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Fractal Relation of Mainstream Length to Catchment Area in River Networks

35 citations

Renzo Rosso and Baldassare Bacchi

Institute of Hydraulics, Politecnico di Milano, Italy

Paolo La Barbera

Institute of Hydraulics, University of Genoa, Italy

Mandelbrot's (1982) hypothesis that river length is fractal has been recently substantiated by Hjelmfelt (1988) using eight rivers in Missouri. The fractal dimension of river length, d, is derived here from the Horton's laws of network composition. This results in a simple function of stream length and stream area ratios, that is, $d = \max(1, 2 \log R_L/\log R_A)$. Three case studies are reported showing this estimate to be coherent with measurements of d obtained from map analysis. The scaling properties of the network as a whole are also investigated, showing the fractal dimension of river network, D, to depend upon bifurcation and stream area ratios according to $D = \min(2, 2 \log R_B/\log R_A)$. These results provide a linkage between quantitative analysis of drainage network composition and scaling properties of river networks.

Introduction

The relation of basin area A to the length L of the water ›urse is one of the major geomorphologic characteristics of ainage basins. Dimensional analysis indicates that this lation should be

$$L \sim A^{1/2} \tag{1}$$

r geometrically similar catchments. *Mesa and Gupta* 987] showed that this relationship also held for river tworks by using the random model postulates and the lditional assumption that link lengths have a common ponential probability distribution.

If the relationship between L and A is taken in the form

$$L \sim A^{\alpha} \tag{2}$$

ıere α is a fitted exponent, empirical studies yield estiated α values larger than 1/2, as predicted by (1). *Hack* 957] first demonstrated the applicability of the power ınction relating length and area, and his data and those of *ıngbein* [1947] satisfied (2), with $\alpha = 0.6 > 1/2$. *Gray* [1961] ther showed that his data and those of *Taylor and hwartz* [1952] satisfied (2), with $\alpha = 0.568 > 1/2$. On the sis of a stratified sample of 250 basins, *Mueller* [1973] served that $\alpha = 0.554 > 1/2$ over the entire population of 0 basins, ranging in size from 10^{-2} to 10^5 km^2. On the sis of the same data, *Mesa and Gupta* [1987] observed that emains rigid at 0.6 for basins less than 2×10^4 km^2, nges suddenly to 1/2 for basins between 2×10^4 and 2.5 10^5 km^2, and finally drops to 0.47 for larger basins. erefore $\alpha = 1/2$ tends to overpredict the mainstream ıgth for smaller basins, and it tends to underpredict for ger basins [e.g., *Eagleson*, 1970, Figure 16-6].

One could observe that the difference in exponents may be e to lack of similarity. However, *Mandelbrot* [1982] suggested that this difference can be due to measuring river lengths on maps of different scales. Assuming $\alpha > 1/2$, *Mandelbrot* [1982] further indicated that mainstream length should be viewed as a fractal measure with dimension $d = 2\alpha$. This result has been recently substantiated by *Hjelmfelt* [1988] using eight rivers in Missouri.

The fractal properties of rivers can be related to quantitative analysis of river networks, following the set of empirical laws collectively referred to as Horton's laws of network composition. These are geometric-scaling relationships, since they hold no matter at what order or resolution the network is viewed, and they yield self-similarity of the catchment-stream system. Although systematic deviations have been observed between Horton's laws and empirical observations for "large" basins [see *Abrahams*, 1984; *Andah et al.*, 1987*b*], these laws generally hold quite exactly for a wide range of scales in nature. *La Barbera and Rosso* [1987, 1989], *Feder* [1988], *Tarboton et al.* [1988], and *Nikora* [1989] developed this approach to investigate fractal geometry of river networks.

On the basis of heuristic arguments, *Feder* [1988] recently indicated that the fractal dimension of river length should be related to the bifurcation and stream length ratios of the river network, R_B and R_L, respectively. Under the assumption that the length of overland flow is about the same for all streams [*Hack*, 1957], *Feder* [1988] derived

$$d = 2 \log R_L/\log R_B \tag{3}$$

D_*

Unfortunately, (3) is not able to provide satisfactory estimates of d in many cases; this is demonstrated in the third section of this paper. Moreover, the above mentioned underlying assumption yields drainage density to be constant with area; this contrasts with the statement that "drainage density should vary inversely as the drainage area, other things equal" [*Horton*, 1945, p. 294].

