e. $a \propto A^{\gamma}$, but that will not change our basic conclusion.) The dynamics of suitable habitat area for those species dynamics have similar equation as $\mathrm{d}a/\mathrm{d}t = ma^{\beta} - ea^{\beta'}$, where $\beta$ and $\beta'$ are similarly scaling exponents for regulating dynamics of suitable habitat heterogeneity, and $m$ and $e$ are constant coefficients. And $\beta' > \beta$ is for limited growth of suitable habitat.

In equilibrium situations, both the species number and suitable habitat area (as well as the total sampled area $A$) become

$$\ln \widehat{S} = \frac{\ln(d/c)}{\nu - \nu'} \quad \text{and} \qquad \ln \hat{a} = \frac{\ln(e/m)}{\beta - \beta'} \propto \ln \widehat{A} \tag{6}$$

Combining equations (6), and $\ln S \propto z \ln A$, we have

$$z = z(c, d, e, m, \beta, \beta', \nu, \nu') = \frac{\left[\dfrac{\ln(d/c)}{\nu - \nu'}\right]}{\left[\dfrac{\ln(e/m)}{\beta - \beta'}\right]} = \frac{(\beta - \beta')\ln(d/c)}{(\nu - \nu')\ln(e/m)} \tag{7}$$

Therefore, the $z$-value appears related to the two mechanisms of regulation at the species and their corresponding habitat levels, and is the function of parameters of species dynamics and self-regulation processes coupling parameters from dynamics of suitable habitat area occupied by those species. When we remove the sampling effect, the $z$-value could be an excellent index of diversity (*see* **Diversity measures**) because it measures how fast the number of species increases in relation to the increase of total suitable habitat area. At different scales in space and/or over time, such as local, regional, and global scales, parameters in the dynamics of species and habitat diversity can be different. Equation (7) predicts that the wide range of the $z$-values of species–area curve is due to the differences in the rates of dynamics at species and habitat levels.

The fractal approach is a powerful tool in ecological studies. By 'scale invariance' in ecology, we mean that scales are ecologically equivalent so that the same ecological conclusions may be drawn from any scale statistically. However, ecological systems are complex. Since a log–log plot of many ecological data or power-law distributions displays a straight line only over a certain range of parameters, such fractal measures alone may not give the whole system information. We have to go beyond scale invariance to scale covariance and scale dynamics [7]. Studies of fractals in ecology have now evolved from the purely descriptive to the beginning of understanding.

*References*

[1] Bak, P., Tang, C. & Wiesenfeld, K. (1987). Self-organized criticality: an explanation of $1/f$ noise, *Physical Review Letters* **59**, 381–384.
[2] Barenblatt, G.I. (1996). *Scaling, Self-similarity, and Intermediate Asymptotics*, Cambridge University Press, Cambridge.
[3] Charnov, E.L. (1993). *Life History Invariants: Some Explorations of Symmetry in Evolutionary Ecology*, Oxford University Press, Oxford.
[4] Frantziskonis, G. & Hansen, A. (2000). Wavelet-based multiscaling in self-affine random media, *Fractals* **8**, 403–411.
[5] Jaggi, S., Quattrochi, D.A. & Lam, N.S. (1993). Implementation and operation of three fractal measurement algorithms for analysis of remote-sensing data, *Computers and Geosciences* **19**, 745–767.
[6] Krummel, J.R., Gardner, R.H., Sugihara, G., O'Neill, R.V. & Coleman, P.R. (1987). Landscape patterns in disturbed environment, *Oikos* **48**, 321–324.
[7] Li, B.L. (2000). Fractal geometry applications in description and analysis of patch patterns and patch dynamics, *Ecological Modelling* **132**, 33–50.
[8] Li, B.L., Loehle, C. & Malon, D. (1996). Microbial transport through heterogeneous porous media: Random walk, fractal, and percolation approaches, *Ecological Modelling* **85**, 285–302.
[9] Li, B.L., Wu, H. & Zou, G. (2000). Self-thinning rule: a causal interpretation from ecological field theory, *Ecological Modelling* **132**, 167–173.
[10] Lin, Z.S. & Li, B.L. (2001). Scaling laws for the prey–predator interaction rates, in *Integrative Systems Approaches to Natural and Social Dynamics – Systems Science 2000*, M. Matthies, H. Malchow & J. Kriz, eds, Springer-Verlag, Berlin, pp. 111–118.
[11] Loehle, C., Li, B.L. & Sundell, R.C. (1996). Forest spread and phase transitions at forest–prairie ecotones in Kansas, USA, *Landscape Ecology* **11**, 225–235.
[12] Malcai, O., Lidar, D.A. & Biham, O. (1997). Scaling range and cutoffs in empirical fractals, *Physical Review E* **56**, 2817–2828.
[13] Mandelbrot, B.B. (1983). *The Fractal Geometry of Nature*, W.H. Freeman, New York.
[14] Milne, B.T. (1997). Applications of fractal geometry in wildlife biology, in *Wildlife and Landscape Ecology*, J.A. Bissonette, ed., Springer-Verlag, New York, pp. 32–69.
[15] Milne, B.T. (1998). Motivation and benefits of complex systems approaches in ecology, *Ecosystems* **1**, 449–456.
[16] Pietronero, L. & Tosatti, E., eds (1986). *Fractals in Physics*, North-Holland, Amsterdam.
[17] Ritchie, M.E. & Olff, H. (1999). Spatial scaling laws yield a synthetic theory of biodiversity, *Nature* **400**, 557–560.
[18] Sugihara, G. & May, R.M. (1990). Applications of fractals in ecology, *Trends in Ecology and Evolution* **5**, 79–86.
[19] Takayasu, H. (1990). *Fractals in Physical Sciences*, Manchester University Press, Manchester.
[20] Tchiguirinskaia, I., Lu, S., Molz, J., Williams, T.M. & Lavallee, D. (2000). Multifractal versus monofractal analysis of wetland topography, *Stochastic Environmental Research and Risk Assessment* **14**, 8–32.
[21] Theiler, J. (1990). Estimating fractal dimension, *Journal of the Optical Society of America A: Optics and Image Science* **7**, 1055–1073.
[22] Vicsek, T. (1989). *Fractal Growth Phenomena*, World Scientific, Singapore.
[23] West, G.B., Brown, J.H. & Enquist, B.J. (1997). A general model for the origin of allometric scaling laws in biology, *Science* **276**, 122–126.
[24] Yamasaki, K., Chiba, S. & Nagahama, H. (1999). Geometrical effect of island shape on the species richness, *Fractals* **7**, 353–357.


(*See also* **Criticality, self-organized**; **Point processes, spatial–temporal**; **Population dynamics**)

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