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Vector Spaces of Magic Squares

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Exercise. An $n \times n$ **magic square** is an $n \times n$ matrix of real numbers in which the sum along each row, each column and each diagonal is a constant (called the **line-sum** of the magic square). For example, three 3×3 magic squares with line-sums 15, $3/2$ and 0, respectively, are

$$\begin{matrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{matrix} \qquad \begin{matrix} -1 & 5/2 & 0 \\ 3/2 & 1/2 & -1/2 \\ 1 & -3/2 & 2 \end{matrix} \qquad \begin{matrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{matrix}$$

(i) Show that the (matrix) sum of two $n \times n$ magic squares is an $n \times n$ magic square. If the line-sums of the squares are m_1 and m_2, what is the line-sum of the sum?
(ii) Show that the (matrix) scalar multiple of an $n \times n$ magic square by a real number k is an $n \times n$ magic square. If the original square has line-sum m, what is the line-sum of the scalar multiple?
(iii) Is the set of all $n \times n$ magic squares (with all possible line-sums) a vector space? Why?
(iv) Is the set of all $n \times n$ magic squares with line-sum $m \neq 0$ a vector space? Why?
(v) Is the set of all $n \times n$ magic squares with line-sum zero a vector space? Why?

This exercise, suggested by Fletcher [3], encourages consideration of the algebraic structure of magic squares, as opposed to methods for generating them. In this article we follow Fletcher's suggestion, using familiar linear algebra techniques to determine the dimensions of the vector spaces of magic squares. Then we use these dimensions to establish an upper bound on the number of magic squares.

Magic squares have fascinated people for centuries. A Chinese emperor is supposed to have seen one—on the back of a divine turtle, no less—as early as 2200 B. C. From that time on, mystical properties have been ascribed to them. In the middle ages, a magic square engraved on a silver plate and worn about the neck was thought to ward off the plague [5]. Writing in 1844, Hutton [4] reported:

> These squares have been called magic squares because the ancients ascribed to them great virtues, and because this disposition of numbers formed the basis and principle of many of their talismans. According to this idea a square of one cell, filled up with unity, was the symbol of the Deity, on account of the unity and immutability of God; for they remarked that this square was, by its nature, unique and immutable, the product of unity by itself being always unity. The square of the root two was the symbol of imperfect matter, both on account of the four elements and of the impossibility of arranging this square magically. A square of nine cells was assigned or consecrated to Saturn, that of sixteen to Jupiter, that of twenty-five to Mars, that of thirty-six to the Sun, that of forty-nine to Venus, that of sixty-four to

Mercury, and that of eighty-one, or nine on each side, to the Moon. Those who can find any relation between the planets and such an arrangement of numbers must, no doubt, have minds strongly tinctured with superstition; but such was the tone of the mysterious philosophy of Jamblichus, Porphyry, and their disciples. Modern mathematicians, while they amuse themselves with these arrangements, which require a pretty extensive knowledge of combination, attach to them no more importance than they really deserve.

Nevertheless, mathematicians both before and after 1844 apparently attached enough importance to magic squares to write thousands of pages about them. In 1888, F. A. P. Barnard, then president of Columbia and after whom Barnard College is named, published a 61-page paper **[2]** at the end of which he included an "approximately complete" bibliography of 47 scholarly papers and books on the subject. Today a complete bibliography on magic squares would probably require all 61 pages!

The magic squares which are best known are those $n\times n$ squares which use only the first n^2 positive integers. These square arrays of the numbers 1 to n^2 will be referred to as **classical** magic squares. The first example given in the opening exercise is a 3×3 classical magic square.

Each magic square yields seven other magic squares, obtained from it by rotating it in the plane through angles of 90°, 180°, and 270° and by rotating it in space about its horizontal, vertical and two diagonal axes. These seven, together with the original, constitute the **symmetries** of the magic square. Symmetric magic squares are regarded as being identical. It is easy to show that the first example in the opening exercise is the only 3×3 classical magic square up to symmetry.

More generally, we shall call any $n\times n$ array of n^2 (integral, real, or complex) numbers in which each line-sum is constant a **magic square**. The second and third examples of the opening exercise are real 3×3 magic squares. Note that the same number may appear several times in a magic square, a statement which is not true of classical magic squares.

