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LETTER TO THE EDITOR

On the spreading dimension of percolation and directed percolation clusters

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Abstract. We study the number A_N of sites that are accessible after N steps at most on clusters at the percolation threshold. On a Cayley tree A_N is of order N^2 if the origin belongs to a large cluster, whereas its average over all clusters is of order N. This suggests that the intrinsic spreading dimension d-hat, defined by A_N ~ N^d-hat, is equal to two for fractal percolation clusters in space dimensions d >= 6 and depends on d for d < 6. For directed percolation clusters we argue that d-hat is related to usual critical exponents by d-hat = (beta + gamma) / nu_parallel. Monte Carlo data that support this relation are presented in two dimensions. Analogous results are derived for lattice animals: d-hat = 2 on the Cayley tree and d-hat = 1 / nu_parallel for directed animals in any dimension.

The problem of anomalous diffusion on percolation clusters has made considerable progress recently and has helped to clarify the various notions of dimension for fractal spaces (Alexander and Orbach 1982, Gefen et al 1983, Rammal and Toulouse 1983). In particular the importance of intrinsic properties, i.e. that depend only on the structure of a fractal space and not on its embedding in a particular Euclidean space, has been stressed by Rammal et al (1984). One such intrinsic property is the average number A_N of distinct sites that are accessible from a given origin in at most N steps. N may be viewed as a chemical distance and A_N corresponds for instance to the number of units that have reacted in a model of chain polymerisation, or to the number of infected individuals in the propagation of an epidemic. For Euclidean lattices A_N ~ N^d and an intrinsic dimension d-hat can be defined similarly for a fractal space through the asymptotic behaviour of A_N for large N:

A_N ~ N^d-hat (1)

We propose to call d-hat the spreading dimension of the space (Toulouse 1984): it must in general be different from the fractal dimension d-bar which describes the scaling of the mass M contained in a region of radius R (M ~ R^d-bar), since d-bar is defined in terms of the Euclidean distance and is not an intrinsic quantity. A simple example is given by a random walk on a Euclidean lattice, which is well known to be a fractal object and which has d-bar = 2, d-hat = 1. Another intrinsic dimension is the spectral dimension d-hat, which describes in particular the average number S_N of distinct sites visited by a random walker after N steps on a fractal space (Rammal and Toulouse 1983):

S_N ~ N^d-hat/2 (if d-hat < 2). (2)

It might be expected, for reasons of economy and simplicity, that $\hat{d} = \bar{d}$: this is not the case, as we show explicitly below for percolation clusters. The spreading dimension is therefore a new characteristic property of fractal spaces.

It is found numerically that for the incipient infinite cluster at the percolation threshold the spectral dimension varies very little with the space dimension d ($\bar{d} = \frac{4}{3}$; Alexander and Orbach 1982, Angles d'Auriac *et al* 1983), suggesting that it might even be a 'super universal' constant. It is then crucial to study the dependence of the spreading dimension on d : if \hat{d} were also constant, or nearly so, one might view percolation clusters in different d as essentially the same object embedded in different Euclidean spaces.

This is in fact not the case. We show that $\hat{d} = 2$ on the Cayley tree and since percolation on the tree generally corresponds to mean-field behaviour (Fisher and Essam 1961) we expect that $\hat{d} = 2$ for $d \geq d_c = 6$. As it is easy to see that $\hat{d} \leq \bar{d}$ holds in general, and $\bar{d} = 1.896 \dots$ for percolation in $d = 2$ (Stauffer 1981), \hat{d} cannot be independent of d . Also, accurate Monte Carlo calculations (Herrmann 1984) indicate that $\hat{d} \sim 1.7$ for $d = 2$. The calculations on the tree need some care: an average over all clusters gives $A_N \sim N$, but we argue that the relevant average for fractal properties has to be taken on large clusters and gives $A_N \sim N^2$.

We also study the number of accessible sites on fully directed percolation clusters on cubic lattices. The problem is easier then since the number of new sites at every step is just the number of sites in the N th section along the preferred direction. A simple scaling argument is presented which relates \hat{d} to known critical exponents of directed percolation. This argument predicts $\hat{d} = 2$ for $d = d'_c = 5$, in agreement with the expectation based on the Cayley tree result. Monte Carlo simulations on the directed square lattice are presented, they are in very good agreement with the scaling prediction.

