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Self-organized river basin landscapes: Fractal and multifractal characteristics

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Abstract. In recent years a new theory of the evolution of drainage networks and associated landscapes has emerged, mainly in connection with the development of fractal geometry and of self-organized criticality (SOC) concepts. This theory has much improved our understanding of the mechanisms which determine the structure of natural landscapes and their dynamics of evolution. In the first part of this paper the main ideas in the theory of landscape self-organization are outlined, and some remarkable features of the resulting structures are presented. In the second part we apply theoretical tools developed in the context of multifractal fields to the study of the scaling properties of the field of elevations of SOC landscapes. We observe that such landscapes appear to be more complex than simple self-affine fractals, although in some cases a simple fractal framework may be adequate for their description. We also show that multiple scaling may emerge as a result of heterogeneity in the field properties reflecting climate and geology.

1. Introduction

The global frameworks of fractal analysis [*Mandelbrot*, 1983] of fluvial structures [*La Barbera and Rosso*, 1987, 1989; *Tarboton et al.*, 1988; *Rodriguez-Iturbe et al.*, 1992a, b, c; *Rinaldo et al.*, 1992; *Turcotte*, 1992; *Chase*, 1992; *Dietler and Zhang*, 1992; *Ouchi and Matsushita*, 1992; *Lavallée et al.*, 1993; *Rinaldo et al.*, 1993; *Ijjasz-Vasquez et al.*, 1993; *Rigon et al.*, 1993], and of self-organized criticality [*Bak et al.*, 1987, 1988; *Bak and Creutz*, 1993] have been the foundations of a new theory about the evolution of drainage networks and their associated landscapes. It has been shown [*Rodriguez-Iturbe et al.*, 1992a; *Rinaldo et al.*, 1992] that optimal channel networks (OCNs) obtained by minimizing the local and global rates of energy expenditure evolve automatically from arbitrary initial conditions to network configurations exhibiting fractal and multifractal characteristics indistinguishable from those found in nature. Although denditric structures obtained through optimality conditions had been introduced before [*Howard*, 1971, 1990], we believe this was the first time fractality was observed in optimal structures. *Sun et al.* [1994a] have studied the original model of *Rodriguez-Iturbe et al.* [1992a] and *Rinaldo et al.* [1992], corroborating the fractal aspects of the planar structure of networks developed under this framework.

Chance and the rules of optimum energy expenditure are the elements which control the evolution and final structure of the drainage network. Although the particular outcome of an individual process is given by randomness, the interplay of such elements in the extended and dissipative system which is the drainage network results in the same fundamental principles of evolution. The fractal characters of the systems obtained from these principles are found not to be dependent on the initial conditions, thus suggesting that natural fractal structures like river networks may arise as a joint consequence of optimality and randomness [*Rodriguez-Iturbe et al.*, 1992c].

An important step forward was taken by *Rinaldo et al.* [1993], who observed that OCNs are a particular case of self-organized critical structures. The theory of self-organized criticality [*Bak et al.*, 1987, 1988] provides in fact a broad framework for the dynamics of fractal growth and an appealing scheme for the evolution of drainage networks [*Rinaldo et al.*, 1993] which, unlike the theory of OCNs, does not explicitly state a global optimality principle as the guiding evolution concept. Again, the characteristics of the state toward which the system evolves are independent of the initial conditions and are very similar to those observed in nature. In the mentioned scheme [*Rinaldo et al.*, 1993; *Rigon et al.*, 1994], landscapes evolve through the interplay of fluvial (erosive) and hillslope (diffusive) processes. The erosion process acts on the system, maintaining it in a critical state (i.e., a state in which the shear stress due to water flow cannot exceed a critical value), while a special kind of diffusive process is considered as preserving the fractal nature of the elevation field. We believe that this use of the theory of self-organized criticality provides a clear link between the fractal landforms observed in nature and the dynamics responsible for their growth.

The quantitative characterization of self-organized critical (SOC) landscapes and their comparison with natural terrain properties (including power laws in the probability distribution of cumulated areas [*Rodriguez-Iturbe et al.*, 1992b] and of stream lengths [*Tarboton et al.*, 1988; *Rodriguez-Iturbe et*

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al., 1992b], multifractal spectra of new growth site measures, i.e., width functions [*Rinaldo et al.*, 1992, 1993], and scaling relationship of slopes with cumulated area [*Tarboton et al.*, 1989; *Rigon et al.*, 1993]) was the subject of previous papers [*Rinaldo et al.*, 1993; *Rigon et al.*, 1994]. This paper describes a systematic analysis of the fractal properties of the resulting topographies and a comparison with measures of the roughness of natural terrains.

2. A Conceptual Framework for Landscape Self-Organization

We will now briefly recall a general scheme for the evolution of drainage networks in which the structure of the field of elevations organizes itself, evolving through a sequence of critical states [*Rinaldo et al.*, 1993] under the action of fluvial and hillslope processes.

Fluvial erosion may be dynamically described by an equation like $\partial z/\partial t = \alpha f(\tau - \tau_c)$, with $f(x) = 0$ for $x \leq 0$ (where τ is the local shear stress, τ_c is the critical threshold for erosion, and α provides the timescale for the erosion process). The shear stress τ is proportional to $y\nabla z$, where y is the flow depth and ∇z is the local slope. From local optimality principles and empirical evidence we have that $y \propto Q^{0.5}$ [*Rodriguez-Iturbe et al.*, 1992a]. Thus at the ith site, $\tau_i \propto Q_i^{0.5}\nabla z_i$ and whenever $\tau_i > \tau_c$ erosion exists. The local optimality principle also leads to the scaling of slopes with discharge as $\nabla z_i \approx Q_i^{-0.5}$, or, using drainage area as a surrogate of discharge, as $\nabla z_i \approx A_i^{-0.5}$.

