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MODELS OF NEW PRODUCT DIFFUSION THROUGH ADVERTISING AND WORD-OF-MOUTH*

JOE A. DODSON, JR.† AND EITAN MULLER‡

A model of the diffusion process is developed which recognizes (1) the interaction between adopters and non-adopters and (2) the influence of external information sources such as advertising. The model is extended by incorporating the effects of repeat purchasing. The models written by the following authors are shown to be special cases of this model: Gould, Nerlove and Arrow, Vidale and Wolfe, Palda, Bass, Nicosia, and Glaister. The behavioral assumptions which support the model are made explicit and the implications of these assumptions for the shape of the new product growth curve are derived.
(MARKETING; MARKETING—BUYER BEHAVIOR; HEALTH CARE—EPIDEMIOLOGY)

1. Introduction

A manager seeking to introduce a new product into the marketplace has a limited number of variables under his control. The marketing manager must understand how these decision variables impact the diffusion process if he hopes to use them effectively. The theory of adoption of new products by a social system has been reviewed by Rogers [17]. These ideas have been expressed mathematically in diffusion models which emerged early in epidemology ([1], [2], and [11]).

A general model of the diffusion process which explicitly describes the influence of advertising and word-of-mouth is presented in the next section. §3 introduces the repeat purchase model and §4 shows that it generalizes several models such as those of Nerlove and Arrow [13], Gould [8], Vidale and Wolfe [22], Palda [16], Bass [3], Nicosia [14], and Glaister [7].

2. A General Diffusion Model For Durable Products

This section introduces a model which can be used to predict industry sales of a durable product.

Let the number of people in the market, N, be divided into $x(t)$—the number of people who are unaware of the existence of the product, $y(t)$—the number of potential customers who are aware of the product but have not yet purchased it, and $z(t)$—the number of current customers who have purchased the product. By definition

$$x(t) + y(t) + z(t) = N(t). \tag{1}$$

The variables x, y, z represent states in the diffusion process. At any point in time a consumer will be in one of these states.[1] A behavioral conceptualization of the progress of a consumer from unawareness through attitude change to ultimate purchase has been given by Lavidge and Steiner [10].

One variable influencing the movement of consumers through these states for a new product is the information acquired from contact with prior purchasers, i.e., word-of-mouth [23]. Early adopters of a new product or new idea interact with other less

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† N. W. Ayer ABH International, New York.

‡ University of Pennsylvania.

[1] This representation is similar to but different from that proposed by Urban [21]. This formulation excludes the possibility of purchase without knowledge of the product's existence as is the case with impulse buying.

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innovative members of the group. The potential adopters may be influenced in their purchase timing by the early adopters of the product. Also a certain proportion of potential customers may purchase the product independently of the influence of word-of-mouth. This inducement to purchase will represent the effects of distribution, promotion, advertising, and other forms of marketing effort.

The general model can be formulated as

$$\dot{x}(t) = -\beta x(t)(y(t) + z(t)) - \mu x(t), \tag{2}$$

$$\dot{y}(t) = \beta x(t)(y(t) + z(t)) + \mu x(t) - \gamma y(t), \tag{3}$$

$$\dot{z}(t) = \gamma y(t) \tag{4}$$

where a dot above a variable denotes differentiation with respect to time, β reflects the impact of word-of-mouth, and μ and γ reflect the effects of the marketing efforts of the firm. More precisely, the explanation of this set of equations is:

(a) The people who know, $y(t) + z(t)$, contact and inform a total of $b(y(t) + z(t))$, out of which only a fraction $x(t)/N(t)$ are newly informed. Thus $\beta = b/N(t)$. In addition, out of the total number of people informed through advertising $\mu N(t)$, only a fraction $x(t)/N(t)$ are newly informed.

(b) The number of people who know but did not yet buy is increased by those newly informed $\beta x(t)(y(t) + z(t)) + \mu x(t)$ and decreased by those who buy $\gamma y(t)$.

(c) The number of people who buy the product is $\gamma y(t)$. Note that in the case of a durable, if we assume that each consumer buys exactly one unit, equation (c) describes the sales, i.e., denoting the sales by $s(t)$, we have $s(t) = \dot{z}(t)$.

This description is similar to that used by Gould [8]. It represents a more realistic extension of the simpler cases considered by Gould since it includes both the effect of word-of-mouth and the effect of other sources of information, e.g., advertising. Its relation to Gould's model will be highlighted later. A similar model was presented by Bernhardt and Mackenzie [4]. The model is developed in more detail here to highlight the influence of advertising and word-of-mouth on the diffusion process, to clarify its relationship with previous work in marketing, and to lay the groundwork for an extension of the model to situations which involve repeat purchasing.