In the present paper the above assumption concerning overland flow length is relaxed, and the fractal dimension of river length is derived analytically from the Horton's laws of network composition. This derivation is reported in the next

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section, and the result differs from (3). This result is shown to improve our capability of relating fractal geometry of river networks to their scaling properties, as represented by Horton's laws. For this purpose, data analysis for three case studies is reported in the third section of this paper. Some further implications in river geomorphology are finally discussed in the fourth section of this paper; these deal with the evidence of fractal relationships between Horton's ratios and the estimation of the fractal dimension of a river network as a whole. A short survey of Horton's law as related to the fractal nature of river networks is also reported in the appendix, to which any pertinent issue must be referred.

Fractal Dimension of Mainstream Length

As introduced above, the fractal nature of river length and catchment area measurements has been recently investigated by *Hjelmfelt* [1988] through map analysis. After analyzing eight rivers in Missouri by means of the ruler method [*Richardson*, 1961], *Hjelmfelt* [1988] found the fractal dimensions for length measurements to vary from 1.036 to 1.291, with an average value of 1.158. This is close to the value of 1.136 hypothesized by *Mandelbrot* [1982] under the assumption that

$$d = 2\alpha \qquad d = D\alpha \qquad (4)$$

where d denotes the fractal dimension of the length of the main water course.

If one denotes with ε the length scale used to measure the geometrical features of a basin, the relation of river length $L(\varepsilon)$ to catchment area $A(\varepsilon)$ can be investigated using the general length-area relationship due to *Mandelbrot* [1982]; this is given by

$$L(\varepsilon) = C\varepsilon^{1-d}[A(\varepsilon)]^{d/2} \qquad (5)$$

you can't assume this

with C denoting a constant of proportionality. We assume here that the river network is described by the Horton's laws of network composition (see the appendix), and we further assume that Horton's order ratios are independent of the length scale. Accordingly, (5) yields

$$L_\Omega(\varepsilon) = C\varepsilon^{1-d}[A_\Omega(\varepsilon)]^{d/2} \qquad (6)$$

for a Ωth order basin, with

$$L_\Omega(\varepsilon) = \sum_{k=1}^{\Omega(\varepsilon)} l_k(\varepsilon) = l_1(\varepsilon)[R_L^{\Omega(\varepsilon)} - 1]/(R_L - 1) \qquad (7)$$

$$A_k(\varepsilon) = A_1(\varepsilon)R_A^{\Omega(\varepsilon)-1} \qquad (8)$$

where

- l_k average length of kth order streams, L;
- A_k average drainage area tributary to kth order streams, L^2;
- R_L stream length ratio, dimensionless;
- R_B bifurcation ratio, dimensionless;
- Ω Strahler's order of the basin, dimensionless.

By substituting (7) for $L_\Omega(\varepsilon)$, and (8) for $A_k(\varepsilon)$ in (6), respectively, one gets

$$d = \frac{\log[l_1(\varepsilon)] + \log\{[R_L^{\Omega(\varepsilon)} - 1]/(R_L - 1)\} - \log C - \log\varepsilon}{\log[A_1(\varepsilon)] + [\Omega(\varepsilon) - 1]\log R_A - \log\varepsilon} \qquad (9)$$

In the case where a first-order basin is examined, (6) gives

$$C = l_1(\varepsilon)\varepsilon^{d-1}[A_1(\varepsilon)]^{-d/2} \qquad (10)$$

which can be substituted for C in (9) to obtain

$$d = 2\,\frac{\log\{[R_L^{\Omega(\varepsilon)} - 1]/(R_L - 1)\}}{[\Omega(\varepsilon) - 1]\log R_A} \qquad (11)$$

The concept of fractal dimension is only properly defined when using an asymptotic limit to infinitely small lengths [*Jullien and Botet*, 1987; *Feder*, 1988]. Therefore it is crucial that self-similarity still holds when proceeding downward, that is, from the higher to the lower levels of scale. *La Barbera and Rosso* [1990] associated the physical meaning of this limit with the propagation of channels further up into the drainage basin with decreasing map scale and used this assumption to derive the fractal dimension of a river network, D, as the scaling exponent of the total length of streams in the river network with varying scale (see the appendix). The same assumption can be used here to derive the fractal dimension of mainstream length.

what madness is this?