Rinaldo et al. [1993] studied the dynamic evolution resulting from local exceedance of an erosion threshold. Briefly, the evolution rules are as follows. Each site i in a two-dimensional lattice has two associated variables, the elevation z_i and the total flow Q_i surrogated by the draining area A_i. An arbitrary initial planimetric network is obtained by choosing the elevations of the landscape randomly or in a systematic way. For each site the shear stress is then computed as $\tau_i \propto A_i^{0.5}\Delta z_i$, Δz being the drop along the drainage direction fixed by the steepest descent (since lattice lengths are equal to one, Δz is equal to the slope). The computed τ_i are compared with a given threshold value τ_c, and the exceedances are identified throughout the lattice. The maximum exceedance is placed, say, at the jth site, and the elevation of this site is reduced to the value which yields $\tau = \tau_c$. Drainage directions are then recomputed because of the modified elevation at the site j. The process is repeated until the system evolves to a critical state in which there are no exceedances of τ above τ_c. This critical state is then perturbed at random by adding elevation to a node. The perturbation may lead to a readjustment of the structure, and this is repeated until further perturbations do not induce variations in the configuration of the system.

It is important to note that the total energy dissipation of the system decreases during the evolution and stabilizes at values very close to those obtained in OCNs, whose dynamics are controlled by the explicit minimization of the total rate of energy expenditure. An explanation of the equivalence of OCNs and self-organized critical networks (SOCs) relies on the equivalence of minimizing the total rate of energy expenditure and the total potential energy of the system [*Ijjasz-Vasquez et al.*, 1993]. In fact, the evolution of self-organized networks leads the system to the state of minimum possible potential energy compatible with the restriction that slopes scale with area as $\nabla z_i \approx A_i^{-0.5}$. The role of randomness is crucial as the perturbations are the driving mechanism that maintains the evolution of the system.

As described above, the evolutionary strategy from any initial condition used by *Rinaldo et al.* [1993] is one in which the maximum exceedance of the threshold shear stress is identified and the elevation of the site is reduced so that the local shear is at the critical value. Two different evolutionary strategies have also been implemented. The first one is a random strategy in which a site is chosen at random among all those τ exceeding τ_c and the elevation is then reduced to make $\tau = \tau_c$. The dynamics then continues as in the original strategy described before. The second strategy is a parallel processing one in which all sites with $\tau > \tau_c$ have their elevations simultaneously updated to make $\tau = \tau_c$. Although conceptually different, the three evolutionary strategies produce self-organized networks with identical fractal and multifractal characteristics. Furthermore, in all cases the total energy expenditure is very close to that obtained for OCNs in similar domains [*Rinaldo et al.*, 1993].

Threshold dependent (TD) processes as in the case of critical shear stress are most important in the carving and evolution of river channels. Nevertheless, often threshold independent (TI) processes act on the formation of the landscape of a river basin, especially at the hillslope scale. The lowering of elevations throughout the basin may in this case be described by a field equation like

$$\frac{\partial z}{\partial t} = \alpha f(\tau - \tau_c) + \beta F(\nabla z). \tag{1}$$

The first term of the right-hand side describes TD fluvial erosion which is zero when $\tau < \tau_c$. The second term models the TI processes which are essentially related to the gradient of the elevation field, ∇z. The TI processes act everywhere, but since $\alpha \gg \beta$ [*Rigon et al.*, 1994] the first term plays the central role whenever erosion forces are acting.

Rigon et al. [1994] present a scheme for landscape self-organization which accounts for the presence of TI processes and is an extension of the one described in the previous section. After the readjustments following a number of perturbations leading to a SOC landscape a diffusion component (TI) is introduced in the scheme. The site i with the largest drop to one of its neighbors j, Δz_{ij}, is identified and its elevation z_i decreased by a quantity proportional to Δz_{ij}. The mass removed from z_i is redistributed among the neighbors j with elevation z_j lower than z_i. Mass redistribution is done according to

$$z_j^{\text{new}} - z_j = \nu \frac{z_i - z_j}{\sum_{k \in nn} (z_i - z_k)} \qquad 0 < \nu < 1, \tag{2}$$

where $k \in nn$ indicates that the sum is over the nearest neighbors lower than i. The points draining into i change their ∇z while the points j change both z and ∇z, and thus a whole set of adjustments occurs following a TI activity at any point. The above procedure produces a lowering of the slopes which results in smoother hillslopes and gentler landscapes when the iterations continue in time. Indeed, without the TI processes the hillslopes of the resulting landscapes look unrealistically rough or at least highly