Consider the case where there are a fixed number, N, of potential cusumers of a product.[2] After buying the product, a purchaser is removed from the group of potential consumers. This is a restrictive assumption which does not allow for repeated purchases. Nonetheless this model may be predictive of the diffusion of such products as color television sets, refrigerators, washing machines, and other types of consumer durables *in their early stages*. The time frame over which sales would be predicted would have to be shorter than the replacement cycle for this model to be considered descriptive of such a market. And, the size of the market N would have to be stable. Full discussion of the model is postponed until repeat sales is introduced in the next section. However, it is useful at this point to consider some special cases of the model in order to clarify its relationship to earlier work and to make explicit the impact of the firm's decisions on the diffusion process.

(A) *Case* I. Consider first the case in which $y(t) = 0$ for all t. This might be the case when the new introduction provides such a significant improvement over existing alternatives that everyone who becomes aware of its existence adopts it. (2) becomes

$$\dot{x}(t) = -\beta x(t)z(t) - \mu x(t). \tag{5}$$

[2] Suggestions for incorporating a constant rate of entry and exit from the population of potential consumers in reduced versions of the model are presented by Bernhardt and Mackenzie [4].

Substituting $x(t) + z(t) = N$ into (5) yields

$$\dot{z}(t) = \beta(N - z(t))z(t) + \mu(N - z(t)). \tag{6}$$

Since $\dot{z}(t)$ is the rate of sales at time t, $s(t)$, the last equations can be written as

$$s(t) = a' + b'z(t) + c'z^2(t), \tag{7}$$

where $a' = \mu N$, $b' = \beta N - \mu$, and $c' = -\beta$. This is equivalent to the equation estimated by Bass [3]. Bass' coefficients of innovation and imitation are equivalent to μ and β. Bass showed that a discrete analogue of equation (7) gave good predictions of initial purchase timing for eleven consumer durables.

The general solution of (6) is

$$z(t) = N(1 - \exp(-\rho t))/(1 + \beta N \exp(-\rho t)/\mu) \tag{8}$$

where $\rho = \beta N + \mu$. This solution provides insight into the relationship between the penetration curve and the relative values of b, the contact coefficient, and μ, which represents the influence of a firm's promotional activities. When $b > \mu$ the growth in penetration follows the penetration curve shown in Figure 1. However, when $b < \mu$, i.e., when promotional activities dominate the market conversion mechanism, then penetration follows the curve shown in Figure 2.

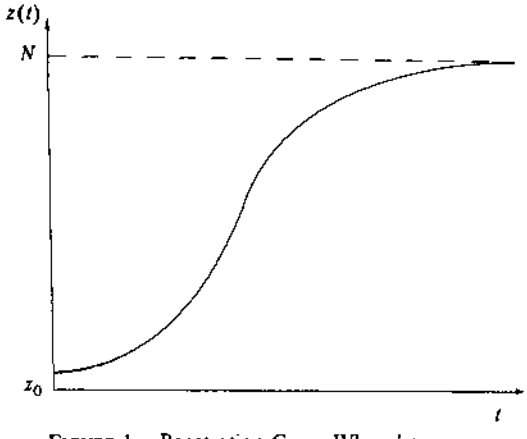


FIGURE 1. Penetration Curve When $b > \mu$.

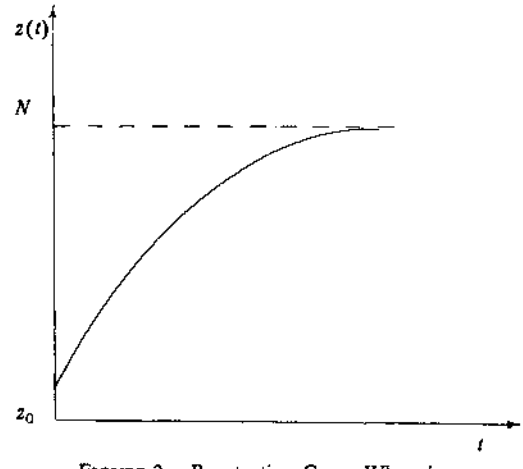


FIGURE 2. Penetration Curve When $b < \mu$.