Taking the limit of the left side of (11) for ε tending to 0 therefore yields Ω tending to infinity; one thus obtains

$$d = \max(1, 2\log R_L/\log R_A) \qquad (12)$$

which provides the fractal dimension of the mainstream length of a river as a function of the stream length and the stream area ratios of a basin. However, *Nikora* [1989] indicated that self-similarity occurs in nature, and "a fortiori" in hydrological objects, only within a certain range of scales. Therefore the above derivation should be regarded as a mathematical description which applies to the range of scales associated with Horton's laws of network composition. This range could be assessed from those investigations on the erosional development of drainage networks which have been carried out by means of field observations [*Montgomery and Dietrich*, 1988, 1989], laboratory experiments [*Sawai et al.*, 1986; *Shumm et al.*, 1987], and mathematical models [*Smith and Bretherton*, 1972; *Cordova et al.*, 1982; *Roth et al.*, 1989].

Data Analysis

In Table 1 we report the estimates of Horton's order ratios for five basins in Italy. Equation (12) has been used to derive the fractal dimension of river length for the mainstream of these drainage basins. This result has been compared with the measure of d which has been obtained from map analysis using the box-counting method [*Hentschel and Procaccia*, 1983; *Parker and Chua*, 1989]. This method provides the value of d as the slope of the straight line which is fitted to the log-transformed pairs of observed box number and box size values (see Figure 1). From the results reported in Table 1 it can be observed that the estimates of d obtained from (12) fit the measured values quite satisfactorily. Therefore equation (12) seems to be able to provide quite reliable estimates of the fractal dimension of these rivers, as compared to the ones which can be obtained from map analysis.

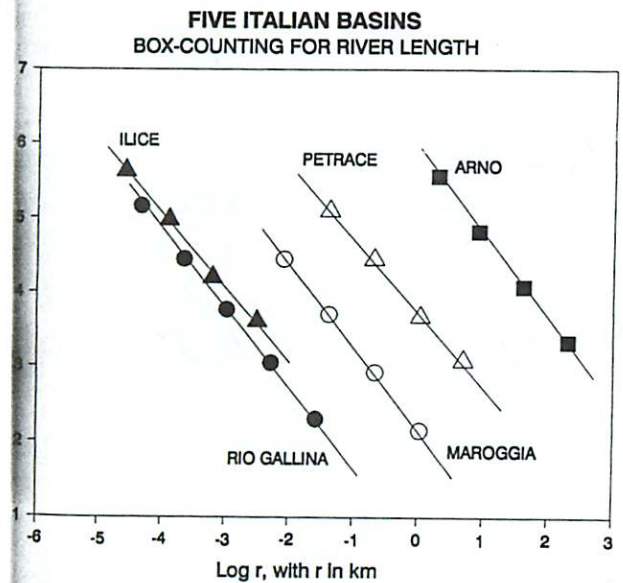
TABLE 1. Fractal Dimension of River Length for Five Basins in Italy: Comparison Between Box-Counting Measures and the Estimates From Horton's Order Ratios

River	Area, km^2	Horton Order Ratios			Fractal Dimension, d	
		R_B	R_L	R_A	Estimated*	Measured†
Rio Gallina	1.1	3.04	2.03	3.92	1.04	1.03 ± 0.01
Ilice	4.7	2.7	2.0	5.1	1.00	1.01 ± 0.03
Maroggia	35.8	3.51	2.02	4.07	1.12	1.10 ± 0.02
Petrace	410	4.1	2.1	4.5	1.00	0.99 ± 0.03
Arno	8229	4.7	2.5	5.2	1.11	1.08 ± 0.04

*Computed from $d = \max [1, \min (2 \log R_L/\log R_A)]$.
†Measured using the box-counting method.

ı the contrary, if one computes d from (3) using the values R_B and R_L reported in Table 1, it can be immediately alized that (3) tends to overpredict d in all cases.