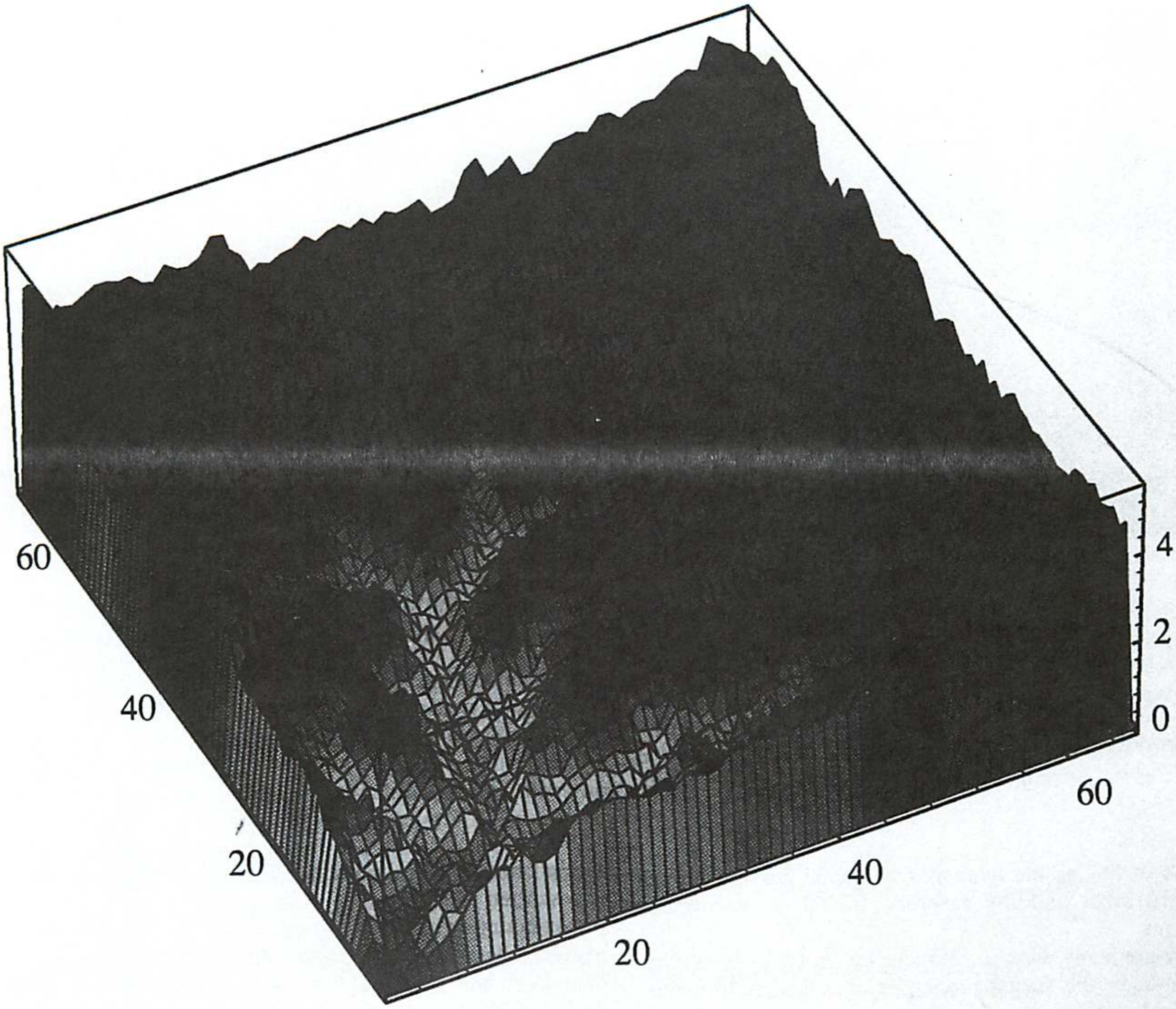

Plate 1. Self-organized landscape resulting from both TD and TI processes [after *Rigon et al.*, 1994].

uncommonly so. One of the relevant effects of TI processes is the lowering of the fractal dimension of the landscape [*Rigon et al.*, 1994]. An example of the effect of repeated cycles of TD and TI processes is shown in Plate 1. Both ν and the length of the iterative process are important, and through their effects a large variety of landscapes can be obtained.

3. Fractal and Multifractal Descriptors of Landscapes

Let us consider the asymptotic field of elevations $z(\mathbf{x}, t = \infty) = z(\mathbf{x})$ produced by the above processes, defined at every point $\mathbf{x}$ in a subset of $\mathbb{R}^2$. If $z(\mathbf{x})$ has the property

$$z(\mathbf{x}_1 + \lambda \mathbf{e}) - z(\mathbf{x}_1) \overset{d}{=} \lambda^H [z(\mathbf{x}_2 + \mathbf{e}) - z(\mathbf{x}_2)], \quad (3)$$

where $\mathbf{e}$ is a unit vector, λ is a separation distance, $\mathbf{x}_1$, $\mathbf{x}_2$ are two arbitrary points, and the d above the equals sign indicates equality in distribution (the exponent H is called the Hurst exponent [*Mandelbrot*, 1983; *Feder*, 1988]), it is said to be simple scaling [*Lavallée et al.*, 1993]. The unique fractal dimension D characterizing the simple scaling field is related to Hurst exponent H via the relationship $D = 3 - H$ [*Feder*, 1988]. It is important to note that a simple scaling field is postulated to have constant statistical characteristics with respect to position and orientation. Equation (3) holds, in fact, for any choice of $\mathbf{x}_{1,2}$ and $\mathbf{e}$.

Many examples of fractal analyses of topography and bathymetry measured along a linear track can be found in the literature [e.g., *Malinverno*, 1989; *Turcotte*, 1992] documenting the fractal nature of natural landscapes. Most of these studies are based on monofractal (or, simple scaling) models that do not seem entirely consistent with the properties of measured field data. Recent contributions [*Schertzer and Lovejoy*, 1989; *Lavallée et al.*, 1993] give a different interpretation of the fractal characters observed in real topographies, arguing that geographical fields are generally multifractal and that inconsistencies are inevitable when these entities are forced into narrower geometric frameworks involving a single fractal dimension. In view of these considerations the characteristics of the SOC landscapes introduced in the previous sections will be analyzed with the

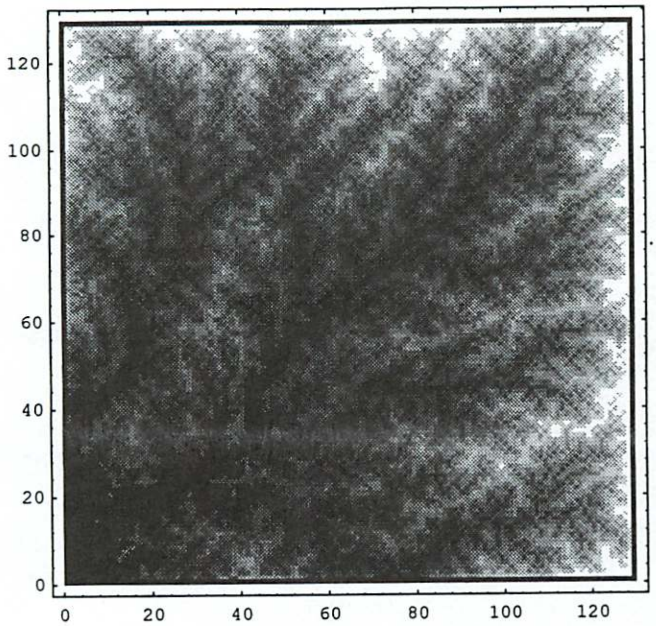

Figure 1. Plot of the set of points S that belong to predetermined ranges of elevations. Isolines are identified at different heights for an elevation field resulting from TD processes only. The solid area refers to the set $S_{\leq 2}$, i.e., the set of points whose elevation is lower than 2 ($z(\mathbf{x}) \leq 2$). Different levels of shading indicate sets S where $2 \leq z(\mathbf{x}) \leq 3$ and so on.