To find the time t^*, at which the sales rate reaches its peak, we differentiate $s(t)$ and set it equal to zero, yielding

$$t^* = (1/\rho)\ln(\beta N/\mu) \quad \text{and} \quad s(t^*) = \rho^2/4\beta. \tag{9}$$

Since $\rho = \beta N + \mu$, the following observation can readily be made. The higher the advertising effort μ, the sooner the peak will arrive and the larger the peak will be.

(B) *Case* II. The second case considered is one for which $\mu = 0$ and $z(t) \ll y(t)$. This occurs when the number of people who actually buy the new product is much smaller than those who know about the product but did not yet buy. This may be representative of a new introduction which faces considerable resistance to trial. Equation (3) then becomes

$$\dot{x}(t) = -\beta x(t) y(t). \tag{10}$$

which gives[3]

$$s(t) = \gamma N - \gamma x_0 \exp((-\beta/\gamma)z(t)) - \gamma z(t). \tag{11}$$

[3] General references which describe the mathematics of differential models include Lotka [11] and Ross [18].

This equation is a generalization of the Bass model since approximation of the exponential function by a second degree polynomial will produce an equation which is equivalent to (7).

(C) *Case* III. The assumption that $y(t) = 0$, i.e., everyone who becomes aware of the product adopts it, is combined with the assumption that there is no influence from prior adopters, i.e., $\beta = 0$. (3) then becomes

$$\dot{x}(t) = -\mu x(t) \tag{12}$$

that is, a constant proportion of the potential consumers purchase the product each period. This implies

$$z(t) = N(1 - \exp(-\mu t)) \tag{13}$$

given that $x(0) = N$. This type of exponential growth to some asymptote as well as a modified version allowing for growth in N was utilized in the sales models of Fourt and Woodlock [6].

Note that equations (8), (11) and (13) were derived under completely different assumptions and thus are applicable to different situations. Each represents a special case of the model developed from equations (2), (3) and (4) which includes the effect of word-of-mouth and advertising within a multi-stage model of the adoption process. Note also that the models presented here are only concerned with the timing of initial purchase. The next section extends the general model to incorporate repeat purchasing.

3. Growth Model with Repeat Sales

The discussion to this point has concentrated on a model which represents the growth in first purchases of a product. It is worthwhile to consider how one might incorporate repeat sales into the model of new product penetration. Repeat sales becomes an important consideration when one is dealing with low-priced, frequently-purchased, branded products, or durable products for a long enough time frame for repeat sales to become a significant proportion of total sales.

Let $x(t)$, $y(t)$ and $z(t)$ be defined as before.[4] The new set of transition equations is given by

$$\dot{x}(t) = -\beta x(t)(N - x(t)) - \mu x(t) + \phi(N - x(t)), \tag{14}$$

$$\dot{y}(t) = \beta x(t)(N - x(t)) + \mu x(t) - (\gamma + \phi)y(t) + \theta z, \tag{15}$$

$$\dot{z}(t) = \gamma y(t) - (\phi + \theta)z(t) \tag{16}$$

where

ϕ is a forgetting parameter,

θ is a switching constant, i.e., the rate at which consumers purchase rivals' brands.

The flows are summarized in Figure 3.

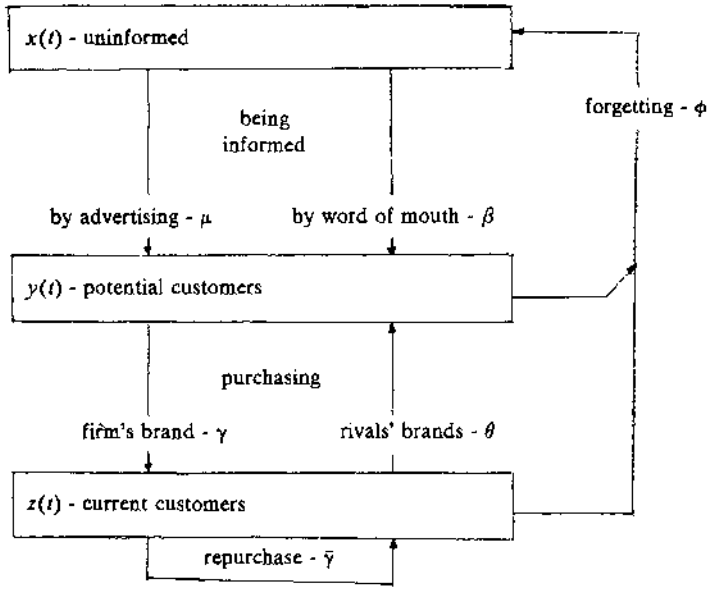
Note that the sales rate, denoted by $s(t)$ is given by