In Table 2 the estimates of d obtained from (12) are mpared with the empirical ones performed by *Hjelmfelt* 988] for eight rivers in Missouri using the Richardson's ethod [*Richardson*, 1961]. We computed the values of furcation and stream length ratios from the maps reported the Missouri Water Atlas [*Schroeder*, 1982]; unfortutely, it was not possible to compute the value of stream ea ratio from these maps because the divides are not dicated there. Therefore we had to use the estimator of R_A ggested by *Hack* [1957] in terms of R_B, R_L, and basin der. Although the procedure used to estimate R_A is biased, can be observed that the estimated dimensions generally produce the empirical values. However, higher deviations e found in this case, as compared with the above reported se study. These deviations may be due to the combined ect of the above mentioned bias of R_A estimates and the wer accuracy of the Richardson's method to measure the ctal dimension of an object, as compared with the boxunting method [*Parker and Chua*, 1989]. Finally, we also mputed d from (3) as proposed by *Feder* [1988]; from the sults reported in Table 2 it can be noticed that these latter timates are very poor.



. 1. Box-counting estimates of the fractal dimension of river length for five rivers in Italy.

The third case study deals with a comparison of the indirect estimates of d, which can be obtained from (12), with the direct estimates performed by using the length area method [*Mandelbrot*, 1982]. For this purpose we examined Alto Liri basin; this catchment, which is located in Southern Italy, is about 519 km^2 in area, with a relief of 1920 m. Fluvial geomorphology of this basin was studied in detail by *Avena and Lupia Palmieri* [1969] using 1:25,000 scale maps jointly with 1:33,000 scale aerial photographs and field surveys for data validation. We used their data to fit the values of Horton's order ratios reported in Table 3 (also see Figure 2). Moreover, we used the data of mainstream length and catchment area for 60 subbasins with Strahler's order ranging from 3 to 6, the order of the basin (see Figure 3). By regression analysis we fitted a power law to these data in order to evaluate α in (2); hence the fractal dimension of river length has been determined from (4), resulting in $d = 1.16 \pm 0.07$ (see Table 3). This result is found to be coherent with the value of $d = 1.12 \pm 0.08$, which can be obtained from (12).

Finally, in Figure 4*a* we reported the frequency distribution of d estimates for 30 rivers in the world. The data used include the above reported case studies and other data of bifurcation, stream length, and stream area either retrieved from literature [*Valdes et al.*, 1979; *Rosso*, 1984; *Sivaplan et al.*, 1990] or directly measured from maps, and the value of d has been estimated from (12). The scattering of the resulting estimates indicates that it is quite arbitrary to assign an invariant fractal dimension to river length.

DISCUSSION

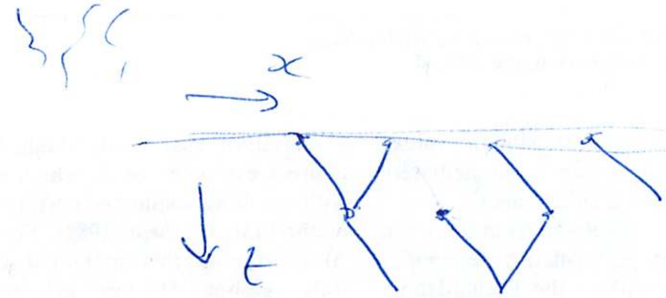
Fractal Relationships of Horton's Order Ratios

The combined range of R_A and R_L values observed in nature is generally confined into a quite narrow band because the joint observations of these quantities tend to display a certain degree of correlation. Equation (12) shows the fractal dimension of mainstream length to increase with stream length ratio for a constant stream area ratio; on the contrary, increasing R_A yields decreasing d for constant R_L. By solving (12) for R_L as a function of R_A and d, one gets R_L to equal R_A at the power of $d/2$, that is,

$$R_L = R_A^{d/2} \tag{13}$$

for $R_L \leq R_A$. Therefore the range of variation of d observed in nature reflects the observed scattering of R_L and R_A values.