This paper will consider only real magic squares, although all of the results are true for complex magic squares as well. If the entries are restricted to the integers, all of the results hold if "vector space" is replaced judiciously by "$\mathbb{Z}$-module." From now on, all magic squares will be real magic squares unless it is specifically stated otherwise. Because 1×1 and 2×2 magic squares are not very interesting and because they bog down the proofs with special cases, it will be assumed that $n\geqslant 3$.

We shall denote by **MS(*n*)** the set of all $n\times n$ magic squares; by ***m*MS(*n*)** the set of all $n\times n$ magic squares with line-sum m; and by **0MS(*n*)** the set of all $n\times n$ magic squares with line-sum zero. The opening exercise reveals that MS(n) and 0MS(n) are vector spaces but that mMS(n) for $m\neq 0$ is not. The space 0MS(n) is a subspace of MS(n) which is, in turn, a subspace of the n^2-dimensional vector space of all $n\times n$ real matrices. Thus the dimension of MS(n) is at most n^2. (Note that mMS(n) is never empty: it always contains the square in which each entry is m/n.)

Let us call each magic square in 0MS(n), whose line-sums are all zero, a **zero magic square**. We will call two $n\times n$ magic squares **equivalent** if one can be obtained from the other by adding the same real number to each entry. It follows, trivially, that each magic square is equivalent to one and only one zero magic square: if an $n\times n$ magic square has line-sum m, it can " zeroed" by subtracting m/n from each entry. Thus there is a one-to-one correspondence between the set mMS(n) for a fixed m and the vector space 0MS(n).

This sets the stage for the main result of this paper.

THEOREM. *The dimension of* $0MS(n)$ *is* n^2-2n-1.

Proof. If an $n\times n$ matrix $A=(a_{ij})$ is in 0MS(n), its $2n+2$ line-sums are all zero. Thus there are $2n+2$ homogeneous linear equations in the n^2 variables a_{ij}, $1\leqslant i,j\leqslant n$. Write these equations in the following order, called the **standard order**: first the n row sums in order, then the n column sums in order, then the NW-SE diagonal sum, and, last, the SW-NE diagonal sum. The resulting coefficient matrix will be a $(2n+2)\times n^2$ matrix of 0's and 1's. When $n=3$, it is the 8×9 matrix

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0, \end{matrix}$$

where the elements in the jth column are the coefficients of the jth variable in the list $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.

In the coefficient matrix determined by A, the first $2n-1$ rows are clearly linearly independent. But the $2n$th row is a linear combination of the first $2n-1$, being the sum of the first n rows minus the sum of rows $n+1$ through $2n-1$. Moreover, the last two rows are both linearly independent of the first $2n-1$ rows: the nth column of the coefficient matrix has 1's in the first, $2n$th and $(2n+2)$nd rows, and zeroes everywhere else, and the n^2th column has 1's in the nth, $2n$th and $(2n+1)$st rows, and zeroes everywhere else, making it impossible to find a nontrivial zero linear combination of the first $2n-1$ rows with either of the last two rows. Finally, it is clear that neither of the last two rows is a scalar multiple of the other. Thus the matrix of coefficients has exactly $2n+1$ linearly independent rows and hence has rank $2n+1$. By the rank and nullity theorem [**1**], the dimension of $0MS(n)$, which is the nullity of the coefficient matrix, is $n^2-(2n+1)$.

COROLLARY. *The dimension of $MS(n)$ is $n^2-2n=n(n-2)$.*

Proof. Let $q=n^2-2n-1$ and $S_1,\ldots,S_q$ be a basis for $0MS(n)$, a subspace of $MS(n)$. Let I be the magic square in $MS(n)$ with 1 in every position, and consider the set $B=\{S_1,\ldots,S_q,I\}$ consisting of n^2-2n vectors of $MS(n)$.

The set B spans $MS(n)$, for if M is any magic square in $MS(n)$ and M has line-sum m, then M is equivalent to the zero magic square $M_0=M-(m/n)I$ of $0MS(n)$. As $S_1,\ldots,S_q$ is a basis for $0MS(n)$, $M_0=c_1S_1+\cdots+c_qS_q$ for some scalars $c_1,\ldots,c_q$, so $M=c_1S_1+\cdots+c_qS_q+(m/n)I$. Moreover, B is linearly independent, for the line-sum of the vector $c_1S_1+\cdots+c_qS_q+c_{q+1}I$, where the c_i's are scalars, is nc_{q+1}. [See parts (i) and (ii) of the opening exercise.] If this vector is to equal the zero vector, which has line-sum zero, we must have $nc_{q+1}=0$ or $c_{q+1}=0$. Then the linear independence of $S_1,\ldots,S_q$ implies that $c_1=\cdots=c_q=0$ as well. Thus B is a basis for $MS(n)$.