$$s(t) = \gamma y(t) + \bar{\gamma} z(t) \tag{17}$$

where $\bar{\gamma}$ is the repeat purchase parameter. Sales result from trial by new customers and repeat purchasing from current customers. The previous model is extended to include repeat purchasing, $\bar{\gamma} z(t)$, and switching $\theta z(t)$. Thus the model need no longer be restricted to situations involving a single adoption. This extension enables us to use the model to describe the diffusion process for frequently purchased goods and to extend our prediction for durable goods further into the life cycle, where repeat

[4] The definitions of a current customer has to be somewhat changed to take into account the possibility of switching. Thus we define z to be the number of customers who are aware of the product and whose last purchase was of the firm's brand. y, as before, can be defined as $N - x - z$.

purchasing occurs. These extensions also indicate the need for a forgetting parameter, ϕ, since it represents forgetting of a specific brand.



It should be noted that there exists a closed loop from y to itself due to purchase of rivals' brands. It is ignored here since it does not affect the transition equations.

FIGURE 3

Although a closed form solution of (14)–(17) is not possible to achieve (because of the appearance of an unintegrable term in the solution) it is possible to characterize the solution by means of phase diagrams. This is done in the appendix. The main result is summarized by the following proposition.

PROPOSITION. *The solution of equations* (14)–(17) *is monotonic increasing in the number of informed persons* $(N - x)$ *and the number of current customers* z, *and is either monotonic increasing or single peaked in the potential customers* y *and the sales* s.

Examples are provided in Figures 4–7. Figure 4 illustrates the growth in sales for a product where the contact coefficient β is relatively "small" and the repeat purchase rate $\bar{\gamma}$ is large relative to the trial rate γ, i.e., $\bar{\gamma} > \gamma$. The definition of a "small" or "large" contact coefficient is given in the appendix. It should be noted that all the

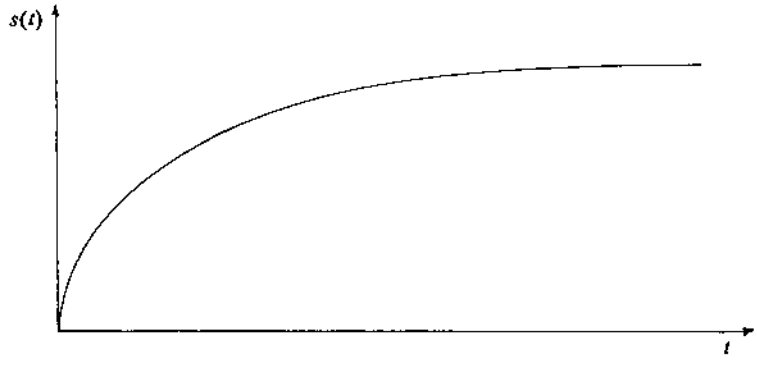


FIGURE 4. Sales When the Repeat Purchase Rate Is Larger Than the Trial Rate and the Contact Coefficient Is Relatively Small.

parameters enter the solution, but the repeat rate $\bar{\gamma}$ and the contact coefficient β are the critical determinants of the success of a new brand, the timing and the maximum sales level the brand will achieve. When the contact coefficient is relatively "large" and the repeat rate is larger than the trial rate the sales growth is plotted in Figure 5. When the trial rate is larger than the repeat rate the sales curve will peak and then fall below the peak to some lower level of sales, Figures 6 and 7. The long term stable level of sales is a function of the relative values of the repeat rate $\bar{\gamma}$ and the switching rate ϕ.

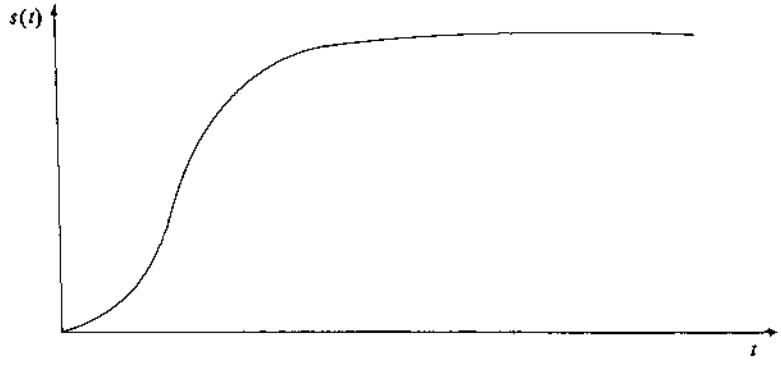


FIGURE 5. Sales When the Repeat Purchase Rate Is Larger Than the Trial Rate and the Contact Coefficient Is Relatively Large.