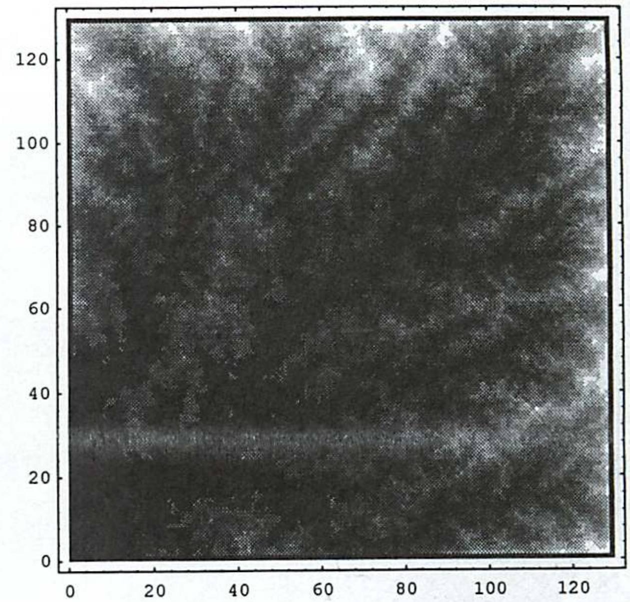

Figure 2. As in Figure 1, for the elevation field in Plate 1, where both TD and TI processes were allowed to occur.

purpose of linking the dynamics of fractal growth through SOC processes and the resulting fractal or multifractal structures.

There are many ways to characterize rough surfaces. One way to study the fractal properties of a field is to define, through the introduction of a threshold value of elevation T, sets that have theoretically determined characteristics with respect to fractal geometry. One such set is the exceedance set $S_{\geq T}$, introduced as the set of all the points $\mathbf{x}$ satisfying $z(\mathbf{x}) \geq T$. Likewise, one can define the perimeter set P_T as the subset of the points of $S_{\geq T}$ having at least one nearest neighbor belonging to the set $S_{<T}$. In general, since P_T is a subset of $S_{\geq T}$, then $D(P_T) \leq D(S_{\geq T})$. In the case of a simple scaling structure the sets $S_{\geq T}$ are characterized by a single fractal dimension $D(S_{\geq T}) = D$ while $D(P_T) = 2 - H$, both values being independent of T [*Lavallée et al.*, 1993]. These properties provide a means of checking whether simple scaling arises for $z(\mathbf{x})$.

Figure 1 shows the sets of points which belong to different ranges in elevation, from which we have obtained the above sets S defining the elevation field $z(\mathbf{x}, t_\alpha)$ for a SOC landscape obtained when only TD actions are operating, i.e., whose field equation is

$$\frac{\partial z}{\partial t} = \alpha f(\tau - \tau_c). \qquad (4)$$

The initial condition $z(\mathbf{x}, 0)$ was obtained by *Eden* [1961] growth [*Rinaldo et al.*, 1993]. Here $\tau_c = 1$ on a 128×128 lattice. Elsewhere [*Rigon et al.*, 1994], we have shown that the statistical features of the landscape are independent of the threshold value τ_c. Figure 2 shows contour lines of the landscape $z(\mathbf{x}, t_\beta = \infty)$ shown in Plate 1, obtained by a cycle of TI processes defined by the field equation

$$\frac{\partial z}{\partial t} = \beta F(\nabla z), \qquad (5)$$

acting from the initial condition $z(\mathbf{x}, t_\alpha)$. The resulting field is described in Figure 1.

Box counting has been applied to the graphs in Figures 1 and 2 to determine the fractal dimension of each contour line. For each contour, box-counting [*Mandelbrot*, 1983; *Feder*, 1988] data are represented reasonably well (Figure 3) by a straight line whose slope is the fractal dimension $D(P_T)$ in a log-log plot of the number of boxes $N(\delta)$ that cover the whole contour versus the box size δ. This indicates that any contour of a self-organized landscape may be considered a self-similar fractal. The main results of the analyses are illustrated in Table 1. We note that the fractal dimensions of the isolevel lines obtained for different contour elevations

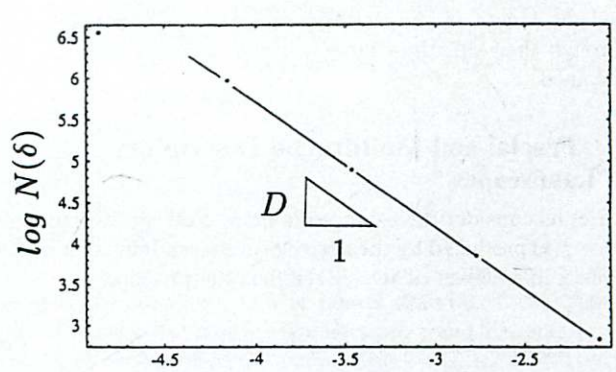

Figure 3. Typical dependence of $N(\delta)$ on δ found in the box-counting procedure applied to one of the isolines of Figure 1 (TD processes only).

Table 1. Elevations, Fractal Dimensions (Box Counting) of the Border Sets, and Correlation Coefficients of Box-Counting Scaling for TD and TI SOC Landscapes

Contour Elevation $z(\mathbf{x}) = T$	TD Actions		TI Actions	
	$D(P_T)_{TD}$	r	$D(P_T)_{TI}$	r
1	1.38	0.99	1.30	0.99
1.5	1.42	0.96	1.30	0.98
2	1.43	0.98	1.30	0.98
2.5	1.43	0.94	1.29	0.97
3	1.39	0.96	1.26	0.97
3.5	1.37	0.99	1.22	0.96
4	1.35	0.92	1.21	0.94

are slightly different. Similar results were observed for the optimal topographies obtained by *Sun et al.* [1994b] by enforcing an artificial perfect scaling of slopes to an optimal channel network [*Rodriguez-Iturbe et al.*, 1992b, c; *Rinaldo et al.*, 1992]. The contours near the average height have very similar fractal dimensions, and, overall, the surfaces produced seem to show the signs of a relatively complex self-affine fractal. However, in the above analysis, sample size effects are important both at the higher and at the lower elevations, thus partially impairing the related results. In the range of elevations where the sample size is constant we observe a consistent behavior analogous to the topographies obtained by *Sun et al.* [1994b]. Moreover, we will show later that other analyses suggest that multiple scaling does not occur in the topographic field obtained via SOC procedures with space-constant field properties. Fast Fourier transform analyses [e.g., *Cochran et al.*, 1967; *Feder*, 1988; *Malinverno*, 1989], roughening measurements [e.g., *Mandelbrot*, 1983; *Feder*, 1988; *Matsushita and Ouchi*, 1989; *Dubuc et al.*, 1989a, b] and height-height correlation measurements [e.g., *Mandelbrot*, 1983; *Feder*, 1988; *Sun et al.*, 1994a, b] also agree with the results shown by the contours near the average elevation.