It is easy to see that the central entry of any magic square in 0MS(3) is zero. This means that 0MS(3) magic squares are anti-symmetric with respect to the diagonals. By the Theorem, the dimension of 0MS(3) is 2; thus a magic square in 0MS(3) is uniquely determined by specifying any two entries not collinear with the central zero. If we choose the first two entries in the first row to be 1, 0 and 0, 1, we get the following basis for 0MS(3):

$$\begin{matrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{matrix} \qquad \begin{matrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix}$$

According to the Theorem, the dimension of 0MS(4) is 7. Using an argument of the same nature as that which shows that there is a unique 3×3 classical magic square up to symmetry, it can be established that the sum of the four corner entries and the sum of the four central entries of a magic square in 0MS(4) are both zero. With these facts, it is easy to find seven entries which, when specified, completely determine a 4×4 zero magic square. Two examples are:

$$\begin{matrix} x & x & x & - \\ x & x & x & - \\ x & - & - & - \\ - & - & - & - \end{matrix} \qquad \begin{matrix} - & - & - & - \\ - & - & x & - \\ x & - & x & x \\ x & - & x & x \end{matrix}$$

The seven squares obtained by putting 1 in one of the designated positions of either pattern and 0's in the other six in all possible ways constitute a basis for 0MS(4). For instance, the seven magic squares which form a basis for 0MS(4) according to the first pattern are:

$$\begin{array}{rrrr} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -2 & 2 & 1 \end{array} \qquad \begin{array}{rrrr} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & 1 \end{array} \qquad \begin{array}{rrrr} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \qquad \begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 \end{array}$$

$$\begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{array} \qquad \begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array} \qquad \begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{array}$$

Motivated by this example, it is natural to make the following definition. A selection of n^2-2n-1 positions in an $n\times n$ matrix is called a **skeleton** of 0MS(n) if the assignment of real numbers to those positions uniquely determines entries in all other positions using the zero magic square conditions. Given a skeleton of 0MS(n), the array of the $2n+1$ positions *not* specified is called the **frame** of that skeleton. In the 0MS(4) examples above, each array of x's is a skeleton and each array of dashes is a frame. When n is large, the number of positions in a skeleton is much greater than the number of positions in its frame, so it is convenient to think of skeletons in terms of the frames they determine.

Every skeleton of 0MS(n) leads to a basis of 0MS(n) in a natural way, by assigning 1 to one skeletal position and 0's to the rest in all n^2-2n-1 possible ways. We shall call this basis the **natural basis** associated with that skeleton. Thus we could determine a canonical basis for 0MS(n) if we could agree on a canonical skeleton of 0MS(n). Unfortunately, there does not appear to be any one skeleton of 0MS(n) which is superior to, or more natural than, all the others. Some skeletons possess certain kinds of symmetry or near-symmetry, while others guarantee the presence of a large number of zeroes in the magic squares of the natural bases they determine. Preference for one skeleton over another seems to be largely a matter of taste.

The preceding ideas can be used to determine a crude upper bound on the number of $n\times n$ classical magic squares. Since the sum of the first n^2 integers is $n^2(n^2+1)/2$, each line-sum of an $n\times n$ classical magic square must be $n(n^2+1)/2$. Letting $l=n(n^2+1)/2$, an $n\times n$ classical magic square can be zeroed by subtracting l/n from each entry. A skeleton of this $n\times n$ zero magic square consists of n^2-2n-1 positions and determines the zero magic square, and hence the classical magic square, completely. Thus the number of $n\times n$ classical magic squares is the number of ways the n^2-2n-1 positions of this skeleton can be chosen from the n^2 numbers $1-l/n, 2-l/n, \ldots, n^2-l/n$, choosing each number no more than once. Since the number of permutations of n^2-2n-1 elements which can be formed from a set of n^2 elements is $(n^2)!/(2n+1)!$, the maximum number of $n\times n$ classical magic squares, taking into account the 8 symmetries of a magic square, is $(n^2)!/8(2n+1)!$. The imprecision of this bound is revealed even when $n=3$: in that case, the bound says that there are at most nine 3×3 classical magic squares, while, as suggested earlier in this paper, it is easy to show that there is, in fact, only one.

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