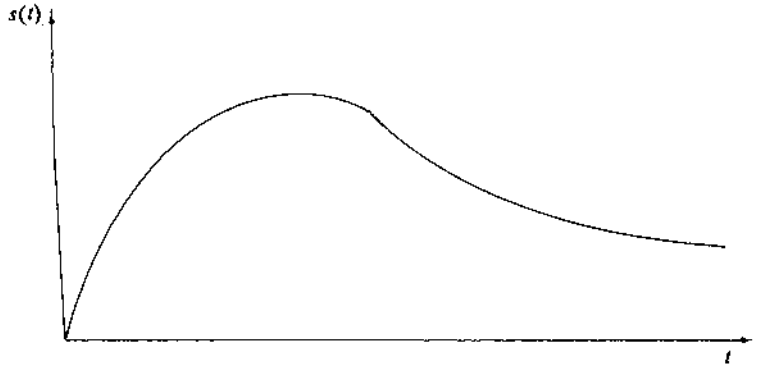


FIGURE 6. Sales When the Trial Rate Is Larger Than the Repeat Rate and the Contact Coefficient Is Relatively Small.

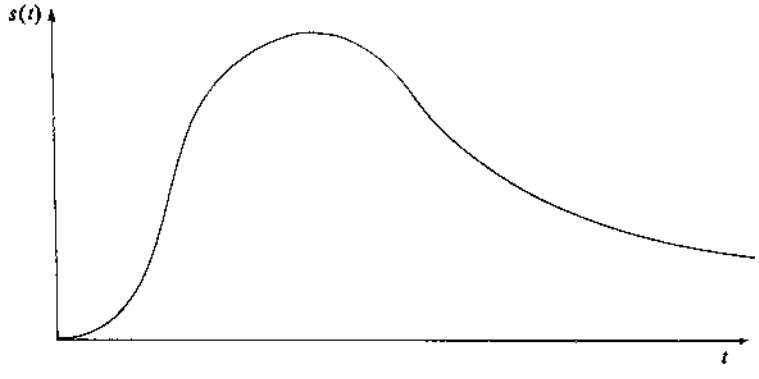


FIGURE 7. Sales When the Trial Rate Is Larger Than the Repeat Rate and the Contact Coefficient Is Relatively Large.

A closed form solution to equations (14)–(16) can be obtained for special cases of the model but this requires some assumptions about the values of the model's parameters. Dodson and Muller [5] derive solutions for some special cases of this model and illustrate a procedure for obtaining empirical estimates of these models' parameters. The latter can also be found in the unabridged version of this paper. The relationship between the diffusion model developed here and earlier modeling of the effects of advertising can be seen by considering the special case where the trial rate and the repurchase rate are equal, $\bar{\gamma} = \gamma = \hat{\gamma}$, and $\beta = \theta = 0$. This represents a situation where awareness is generated directly from advertising. Assume the effect of competitive activity is constant and is reflected in the repeat purchase rate $\bar{\gamma}$. Then equations (14)–(16) become

$$\dot{x}(t) = -\mu x(t) + \phi(N - x(t)), \tag{18}$$

$$\dot{y}(t) = \mu x(t) - (\hat{\gamma} + \phi)y(t), \tag{19}$$

$$\dot{z}(t) = \hat{\gamma}y(t) - \phi z(t) \tag{20}$$

and sales is given by

$$s(t) = \hat{\gamma}(N - x(t)). \tag{21}$$

Differentiating with respect to time and substituting from (18) yields

$$\dot{s}(t) = \mu(\hat{\gamma}N - s(t)) - \phi s(t). \tag{22}$$

Since μ represents the rate of conversion from unawareness to awareness, i.e., movement from state x to state y, it is naturally a function of the firm's advertising expenditure. If μ is to be linearly related to advertising costs, then (22) is equivalent to the Vidale-Wolfe model [22] with the appropriate advertising response constant. Horsky [9] has shown Palda's model [16] to be a special case of Vidale-Wolfe's formulation.