Figure 4 shows the log-log plot of the power spectrum $\langle S(k)\rangle$ (ensemble-)averaged over all south-north (S-N) and east-west (E-W) topographic profiles obtained from vertical cuts (or transects) through a SOC landscape produced by TD actions (equation (4)). The data can be fitted by a straight line

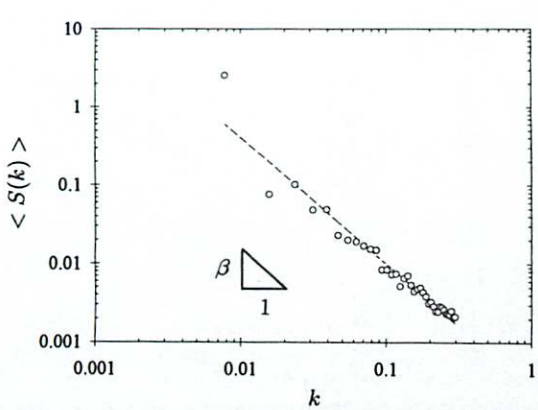

Figure 4. Power spectrum averaged over all S-N and E-W transects of the TD 128 × 128 SOC landscape considered. The slope in the log-log plot is $-\beta = -1.6$, yielding $D = 1.7$ for the (mean) fractal dimension of the transects.

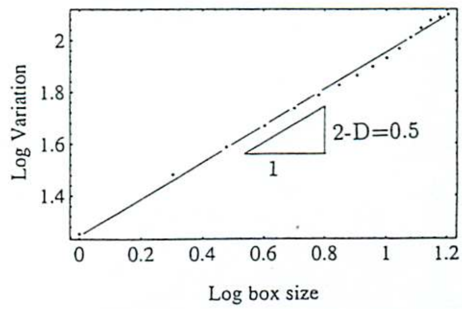

Figure 5. Results of the variation method analysis applied to a transect of the TD 128 × 128 SOC landscape considered. Averaging over all transects gives $D = 1.65$ for the mean fractal dimension [after *Rigon et al.*, 1994].

(i.e., $S(k) \propto k^{-\beta}$) with slope $-\beta$ ($\beta = 1.6$) giving, as shown later, $D = (5 - \beta)/2 = 1.7 \pm 0.1$ [*Feder*, 1988].

Figure 5 shows the results of the variation method analysis [*Dubuc et al.*, 1989a, b] on a single vertical cut of the surface of a TD 128 × 128 SOC landscape with $\tau_c = 1$. The range variation of the height z within a box of size L scales as proportional to L^H ($D = 2 - H$) if the object is simple scaling. Once ensemble averaging is performed, the resulting fractal dimension is $D = 1.65 \pm 0.05$.

The analysis of mean deviation, first introduced for self-affine fractals by *Matsushita and Ouchi* [1989], studies the deviation $\sigma^2(L)$ as a function of the horizontal length scale L as

$$\sigma^2(L) = \left\langle \frac{1}{L^2} \sum_{i,j=1}^{L} (z_{ij} - \langle z\rangle_L)^2 \right\rangle, \qquad (6)$$

where z_{ij} is the elevation at the lattice sites $\mathbf{x} = (i, j)$ and $\langle z\rangle_L$ is the average height for the surface within a grid of size $L \times L$. For a self-affine fractal, σ is related to L by a power law $\sigma \propto L^H$. Figure 6 shows the log dependence of σ on L from which we derive $D = 1.66 \pm 0.05$. We have repeated all the above analyses for the TD + TI processes described in (1) leading to the landscape in Plate 1 and have obtained

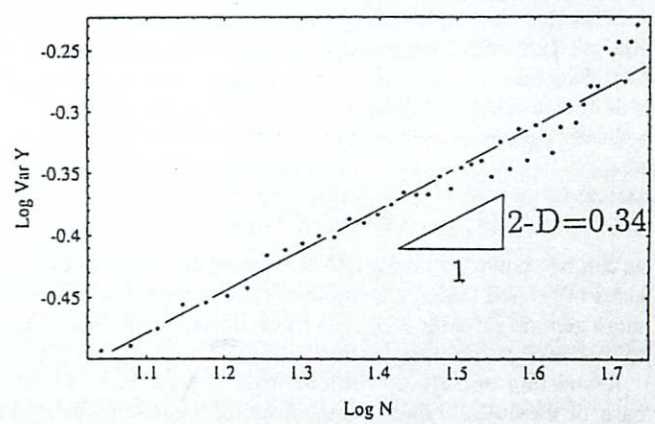

Figure 6. Dependence of mean deviation $\sigma(L)$ on the horizontal length scale L obtained applying the *Matsushita and Ouchi* [1989] analysis to the TD 128 × 128 SOC landscape considered. This procedure gives the value $D = 1.66$ for the mean fractal dimension [after *Rigon et al.*, 1994].

