Social Structure from Multiple Networks. II. Role Structures[1]

Scott A. Boorman
University of Pennsylvania

Harrison C. White
Harvard University

Role structures in small populations are given operational meaning as algebras generated from the sociometric blockmodels of Part I by Boolean multiplication (matrix multiplication employing binary arithmetic). Many different sociometric structures can yield the same algebraic multiplication table, which captures a different level of social structure. Elements of the algebras are interpreted concretely as compound roles, and interlock among these roles is studied through investigation of their algebraic properties (equations and inclusions). Similarities and differences among algebras from six case studies are explored by means of homomorphisms as well as by multidimensional scaling on a derivative numerical distance measure. Results for particular populations, including reliability and stability tests, are summarized through simple target tables reporting aggregations of more complicated role structures.

Roles have long fascinated sociologists and lay people alike. Particular kinds and classes of role have received meticulous and often insightful treatments. Many of these treatments have been by professional sociologists and anthropologists (for sick people, Parsons [1949]; for Italian immigrants in an American city, Whyte [1943]; for members of the International Typographical Union, Lipset, Trow, and Coleman [1956]). Some of the best analyses have been by amateurs (for diplomats, Nicolson [1932]; for federal judges, Wyzanski [1952]; for soldiers, Hackett [1963]; for bureaucrats, Tullock [1965]; for scientists, Davis [1968]).

Within this vast body of work, almost all useful insights have remained particular. There is no model for roles comparable with the fundamental models in other areas of analytic sociology: the industrial mobility of

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labor (Blumen, Kogan, and McCarthy 1955); social hierarchies (Landau 1965); population dynamics (Lotka 1925); attitude change (Coleman 1965); social networks (Moreno and Jennings 1945). In dealing with roles, anthropologists more than sociologists have tried to break beyond the limits of their specific examples (Rivers 1924); however, their chief success in constructing general role theories has been in the specialized areas of kinship and kinship semantics (e.g., Lévi-Strauss 1949; Romney 1965; but see Schneider 1965). Almost alone, Nadel (1957) attempted to set up a formalism for describing roles and role interlock in greater generality. Nevertheless, his effort also remains unsatisfying, mainly because his formal descriptions have little bearing on concrete populations.[2]

Any attempt to develop a general theory of roles is audacious by definition. An excellent a priori case can be made that such generality is in fact not possible: for quite different reasons economists, social psychologists, and lawyers would feel sympathetic to such a viewpoint. By comparison, the vision of the classical sociologists, particularly Durkheim and Simmel, is notably more optimistic. These early theorists were committed to the possibility of a unified treatment of roles, crystallized from the multiple relationships among men in groups.

This paper is a direct theoretical attack on the problem of role structure. As in the tradition of French structural anthropology through Lévi-Strauss and Lorrain (and as also influenced by Jakobson and the linguists), the approach is algebraic.[3] Role structure is modeled as an algebra, the elements of which are compound roles formed from a fixed set of primitive roles; different role structures are compared by algebraic mappings (homomorphisms). The quest is for a dual description of social structure of the kind which classical balance theory suggests in principle but is unable to extract from data. On one level are networks, which Part I aggregates into blockmodel images. On a second and distinct level are roles. Connecting the two levels are statements (equations and inclusions) in the appropriate algebra. These statements formally describe role interlock. Transitivity is a familiar example, though a very restrictive one.

Throughout the paper, all developments proceed from a central belief that generalizations about role structure should be inductively obtained

[2] Nadel's formalism shows him to have been deeply influenced by symbolic calculi in the tradition of Russell and the young Keynes (1921). This bias was unfortunate, since it distracted him from the development of techniques with computational power. Poincaré (1913) is a biting criticism of formal calculi possessing chiefly definitional uses; see also Lewis (1960) for a fascinating historical review of classical systems of logistic which bears directly on the problems of constructing novel formalisms.

[3] The French structuralist tradition is filled with attempts to translate structural ideas into some kind of algebraic or quasi-algebraic language. In addition to Lévi-Strauss (1949), see Courrège (1965), Foucault (1966), and Lacan (1957). Lorrain's (1975) developments are by far the most sophisticated.

by comparing results based on concretely presented social structures. The key is the word "concrete": adoption of a concrete viewpoint sets the present work apart from the long tradition of algebraic models of kinship which may be grounded on purely cultural constructs not tied to any particular population (Weil 1949; White 1963).[4] Below, we always derive role structure starting from networks of social relations in a concrete population.