To see the relation to Gould's model, one has to use Gould's definition of goodwill denoted by G, as the number of people who know of the product, i.e. $G(t) = N - x(t)$. Substituting that into (18) yields

$$\dot{G}(t) = \mu(N - G(t)) - \phi G(t). \tag{23}$$

This equation is the first diffusion model discussed by Gould [8]. From (23) and (22) it is clear that the dynamics of the models of Gould and Vidale and Wolfe are the same and are equivalent to the reduced model (18)–(21). The fact that the models of Gould and Vidale and Wolfe are equivalent was also found independently by Sethi [19]. While the "first" diffusion model of Gould ignores the influence of word-of-mouth, Gould discussed a second diffusion model which assumes that the firm has direct control over the mechanism of word-of-mouth, and it is the only mechanism of information diffusion. Both assumptions seem unrealistic. At best the control of the firm is very indirect, for example showing women talking about deodorant and washing liquids in the hope that this will indeed induce women to do just that. It seems more reasonable to expect the word of mouth mechanism to be a complementary one to the direct message sending mechanism. Both mechanisms were taken into account in the general diffusion model above.

Gould recognized the fact that his diffusion processes represent extreme cases [8, p. 368]. This fact, however, was ignored in a later paper by Glaister [7]. Although his contention that there must be a critical mass or a threshold phenomenon is appealing and certainly imaginative, it is based on an erroneous notion that the only mechanism

at work is word-of-mouth. A formal proof of that fact (together with a discussion on skimming pricing policy) is given in Muller [12]. To obtain Glaister's model from the general diffusion model, rewrite equations (14)–(16) with $\mu = \phi = \bar{\gamma} = \theta = 0$, and further assume that the current customers' group is negligible with respect to the potential customers group, that is $z \ll y$. This yields:

$$\dot{x}(t) = -\beta x(t)y(t),$$

$$\dot{y}(t) = \beta x(t)y(t) - \gamma y(t),$$

$$\dot{z}(t) = \gamma y(t).$$

This is exactly the Glaister model where β is a function of price.

To see the relation to Nerlove and Arrow's model [13] rewrite equation (23) as

$$\dot{G}(t) = \mu x(t) - \phi G(t). \tag{23.1}$$

Note that $\mu x(t)$ is the effective investment in goodwill, since $\mu x(t)$ is the number of people newly informed, while the goodwill is the total number of informed people. Denote the effective investment by $I(t)$, that is:

$$I(t) = \mu x(t) \tag{24}$$

and substitute into (22) to get:

$$\dot{G}(t) = I(t) - \phi G(t). \tag{25}$$

This is a standard capital accumulation model. It is also the Nerlove and Arrow model when the capital in question is advertising goodwill.

It should be noted that except for Gould, who based his model on the early diffusion models of Ozga [15] and Stigler [20], the earlier models were built on different premises. Vidale and Wolfe's is an empirically oriented model which was not built around a diffusion process. The same is true with respect to the Bass model. His basic assumption was that the conditional probability of purchase (given that no purchase has yet been made) is linear in the number of people who already bought the product. It is interesting to note that such differing approaches can bring about similar formal models. This is even more striking when comparing the general diffusion model to Nicosia's [14]. His model can be summarized by the following equations:

$$\dot{B} = a_1 A - a_2 B, \tag{26}$$

$$\dot{A} = a_3 B - a_4 A + a_5 U \tag{27}$$

where

B = buying behavior (sales),
A = attitude,
U = advertising,
a_1 to a_5 = known constants.

Equations (26) and (27) imply that sales and attitude are positively correlated. The depreciation in both sales and attitude is proportional to the level of those variables. The attitude tends to increase with an increase in the level of advertising. A formally equivalent model (the variables have to be reinterpreted as well as the parameters) can be defined by calling the effective investment $I(t)$, that is, recall equation (27), and set $\beta = 0$. Equations (16) and (15) become:

$$\dot{z} = \gamma y(t) - (\phi + \theta)z(t), \tag{28}$$

$$\dot{y} = \theta z(t) - (\gamma + \phi)y(t) + I(t). \tag{29}$$

This is formally equivalent to the (26) and (27), where the buying behavior variable is the number of current customers, and the attitude is summarized by the number of potential customers. The parameters conform to the "over-damped" case discussed by Nicosia.[5]

4. Conclusion

The general diffusion model is a powerful model in that it unifies several theories and models both in economics and in marketing. Specifically, the models written by the following authors are special cases (or closely related to one) of this model: Gould, Nerlove and Arrow, Vidale and Wolfe, Palda, Bass, Nicosia, and Glaister.