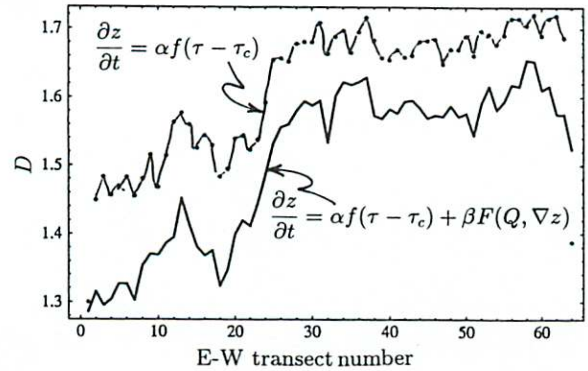

Figure 7. Values of the fractal dimension for different E-W transects of TD and TD + TI topographies.

consistent results for the mean fractal dimensions. Nevertheless, we observed a nonnegligible range in the individual fractal dimensions of the vertical cuts of the topography and in the contour lines at different elevations. Figure 7 shows a plot of the fractal dimensions for different E-W one-dimensional tracks of the topographies of TD and TD + TI topographies. One notices that the self-affine characteristics of the evolving surface rendered by the TI type of self-organization are less rough. The above discussion suggested that the complexity of the self-affine fractal surface needed further analysis.

There are other ways to determine the scaling nature of a field, namely those related to the definition of generalized variograms [*Lavallée et al.*, 1993], which are essentially a generalization of height-height correlation analyses. The qth moment of a range, or generalized variogram, $C_q(\lambda)$, is defined as

$$C_q(\lambda) = \langle |z(\mathbf{x}) - z(\mathbf{x} + \mathbf{r})|^q \rangle_{|\mathbf{r}|=\lambda}, \tag{7}$$

where the angle brackets are the ensemble averaging operator and λ is the separation distance. The qth moment will be said to be scaling if the following relation holds:

$$C_q(\lambda) = \lambda^{K(q)} C_q(1) \propto \lambda^{K(q)}. \tag{8}$$

It is worth noting that the definition of $C_q(\lambda)$ given above implies that the random field considered is stationary, since the qth moment is supposed to depend only on the separation distance, or lag, λ. The implications of this hypothesis will be considered in greater detail later on in this paper.

In the case of simple scaling the exponent $K(q)$ is linear, i.e.,

$$K(q) = qH, \tag{9}$$

as can be shown by taking qth powers of the moduli of both sides of (3) and taking ensemble averages. In multifractals a more general form for $K(q)$ has to be assumed: We will refer to this case as multiple scaling.

Restricting ourselves, without loss of generality, to the case of elevation values sampled along a one-dimensional track, we will consider the relation between the form of the power spectrum obtained from the Fourier transform of the data and the exponent $K(q)$ governing the scaling of the moments. It is known that for a series whose graph is self-affine the power spectrum has the form of a power law, i.e., $E(k) \propto k^{-\beta}$, in a suitably chosen interval for the frequency k. The Wiener-Khinchine theorem states that the power spectrum is the Fourier transform of the autocorrelation function $R(\lambda) = \langle z(x + \lambda) z(x) \rangle$ (a function of the lag only, for stationary random processes), which also needs be power law as $R(\lambda) \propto \lambda^{\beta-1}$ [*Falconer*, 1985] for a self-affine graph $z(x)$. The autocorrelation function can, in turn, be related to the second order moment $C_2(\lambda)$ via

$$\langle |z(x + \lambda) - z(x)|^2 \rangle = 2\langle z(x)^2 \rangle - 2R(\lambda), \tag{10}$$

giving $\beta = 1 + K(2)$. In the case of simple scaling $K(2) = 2H$ and $\beta = 1 + 2H$. Since the fractal dimension of the "profile" of $z(x)$ is $D = 2 - H$, we have $D = (5 - \beta)/2$. These relations are commonly used in evaluating the fractal dimension of a two-dimensional profile and can be useful in determining the fractal or multifractal nature of such a profile. *Lavallée et al.* [1993] have contended, in fact, that since for multifractals $K(2) \neq 2K(1) = 2H$, the above relation between D and β will not hold in virtually all applications to real geophysical profiles. In fact, to justify $D = 2 - H$ we only need $C_1(\lambda) = \lambda^H C_1(1)$, while $D = (5 - \beta)/2$ also requires that $C_2(\lambda) = \lambda^{2H} C_2(1)$.

We have applied the analysis of moments to the SOC landscapes introduced earlier, computing qth moments from averages along both S-N and E-W transects. This amounts to considering the field as isotropic and transects as realizations of a (stationary and ergodic) random process. Statistical parameters of the process itself can thus be computed on these realizations. The scaling properties of the moments obtained from the usual 128 × 128 TD landscape are shown in Figure 8. The related $K(q)$ versus q graph is shown in Figure 9. It can be seen that the moments follow a scaling behavior remarkably well and that an almost perfect simple scaling behavior is exhibited by the $K(q)$ versus q curves. The exponent that characterizes the simple scaling behavior was found to be $H = 0.303 \pm 0.001$ in the case of TD processes, and $H = 0.368 \pm 0.004$ for TD + TI processes, showing that TI processes are responsible for a lowering of the fractal dimension of landscapes (recall that $D = 2 - H$) [*Rigon et al.*, 1994]. These facts suggest that transects from SOC elevation fields obtained through TD and TI actions can be considered as realizations of a stationary random process and that, as such, their study is relevant to the fractal characterization of the fields themselves. The results ob-

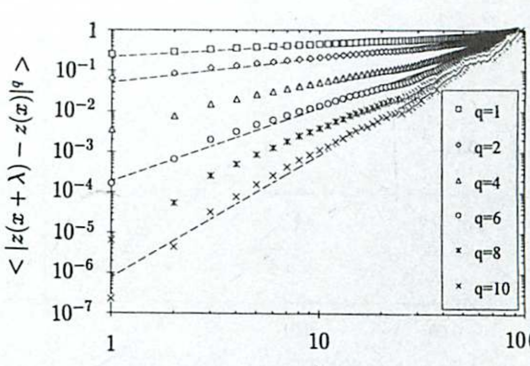

Figure 8. Scaling of qth moments in a SO elevation field on which only TD processes were allowed to occur. The lowest correlation coefficient was found to be $R = 0.99$.

tained through the study of qth moments are consistent with our previous fractal analysis of various contour lines of elevation taken from the same SOC landscapes as considered here.