Analogous results may be obtained for lattice animals and we find that $\hat{d} = 2$ for animals on the Cayley tree, both isotropic and directed. For directed animals a scaling argument leads to the simple result $\hat{d} = 1/\nu_{||}$ in any space dimension.

It is well known (Fisher and Essam 1961, Nickel and Wilkinson 1983) that the average height (i.e. number of levels or generations) of a percolation cluster containing s sites, on a Cayley tree of coordination number $1 + \sigma$, is given by

$$\langle n \rangle_s \sim [\pi\sigma/2(\sigma - 1)]^{1/2} s^{1/2} \quad (3)$$

at the percolation threshold (probability of site occupancy $p = p_c = 1/\sigma$). It is then surprising that the average number A_N of sites that can be reached in at most N steps is only of order N , since on a tree there is a direct correspondence between steps and generations. One would rather expect A_N to be of order N^2 , unless the probability distribution of accessible sites is very broad.

Let us consider the generating function for the number $\Omega_{s,k}^{N,n}$ of clusters of s sites which extend exactly up to level N , and contain k sites up to level n . We consider a rooted tree, where the root ($n = 0$) has only σ neighbours. With the convention $\Omega_{0k}^{N,n} = \delta_{N,0}\delta_{k,0}$, one has:

$$g_N^n(y, x) = \sum_{s,k=0} \Omega_{s,k}^{N,n} y^s x^k; g_0^n(y, x) = 1 + xy. \quad (4)$$

For percolation the variable y is related to the site occupation probability p by $y = p(1 - p)^{\sigma-1}$, while for lattice animals y is the fugacity. The generating function for

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Now the generating setting $H_N(y) = \zeta_N^n$

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From these relations $(\sigma - 1)^{\sigma-1}/\sigma^\sigma$:

$$S_0(N) = \sigma^{-1},$$

$$S_n(N) = (1 + H_N^{-1})^n$$

$$a_n(N) = 1 + \sigma[(H_N -$$

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The above derivation on the Cayley tree, since We first remark that directed percolation. Euclidean spaces: for

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clusters extending *at most* up to level N is

$$\zeta_N^n(y, x) = \sum_{e=0}^N g_e^n(y, x). \quad (5)$$

Denoting the mean number of sites between levels 0 and n respectively $a_n(N)$ for clusters extending exactly up to N and $S_n(N)$ for clusters extending at most to N , these are given by:

$$a_n(N) = \partial \ln g_N^n / \partial x|_{x=1}, \quad (6a)$$

$$S_n(N) = \partial \ln \zeta_N^n / \partial x|_{x=1}. \quad (6b)$$

Now the generating functions (4) and (5) are independent of n for $x=1$, and setting $H_N(y) = \zeta_N^n(y, 1)$ one can write the following recursion relations:

$$H_0 = 1 + y, \quad H_N = 1 + y(H_{N-1})^\sigma, \quad (7)$$

$$\zeta_N^0 = 1 + x(H_N - 1), \quad \zeta_N^n = 1 + yx(\zeta_{N-1}^{n-1})^\sigma. \quad (8)$$

From these relations and equation (6) one gets at the critical value $y_c = (\sigma - 1)^{\sigma-1} / \sigma^\sigma$:

$$S_0(N) = \sigma^{-1}, \quad a_0(N) = 1,$$

$$S_n(N) = (1 + H_N^{-1})[1 + \sigma S_{n-1}(N-1)], \quad (9)$$

$$a_n(N) = 1 + \sigma[(H_N - 1)S_{n-1}(N-1) - (H_{N-1} - 1)S_{n-1}(N-2)] / (H_N - H_{N-1}).$$

For $1 \ll n \ll N$ one obtains the asymptotic behaviour:

$$H_N \sim [\sigma / (\sigma - 1)] [1 - 2/N(\sigma - 1)],$$

$$S_n(N) \sim n/\sigma, \quad a_n(N) \sim [(\sigma - 1)/2\sigma]n^2. \quad (10)$$

These results show that if the average is performed over *all* clusters the number $S_n(N)$ of accessible sites grows linearly with the number of steps n : this average is in fact dominated by small clusters that have no occupied site at level n . On the contrary, when the average is performed only over large clusters the average number of accessible sites $a_n(N)$ grows like n^2 . We give in figure 1(a) the exact values of $a_n(N)$ for $N = 10^5$ and $\sigma = 2$ obtained via relations (9) at $p = p_c = \frac{1}{2}$. We noticed that such a large value of N was necessary to reach the asymptotic regime.