Although a river network is represented by nearly linear



$t = 1$

D	h	d_2
$3/2$	$2/3$	1

$$h = \frac{d_2}{2}$$

$$h = \frac{d_2}{D}$$

$D \sim 1.8 \sim 1.9$

TABLE 2. Comparison Between Measured and Estimated Values of the Fractal Dimension of River Length for Eight Basins in Missouri

River	Order	Area, km^2	Horton Order Ratios R_B	R_L	R_A	Fractal Dimension, d: Measured*	Estimated†	Estimated‡
Big	4	2448	3.24	2.52	4.60	1.214	1.212	1.572
Big Piney	3	1950	4.25	3.01	6.32	1.078	1.196	1.523
Blackwater	4	3919	3.31	1.85	4.20	1.036	1.000	1.028
Bourbeuse	3	2233	4.12	3.34	6.47	1.291	1.292	1.704
Gasconade	4	9104	4.18	3.11	5.83	1.155	1.287	1.587
Lamine	4	2893	2.98	1.90	4.08	1.145	1.000	1.176
Meramec	5	10321	3.19	2.18	4.08	1.137	1.108	1.334
Moreau	3	1510	3.46	2.98	5.58	1.211	1.270	1.759

*Measured using Richardson's method [*Hjelmfelt*, 1988].
†Computed from $d = \max\ [1, \min\ (2, 2 \log R_L/\log R_A)]$.
‡Computed from $d = 2 \log R_L/\log R_B$ [*Feder*, 1988].

$\beta = 1+h$?!

Langbein's law.

members, both the mainstream course of a river and the total length of streams in the river network are fractals. For $R_B \geq R_L$ one obtains from (A7)

$$R_B = R_L^{D/d} \tag{14}$$

with D denoting the fractal dimension of the river network, that is, the total length of streams in the network. This can be combined with (13) to get

$$R_B = R_A^{D/2} \tag{15}$$

for $R_A \geq R_B \geq R_L$. Equations (13), (14), and (15) give an insight of the combined role of bifurcation, stream length, and stream area ratios in the characterization of the scaling properties of river networks.

Fractal Dimension of the River Network

The scaling properties of the river network as a whole can be viewed as the product of the structural composition of the drainage system and the effect of small irregularities reflected by d [*Tarboton et al.*, 1990; *La Barbera and Rosso*, 1990]. By combining (12) and (A9) one gets

$$D = \min\ (2, 2 \log R_B/\log R_A) \tag{16}$$

which shows that the fractal dimension of a river network depends on its bifurcation and stream area ratios.

For the Alto Liri basin, (16) yields an estimate of $D = 2 \log (4.75)/\log (5.13) = 1.906$ (see Table 3). This result can be compared with the direct estimate of D, which can be obtained by means of the length-area relationship in the form

x.

$$Z \sim A^{\beta} \tag{17}$$

-oops

where Z denotes the total length of streams in a subbasin of area A, and $\beta = D/2$ is an exponent fitted to the data. By regression analysis we obtained $\beta = 0.95 \pm 0.03$, which yields D to be 1.90 (see Table 3). This value is very close to the estimate of 1.906 obtained from (16).

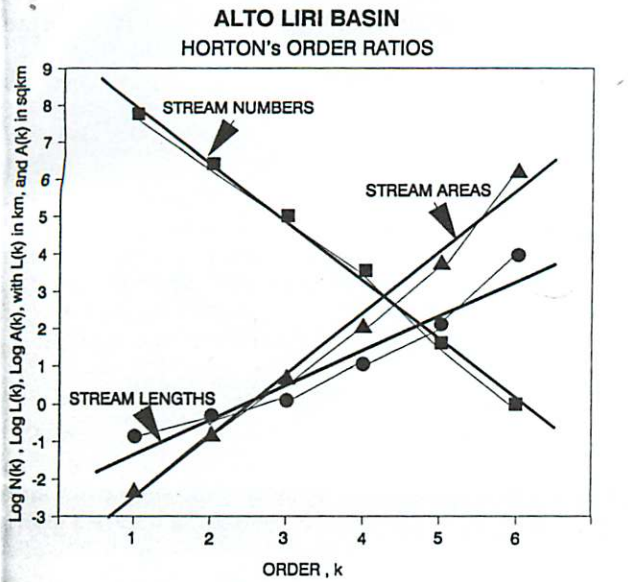
In Figure 4*b* the frequency distribution is displayed for the estimates of D obtained from (16) for the above mentioned 30 rivers in the world. It can be observed that the fractal dimension of river networks also seems to vary from one river to another, and it is therefore quite arbitrary to assign an invariant fractal dimension to the total length of streams in river networks.