4. Discussion

We have studied a 2200-km^2 portion of the digital terrain map (DTM) of northern Italy produced by Servizio Geologico Nazionale (pixel size: 250×310 m^2) to test the scaling properties of real topographies. The results of the literature [*Lavallée et al.*, 1993] and of our computations (Figure 10) show, quite convincingly, that the natural fields studied indeed show the signatures of multiscaling, i.e., a nonlinear relation between $K(q)$ and q. This fact implies that the geometric properties of natural landscapes in general may not be described by specifying a single fractal dimension.

We suggest that multiscaling may be a byproduct of heterogeneity in the erodibility of the soil mantle of natural geological formations as, in nature, this property strongly varies not only with position on the surface but also with depth, i.e., with the degree of erosion achieved. To show this, we will model heterogeneity through a nonuniform spatial distribution of the field parameters. The field equation is in this case

$$\frac{\partial z}{\partial t} = \alpha f(\tau - \tau_c(\mathbf{x})) \qquad (11)$$

and the spatial variability of τ_c considered first is that of a random function with a correlation structure [*Rigon et al.*, 1994]. The results of the generalized covariance analysis applied to the SOC procedure (obtained with a lognormal field $\tau_c(\mathbf{x})$ in (11) with $\langle \tau_c \rangle = 1$, $\sigma_\tau^2 = 0.2$, and exponential correlation structure with integral scale equal to 2 pixels [*Rigon et al.*, 1994]) are shown in Figure 9. The field chosen is known [*Rigon et al.*, 1994] to produce statistics that match those observed in nature. The effects of such "mild" heterogeneities result in significantly rougher landscapes (H = 0.280 ± 0.002 from Figure 9) whose scaling structures may still (although not conclusively) be considered simple. The linear relation between $K(q)$ and q is, in fact, less closely attained than in the case of homogeneous fields, suggesting that departures from simple scaling behavior may take place if a higher degree of heterogeneity be considered.

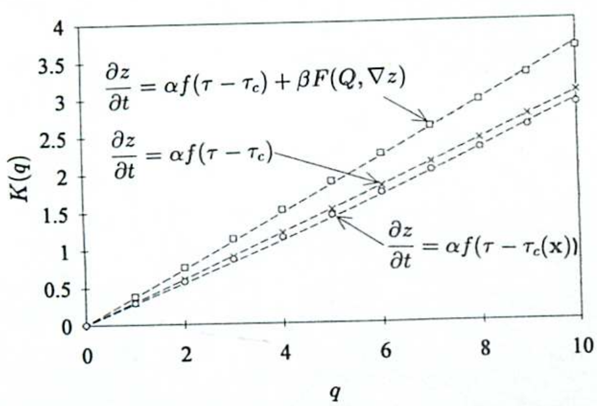

Figure 9. $K(q)$ versus q curves for SOC landscapes resulting respectively from TD, TD + TI, and heterogeneous TD + TI processes. A linear best fit (with zero intercept) has been plotted to make simple scaling behavior more evident. The exponent of the simple scaling relation was found to be $H = 0.303 \pm 0.002$ for TD effects, $H = 0.368 \pm 0.004$ for TD + TI actions, and $H = 0.280 \pm 0.002$ in the case of heterogeneous TD + TI processes.

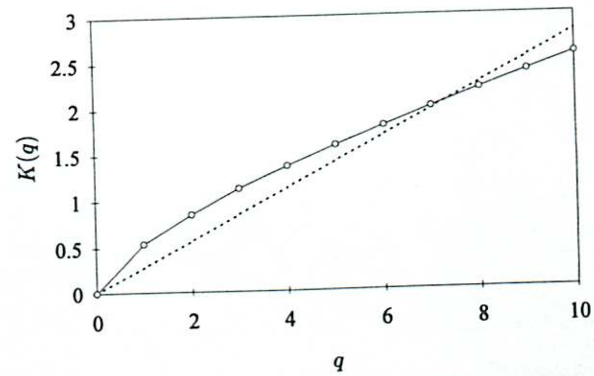

Figure 10. $K(q)$ versus q curve for the natural elevation field in northern Italy considered. A linear best fit (with zero intercept) has been plotted to visualize the multiple-scaling properties evidenced by the field.

To test the effects of greater heterogeneities, the TD erosion algorithm expressed by (11) was run on a 64×64 lattice characterized by a highly bimodal distribution of the critical shear stress. The field chosen is divided along the diagonal through the outlet, and the chosen values for the critical shear stress are $\tau_c = 2$ (upper half) and $\tau_c = 0.5$ (lower half). The elevation field, shown in Plate 2, was analyzed in the usual manner. The form of the moments and the $K(q)$ versus q relation are shown in Figures 11 and 12, respectively. It can be seen that moments match a power law scaling behavior (the minimum correlation coefficient of the linear regression being $R = 0.96$) and that multiscaling is revealed by the form of $K(q)$.

Thus SOC elevation fields obtained through TD and TI processes with spatially homogeneous characteristics are suggested to exhibit simple scaling and statistical stationarity of the increments when sampled along coordinate directions. The single fractal dimensions characterizing each of the resulting fields is in the range observed for the roughness of natural landscapes. A subtle interplay of nonstationarity of the increments of the field values and multiple scaling has been observed. In fact, landscapes self-organization with an underlying constant threshold τ_c in (1) implies nonstationarity in the increments of elevation when sampled along the flow directions, due to scaling of slopes with cumulated drainage area [*Tarboton et al.*, 1989; *Gupta and Waymire*, 1989; *Rinaldo et al.*, 1993; *Rigon et al.*, 1994]. Thus it is the sampling procedure along coordinate axes (i.e., along E-W and N-S tracks) that naturally does not detect the nonstationarity induced by the network structure, and a global stationarity of increments results (Figure 8).