The spreading dimension is therefore $\hat{d} = 2$ for percolation on the Cayley tree and is larger than the spectral dimension $\bar{d} = \frac{4}{3}$ (Angles d'Auriac *et al* 1983). This result is expected to hold when mean-field theory is valid, i.e. for $d \geq d_c = 6$. Since the fractal dimension \bar{d} is a bound to \hat{d} , one has $\hat{d} \leq \bar{d} = \frac{9}{8}$ for $d = 2$: \hat{d} is not independent of d and the fractal percolation clusters have a different intrinsic structure for different d , even though their spectral dimension appears to be $\approx \frac{4}{3}$ for all d .

The above derivation also shows that identical results hold for lattice animals on the Cayley tree, since the critical value y_c is the same for percolation and animals.

We first remark that on a Cayley tree there is no distinction between ordinary and directed percolation. The difference only comes when one makes contact with Euclidean spaces: for isotropic percolation, a number m of generations corresponds

† One can show that the number of accessible sites of clusters extending at least to level n grows like n^2 .

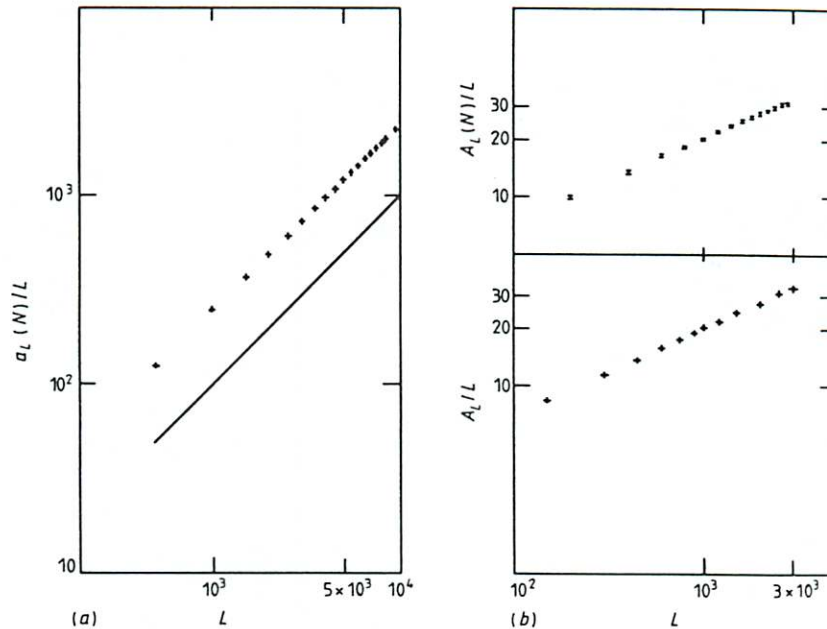


Figure 1. Mean total number of accessible sites after L steps, on percolation clusters at threshold (double logarithmic plots): (a) Exact results on the Cayley tree of coordination number 3, on clusters of length $N = 10^5$ exactly. The full straight line has unit slope. (b) Monte Carlo results for directed percolation on the square lattice. A_L corresponds to an average over clusters extending at least up to L , $A_L(N)$ to clusters extending at least up to $N = 5000$. Both quantities have the same asymptotic behaviour, within numerical accuracy.

to a Euclidean distance $m^{1/2}$, whereas for the directed case the distance is m along the preferred direction and $m^{1/2}$ along the others. The two problems correspond to different (local) embeddings of the tree in Euclidean spaces of very high dimension, but the intrinsic properties like the spectral and spreading dimensions are identical. A direct consequence is that we expect $\hat{d} = 2$ for directed percolation when $d \geq 5$ (Obukhov 1980).