CONCLUSION

The length of a river course can be viewed as a fractal measure with fractional dimension d [*Mandelbrot*, 1982;

TABLE 3. Fractal Dimensions for the River Network of Alto Liri Basin in Southern Italy: Comparison Between Length-Area Measures and the Estimates From Horton's Order Ratios

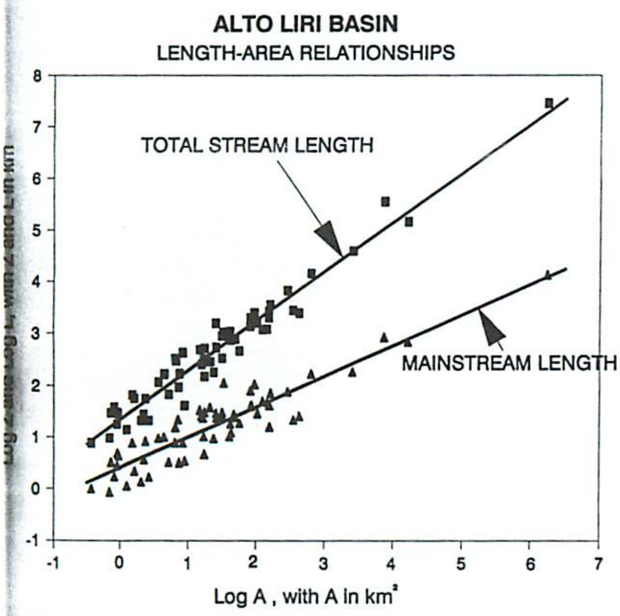
Parameter	Estimated Value	Standard Error	R^2
	Horton's Order Ratios		
Bifurcation ratio R_B	4.75	0.24	0.995
Stream length ratio R_L	2.50	0.30	0.931
Stream area ratio R_A	5.13	0.46	0.987
	Total Stream Length Versus Area		
Constant, $km^{1-2\beta}$	3.86	...	
Exponent β	0.95	0.03	0.952
Fractal dimension $D = 2\beta$	1.90	0.06	
	Mainstream Length Versus Area		
Constant, $km^{1-2\alpha}$	1.52	...	
Exponent α	0.58	0.03	0.842
Fractal dimension $d = 2\alpha$ crap.	1.16	0.07	
	Estimates of Fractal Dimensions From Horton's Order Ratios		
$d = 2 \log R_L/\log R_A$	1.12	0.08	...
$D = 2 \log R_B/\log R_A$	1.90	0.12	...



g. 2. Horton's order ratios, estimated by linear regression for Alto Liri basin in southern Italy.

jelmfelt, 1988]. For an ordered drainage system, d can be rived from the Horton's laws of network composition; cordingly, d results in a simple function of stream length d stream area ratios, which are given by (12). Because d is so related to the fractal dimension D of the network as a hole [*Tarboton et al.*, 1990], the present analysis provides method for determining D as a simple function of the furcation and stream area ratios of a basin; this is given by 6).

These results provide a linkage between the structure of twork composition and the mainstream length catchment ea ratio. For river networks which are space filling (i.e., ith $D = 2$), the stream area ratio equals the bifurcation tio, and the drainage density is thus constant with area.



g. 3. Length-area relationships for total stream length and main-ream length of 60 subbasins of Alto Liri basin in southern Italy.