A spatially variable, statistically stationary, threshold field (Figure 9) creates rougher but still simple scaling structures, at least within the sensitivity limits of the analyses performed. Multiscaling clearly occurred only when a nonstationary threshold field (e.g., bimodal) had been employed. Care should therefore be exerted in evaluating the presence

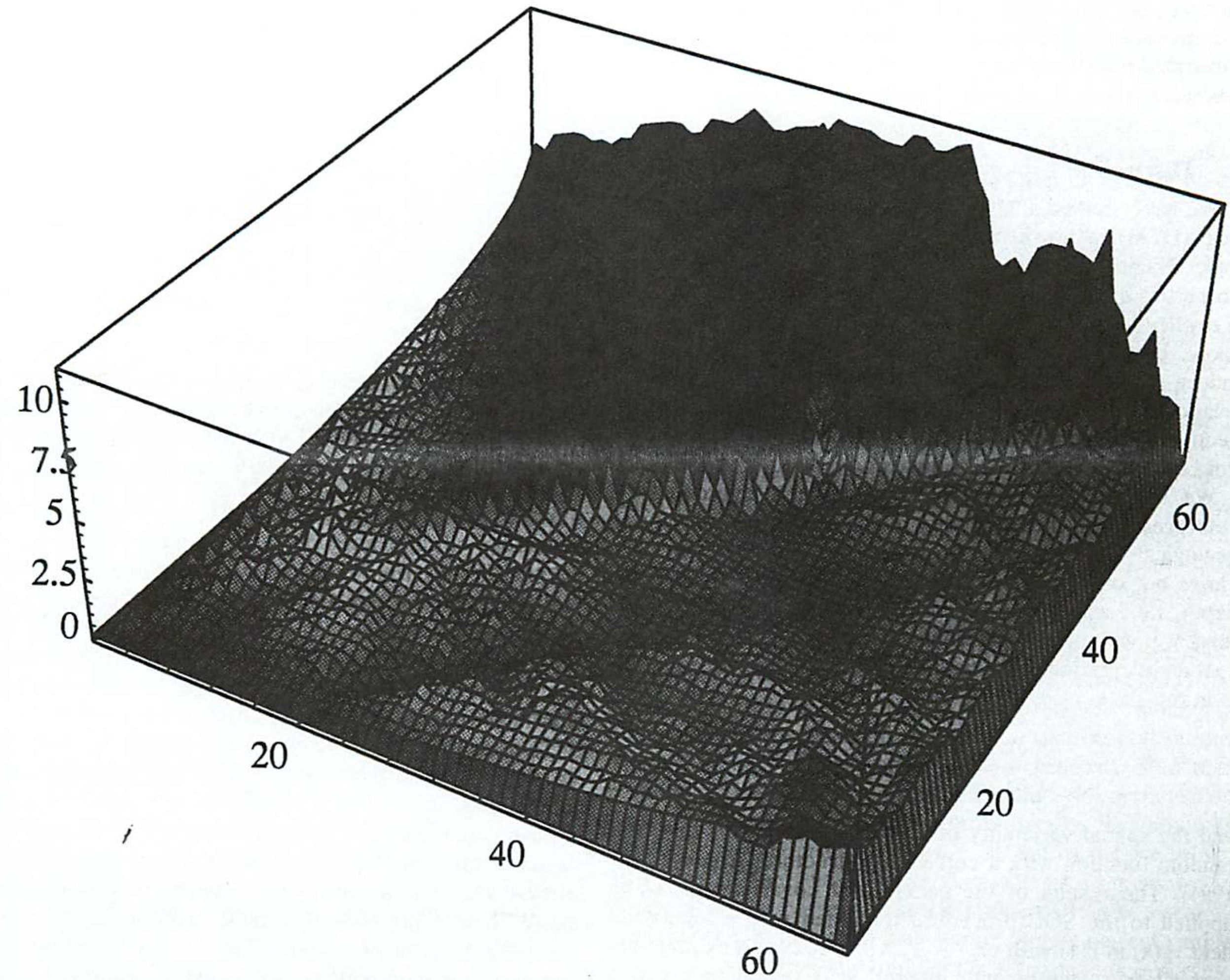

Plate 2. TD SOC landscape resulting from a bimodal distribution of the critical shear stress.

of multiscaling when using only tools that postulate stationarity.

Thus multiple-scaling behavior experimentally found in natural landscapes cannot be explained in terms of spatially homogeneous processes. We have shown that multifractality occurs in a particular case of spatially variable field properties, reflecting natural heterogeneity in the interplay of exposed lithology of the soil mantle, geology, climate, etc. This suggests that heterogeneity of field properties is needed for multiple scaling to emerge.

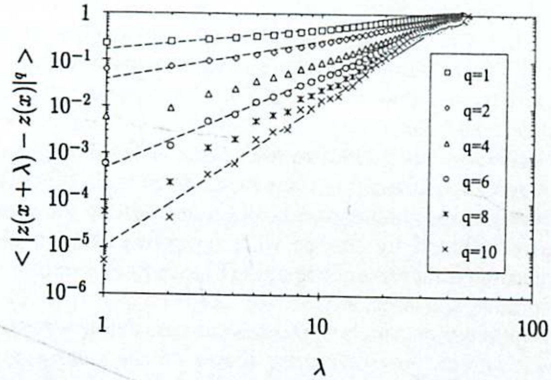

Figure 11. Scaling of qth moments for the SOC landscape shown in Plate 2. The lowest correlation coefficient was found to be $R = 0.96$.

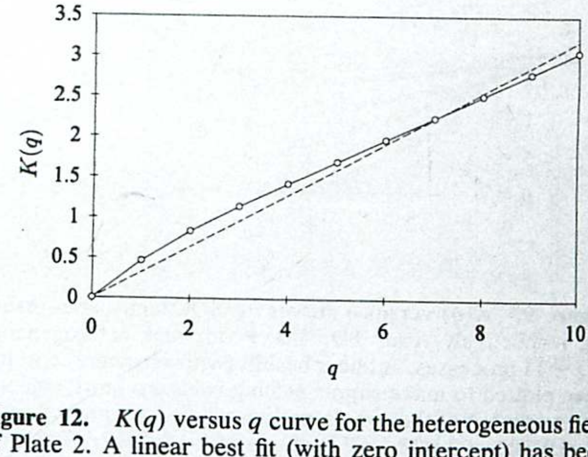

Figure 12. $K(q)$ versus q curve for the heterogeneous field of Plate 2. A linear best fit (with zero intercept) has been plotted, as in Figure 10, to visualize the multiple-scaling properties evidenced by the field.

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