On a cubic lattice the accessible sites in a fully directed cluster ('wetted' sites) are just the sites belonging to that cluster in the N th section from the origin, perpendicularly to the preferred direction. A scaling argument may then be given, above the percolation threshold: the wetted sites of the infinite cluster lie inside a cone of angle

$$\theta \sim \xi_{\perp} \xi_{\parallel} \sim (p - p_c)^{\nu_{\parallel} - \nu_{\perp}}, \tag{11}$$

where ξ_{\perp} and ξ_{\parallel} are the associated critical exponents (Kinzel 1982). At a large distance L from the origin and not too close to the edges of the cluster, the density of wetted sites is given by the probability $P(p) \sim (p - p_c)^{\beta}$ that a site belong to the infinite cluster, so the average number $\langle B_L(p) \rangle$ of wetted sites in a section is:

$$\langle B_L(p) \rangle \sim [L\theta]^{d-1} P(p) \sim L^{d-1} (p - p_c)^{\beta + (\nu_{\parallel} - \nu_{\perp})(d-1)}. \tag{12}$$

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scaling relation holds, with ξ_{\parallel} replaced by L . This gives:

$$\langle B_L \rangle_{p_c} \sim L^{[(d-1)\nu_{\perp} - \beta]/\nu_{\parallel}}$$

and

$$\hat{d} = 1 + [(d-1)\nu_{\perp} - \beta]/\nu_{\parallel} = (\beta + \gamma)/\nu_{\parallel}, \quad (13)$$

using hyperscaling relations for directed systems in the last step.

At the upper critical dimension $d_c = 5$, $\beta = 1$, $\nu_{\parallel} = 1$, $\nu_{\perp} = \frac{1}{2}$, so $\hat{d} = 2$ and (15) agrees with the result on the Cayley tree.

To check prediction (15) we have performed Monte Carlo calculations on the fully directed square lattice, at the percolation threshold $p_c = 0.705$ (Kinzel 1982). A study of the distribution of cluster lengths showed that 30 percent of all clusters reached a length $L = 3000$, and that this length is well into the scaling region where the distribution decays very slowly with a power law. The results are presented in figure 1(b): they give the estimate

$$\hat{d} = 1.46 \pm 0.03$$

for clusters extending at least to $N = 5000$. This is to be compared with the prediction $\hat{d} = 1.47$ from equation (13), using $\nu_{\parallel} \approx 1.73$, $\nu_{\perp} \approx 1.10$ and $\beta \approx 0.28$ (Kinzel 1982), and the agreement is satisfactory.

An interesting remark is that if one sets formally $\nu_{\parallel} = \nu_{\perp}$ in (15), the expression of \hat{d} reduces to $(d - \beta/\nu)$, that is the fractal dimension \bar{d} of the infinite cluster in standard percolation. This is puzzling at first since \bar{d} is not an intrinsic dimension and one may wonder whether (15) is properly intrinsic. In fact, the introduction of a preferred direction along the diagonal of a cubic lattice defines a natural distance, in the sense that the distance between two connected points along a directed path is independent of the particular path followed between them and is invariant under distortions that preserve the local ordering of lattice sites. As defined above, ν_{\parallel} corresponds to that natural distance and is indeed an intrinsic exponent.

For directed lattice animals the reasoning is simpler still: the average length of a large animal containing s sites is proportional to $s^{\nu_{\parallel}}$, but this is also the number L of steps necessary to reach the s sites, so we expect

$$A_L \sim L^{1/\nu_{\parallel}}, \quad \hat{d} = 1/\nu_{\parallel}. \quad (14)$$

Here the problem of small clusters does not arise. The upper critical dimension for directed animals is $d_c = 5$ (Day and Lubensky 1982): for $d \geq 5$, $\nu_{\parallel} = \frac{1}{2}$ and we expect the spreading dimension to remain constant, $\hat{d} = 2$. This agrees with the mean-field value obtained above on the Cayley tree. We note that the generalisation of the fractal dimension \bar{d} to directed objects is not unique (Nadal *et al* 1983), but that the spreading dimension is well defined and has a simple expression.

We thank G Toulouse for initiating our interest in this problem and stimulating our work. To our knowledge, the first public discussion of this number of accessible sites in fractals took place at the Gaithersburg conference on 'Fractals in Physical Sciences' (November 1983) with Dr S Havlin presenting results pertaining to percolation clusters in dimension two and Dr R Rammal reporting on similar calculations for two-dimensional Sierpinski gaskets (unpublished). We are also grateful to Dr H J Herrmann for communicating his numerical results prior to publication. One of us (HM) thanks the CONICET (Republic of Argentina) for its support while this work was performed.

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