The influence of advertising and word-of-mouth in the diffusion process have been incorporated into a model that includes the effects of repeat purchasing. And, the influence of each of these mechanisms, advertising and word-of-mouth, on the shape of the new product growth curve has been described.

Appendix

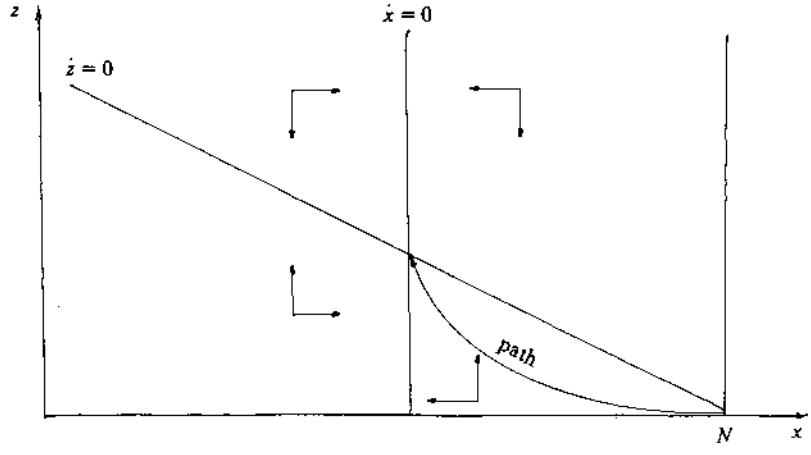
PROOF OF THE PROPOSITION. From the $z \cdot x$ space it is clear that both x and z are monotonic and thus since the sales can be written as

$$s = \gamma(N - x) + (\bar{\gamma} - \gamma)z$$

then if $\bar{\gamma} > \gamma$, s is monotonic increasing.

$\dot{x} = 0$ is $\mu x = (N - x)(\phi - \beta x)$,

$\dot{z} = 0$ is $\gamma(N - x) = (\gamma + \phi + \theta)z$;



checking $\ddot{s}$ at $t = 0$, we find that since $x(0) = N$, $\ddot{s} < 0$ if and only if $\beta < 1/N[\gamma + \phi + \mu - \bar{\gamma}]$, we denote this case by a "relatively small" contact coefficient. From the $y \cdot x$ space it is clear that y is either monotonic or single peaked.

$\dot{x} = 0$ is $\mu x = (N - x)(\phi - \beta x)$,

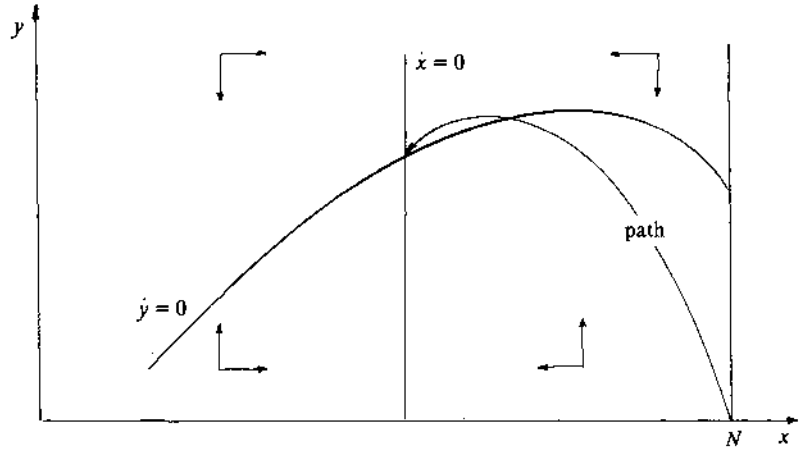
$\dot{y} = 0$ is $(\gamma + \phi + \theta)y = \beta x(N - x) + \mu x + \theta(N - x)$.

Lastly we have to prove that if s is not monotonic then it can only be single peaked (and not, say, with a max. and a min. or multi-maxima.)

[5] See Nicosia [14, p. 228].