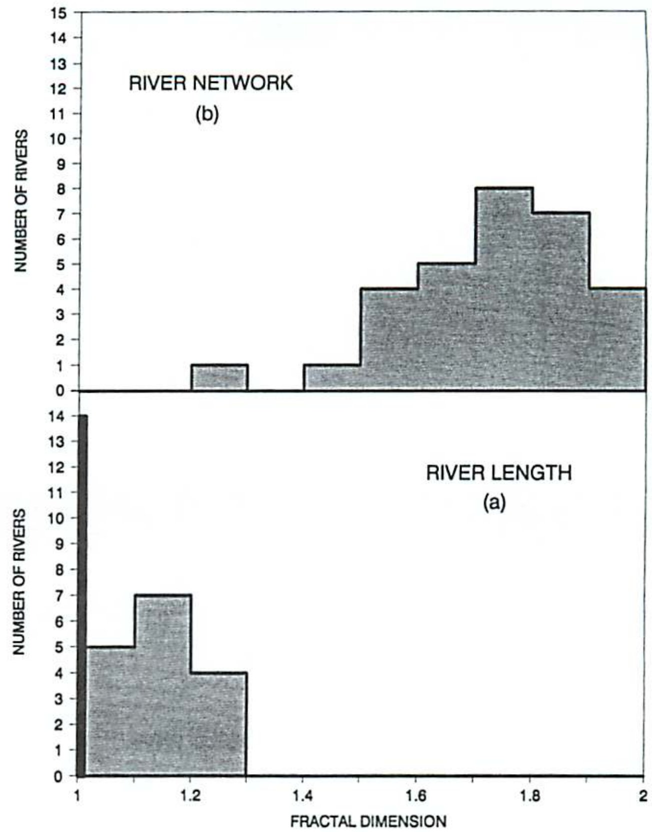


Fig. 4. Histogram of fractal dimension of (*a*) river length and (*b*) river network for 30 rivers in the world.

The assumption of topological randomness [*Shreve*, 1967] corresponds to catchments which are geometrically similar from dimensional analysis because the modal values of $R_A = R_B = 4$ and $R_L = 2$ yield d to be 1 and D to be 2, so that $\alpha = 1/2$ and $\beta = 1$. The departure of α from 1/2 yields the mainstream length to be fractal and the river network to display a fractal dimension less than 2, since $\beta < 1$. The average value of α reported by *Gray* [1961] yields $d = 1.13$, and *Hjelmfelt* [1988] found experimental values of d ranging from 1.036 to 1.291, with an average of 1.158. The corresponding D values obtained from (16) for these rivers in Missouri range from 1.444 to 1.649, with an average of 1.576. The data reported here for six basins in Italy display a large scatter of fractal dimensions of river length ranging from 1 to 1.16; the highest value has been determined by means of the length area method for the Alto Liri basin. In this latter case the length area method and the estimation performed using Horton's order ratios provide very close results for both d and D. Finally, the average value of $\beta = 0.833$ [see *Gregory and Walling*, 1973, Figure 5-12] yields an average value of D equal to 1.67; this value is coherent with the ones reported in Figure 4*b*, although a large scattering is displayed by field observations.

The results presented above provide a simple mathematical representation of the scaling properties of river networks over a range of scales. This is derived from the Horton's laws of network composition. It has been pointed out that these laws rarely apply to "large" basins. For instance, *Andah et al.* [1987*b*] found that the values of the Horton's ratios change abruptly when passing from mountain and piedmont areas to plain areas, for rivers in Ghana. The results by *Mesa and Gupta* [1987] also imply that the average fractal dimension of river networks varies with basin size. Therefore Horton's laws, as well as the random model postulates, need some modification in order to theoretically explain the observed complexity of river networks, and a multifractal approach may be required to analyze the geometry of complex river network systems. As noted by *Andah et al.* [1987*b*] and *Gupta and Mesa* [1988], this remains an important unsolved problem in fluvial geomorphology and basin hydrology.

Further developments are also needed in order to assess the performance of different methods used to determine fractal dimensions from map analysis. The methods which have been referred to in the present work, that is, the ruler method, the box-counting method, and the length area method, can provide different results in some cases, and the reliability of these results still remains to be fully assessed. Other methods (e.g., the correlation integral method) should also be investigated.

Appendix: Horton's Laws and Fractal Nature of River Networks

Horton's laws of network composition are usually stated in terms of the *Strahler*'s [1952] ordering scheme, which is adopted here. A system of stream ordering which recognizes the existence of a hierarchy among the separate branches is therefore assumed to represent the structure of a river network. This postulates that (1) source streams are of order 1 and (2) when two streams of order i and j, respectively, merge, a stream of order k is formed, with

$$k = \max \{i, j, \mathrm{Int}\,[1 + \tfrac{1}{2}(i + j)]\} \qquad \text{(A1)}$$

where function Int [ ] denotes the integer part of the argument. Following (A1), when two streams of equal order join, a stream of one order higher is formed; and when two streams of different order join, the continuing stream retains the order of the higher-order stream.