The sales can be written as

$$s = (\gamma - \bar{\gamma})y + \bar{\gamma}(N - x).$$



Differentiating twice we get that when $\dot{s} = 0$, $\ddot{s} = \dot{x}\{\gamma(\beta(N - 2x) + \mu - \theta) + \bar{\gamma}(\theta - \gamma)\}$. At the first extremum $\ddot{s} < 0$ since this extremum must be a maximum (since initially s is increasing). Since $\dot{x} < 0$ the expression inside the curly bracket is positive when $\dot{s} = 0$. At any subsequent time, since $\dot{x} < 0$ this expression *increases*. Thus if $\dot{s} = 0$ again, $\ddot{s} < 0$ which implies another maximum. This of course is impossible since there must be a minimum in between. After the first maximum, therefore $\dot{s} = 0$ only if $\dot{x} = 0$ which is at the steady state.[6] Q.E.D.

[6] We wish to acknowledge our appreciation for helpful comments by Nancy L. Schwartz and two anonymous referees on earlier versions. Most of this work was done while the authors were at Northwestern University.

This is an abridged version of the paper. Anyone who wishes to obtain the unabridged version which includes empirical estimation can do so (at cost) through the TIMS office, 146 Westminster St., Providence, R.I. 02903.

References

1. Bailey, N. T. J., *The Mathematical Theory of Epidemics*, Hafner, New York, 1957.
2. Bartlett, M. S., *Stochastic Population Models in Ecology and Epidemiology*, Metheun, London, 1960.
3. Bass, F. M., "A New Product Growth Model for Consumer Durables," *Management Sci.*, Vol. 15 (January 1969), pp. 215–227.
4. Bernhardt, I. and Mackenzie, K., "Some Problems in Using Diffusion Models for New Products," *Management Sci.*, Vol. 19 (October 1972), pp. 187–200.
5. Dodson, A. J. and Muller, E., "Models of Product Diffusion Through Advertising and Word of Mouth," Discussion Paper No. 182, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, April 1976.
6. Fourt, L. A. and Woodlock, J. W., "Early Prediction of Market Success for New Grocery Products," *J. Marketing*, Vol. 26 (October 1960).
7. Glaister, S., "Advertising Policy and Returns to Scale," *Economica*, Vol. 41 (May 1974) pp. 139–156.
8. Gould, J. P., "Diffusion Processes and Optimal Advertising Policy," in E.S. Phelps et al., editors, *Microeconomic Foundation of Employment and Inflation Theory*, Norton, New York, 1970.
9. Horsky, D., "Empirical Analysis of Optimal Advertising Policy," *Management Sci.*, Vol. 23 (June 1977), pp. 1037–1049.

10. LAVIDGE, R. J. AND STEINER, G. A., "A Model for Predictive Measurements of Advertising Effectiveness," *J. Marketing*, Vol. 25 (October 1961).
11. LOTKA, A. J., *Elements of Mathematical Biology*, Dover, New York, 1956.
12. MULLER, E., "On Diffusion Processes and Skimming Pricing Policy," Center for the Study of Organizational Innovation, Univ. of Pennsylvania, November 1977.
13. NERLOVE, M. AND ARROW, K. J., "Optimal Advertising Policy under Dynamic Conditions," *Economica*, Vol. 29 (May 1962), pp. 129–142.
14. NICOSIA, F. M., *Consumer Decision Processes*, Prentice-Hall, Englewood Cliffs, N. J., 1966.
15. OZGA, S., "Imperfect Markets Through Lack of Knowledge," *Quart. J. Econom.*, Vol. 74 (February 1960), pp. 29–52.
16. PALDA, K. S., "The Measurement of Cumulative Advertising Effects," *J. Business*, Vol. 38 (April 1965), pp. 162–179.
17. ROGERS, E. M., *Diffusion of Innovations*, The Free Press, New York, 1962.
18. ROSS, S. L., *Introduction to Ordinary Differential Equations*, Blaisdell, London, 1966.
19. SETHI, S. P., "Dynamic Optimal Control Models in Advertising: A Survey," *SIAM Rev.*, Vol. 19 (October 1977).
20. STIGLER, G. J., "The Economics of Information," *J. Political Economy*, Vol. 69 (June 1961), pp. 213–225.
21. URBAN, G. L., "SPRINTER MOD III: A Model for the Analysis of New Frequently Purchased Consumer Products," *Operations Res.*, Vol. 18 (September 1970), pp. 805–854.
22. VIDALE, M. L. AND WOLFE, H. B., "An O.R. Study of Sales Response to Advertising," *Operations Res.*, Vol. 5 (June 1957), pp. 370–381.
23. WHYTE, W. H., JR., "The Web of Word of Mouth," in Lincoln H. Clark (ed.), *Consumer Behavior*, Vol. II, 1958.