The empirical laws of stream numbers and stream lengths [*Horton*, 1945] state that the bifurcation R_B and the stream length R_L ratios are constant for a catchment. Moreover, the empirical law of stream areas [*Schumm*, 1956] states that the stream area ratio R_A is also constant. If we denote with k the order of a stream segment, these ratios are defined as

$$R_B = n_k/n_{k+1} \qquad \text{(A2)}$$

$$R_L = l_{k+1}/l_k \qquad \text{(A3)}$$

$$R_A = A_{k+1}/A_k \qquad \text{(A4)}$$

where n_k is the number of streams of order k, l_k is the mean length of streams of order k, and A_k is the mean tributary area of streams of order k, respectively. Estimates of R_B, R_L, and R_A can be obtained from the slopes of the straight lines resulting from plots of logarithmic-transformed values of n_k, l_k, and A_k versus order k, for k ranging from 1 to Ω, that is, the order of the basin (see Figure 2). These laws yield the following equation for the drainage density:

$$A_\Omega/Z_\Omega = (A_1/l_1)(R_A/R_B)^{\Omega-1}[(R_L/R_B) - 1]/[(R_L/R_B)^\Omega - 1] \qquad \text{(A5)}$$

with Z_Ω denoting the total length of streams in a Ωth order basin ((A5) can be derived by combining the equations (6) and (7) reported by *Hack* [1957]). Because the basin order appears in (A5), it is clear that (A5) predicts that the drainage density varies with area.

The Horton's laws are geometric-scaling relationships which yield self-similarity of the catchment stream system at all scales in the network. On the basis of self-similarity described by laws of stream number and of stream lengths, *La Barbera and Rosso* [1987] first reported that the fractal dimension of river networks is given by log R_B/log R_L for $R_B \geq R_L$; this can be also written in the general form

$$D = \min\,[2, \max\,(1, \log R_B/\log R_L)] \qquad \text{(A6)}$$

Some arguments to support (A6) were also reported by *Tarboton et al.* [1988], and *Nikora* [1989] further substantiated (A6) on the basis of a large number of rivers in the world. From (A6) the fractal dimension of a stream network can take values from 2 to unity for the combined ranges of R_B and R_L values observed in nature. Although it has been observed that river networks display varying values of fractal dimension, D generally lies between 1.5 and 2, with an average of approximately 1.67. *La Barbera and Rosso* [1989] further observed that values of R_B and R_L close to those values of $R_B = 4$ and $R_L = 2$, which descend from the assumption of topological randomness [*Shreve*, 1967], yield the fractal dimension of stream networks to be near 2. Finally, it has been observed that digital filtering techniques used to identify drainage patterns from digital elevation models (DEMs) often tend to reproduce networks with D near 2 [*Andah et al.*, 1987*a*].

On the basis of the results reported by *La Barbera and Rosso* [1989], some further developments of the analysis of the fractal nature of river networks have been recently discussed by *Tarboton et al.* [1990], and *La Barbera and Rosso* [1990]. By introducing the further source of fractal behavior due to the small irregularities of individual streams, *Tarboton et al.* [1990] get

$$\log R_B/\log R_L = D/d \qquad \text{(A7)}$$

with d denoting the fractal dimension of stream length. La Barbera and Rosso [1990] also performed this analysis following a different route in the analytical derivation of the mutual relationship between small irregularities and the structure of network composition, thus obtaining

crapola.

$$\log R_B/\log R_L = D(2 - d) \qquad \text{(A8)}$$

However, (A7) and (A8) give quite close results for the cases of practical interest. Although the influence of small irregularities on the fractal nature of river networks is still an open question from a theoretical point of view, it must be observed that this occurs in the hydrological practice because the evaluation of Horton's order ratios is generally performed measuring stream lengths as segments. Therefore either (a7) or (A8) probably provide a reliable description of the fractal behavior of river networks. So in the case when (A7) is adopted, (A6) can be rearranged as follows

$$D = \min\,[2, \max\,(1, d \log R_B/\log R_L)] \qquad \text{(A9)}$$

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B. Bacchi and R. Rosso, Institute of Hydraulics, Politecnico di Milano, 32 Piazza Leonardo da Vinci, Milan I-20133, Italy.

P. La Barbera, Hydraulic Institute, University of Genoa, Via Montallegro 1, Genoa I-16143, Italy.

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