This paper is more complicated technically than Part I. Several levels must be kept in view: (1) the raw networks on a concrete population; (2) blockmodels of those data, obtained by the methods of Part I; (3) role structures (semigroups) formed from the blockmodels; (4) comparisons of role structures, using homomorphisms and derived measures of structural distance; and (5) sets of algebraic equations (target tables) obtained from the semigroups, which may be given substantive interpretations with reference to the original data. At all times it is crucial to separate formal constructions from their associated interpretations. One of the problems with sociological language is that it is inherently rich, so rich as to evade the capacity of any formalism to capture it entirely (Boudon 1968). We will speak of roles, role structures, and positions both in a specific technical sense and in a broader empirical one. The context should make the intention clear.

In a strictly ancillary (though extremely important) way, we will also make use of certain additional technical methods. Most important is our use of the MDSCAL-5 algorithm based on the work of Shepard and Kruskal (as a means of representing similarities among role structures in a Euclidean space) as well as our use of inclusion partial orders (to suggest simplifications of role structures).

SUMMARY OF BLOCKMODELS (SYNOPSIS OF PART I)

Each of several networks of ties on a concrete population is reported as a square binary (0–1) matrix reporting the incidence of the given type of tie. The rows and columns of each matrix are then identically permuted in a self-consistent search with two objectives. First, the population of

[4] An additional difference of great technical importance is that the algebraic work of Weil and of White is strictly group theoretic. As such, it applies only to a very restricted class of social structures, chiefly represented by the Australian classificatory systems. The crucial step from groups to semigroups was taken in the late 1960s, in independent contributions by Boyd (1966, 1969*a*, 1969*b*) and White (1969). This transition relinquishes the great power of group theory as a branch of mathematics in exchange for the far wider substantive applicability of semigroup algebra. (For mathematical definitions see the section on methods below.) To some extent, a parallel course of evolution has been independently followed by mathematical linguistics, which has increasingly come to trade mathematical power for breadth of application (see Harris 1968; Chomsky and Halle 1968).

persons is to be partitioned into *positions;* second, for each type of tie there is to be a specified network of *bonds* connecting positions. Following rearrangement, the rectangular submatrix reporting ties from persons in one position to persons in a second position is termed a *block* for that type of tie. The specific aim of rearrangement is to reveal *zeroblocks,* blocks containing no ties; all other blocks correspond to bonds between positions. The *image* of a type of tie is a square binary matrix reporting the bonds. A *blockmodel* for a population is the set of images obtained from the separate types of tie. The term *block* will also be used for the set of members of a position.

Mathematically, a blockmodel is a homomorphic image of the original multigraph (Ore 1962).

Thus in Part I the concept of a position (in a social structure) is given operational content as a set of persons who are structurally equivalent in a blockmodel. Furthermore, to each position is associated a well-defined *role set,* which is that position's row and column across all blockmodel images; this definition is consistent with Merton's (1957, p. 369) usage. Both ideas take *simultaneity* into account: the fact that all positions and role sets are determined relative to one another. But the blockmodels of Part I fail to deal with interrelations among different perspectives. A role set may be seen as one position's view of the interlock of roles. But each position has its own viewpoint, and one may hope for global regularities which reflect a structure of roles not limited to any one position.

The task of Part II begins here. A number of case studies are treated, involving six small populations: (1) an industrial work group, (2) the management of an industrial firm, (3) a contemporary American monastery, (4) a research network of biological scientists, (5) and (6) two different experimental college fraternities. Five of the populations were discussed in Part I; one of the fraternity cases is new. Positive and negative affect, in various guises, are the types of tie most commonly represented in these case studies. A major part of our results hence concerns role structures based on sentiment.

Alternative blockmodels for a given case are suggested in Part I (see also Breiger, Boorman, and Arabie 1975), depending on the strictness with which the sociometric data are assessed and on the level of aggregation sought. A major concern of the present paper, and a principal cause of its length, is identifying a nucleus of role structure which is robust across all blockmodels suggested for a given case (see figs. 13 and 14).

Three distinctive properties of the blockmodels of Part I are crucial in the development of role structure models. First, a bond may obtain from one position to another on each of several types of ties—"ambiguity" in the overall quality of relations from one position to another is considered normal. Second, no special weight is given to reciprocated choices—a bond may or may not be reciprocated by a bond of that or another type from the other position. Third, a bond from a position to itself is treated on the same footing as one to another position—"reflexivity" is a separate substantive issue for each position on each type of tie.

The organization of Part II is conventional: Theory, Methods, Results, and Conclusion. Each section is as self-contained as possible.

THEORY

The main ideas build from one axiom, the Axiom of Quality, together with one definition, that of role structure (see below). Mathematical details are deferred to the section on methods; here, only minimal formalism is introduced to allow initial discussion of examples.

The Basic Ideas

Introduction.—Several different contributions to role theory serve as points of reference. Our approach resembles that of Gross, Mason, and McEachern (1958) in its commitment to deriving roles from data on concrete populations. However, they studied only a single, focal position (that of school superintendent) possessing a single incumbent. Their problem was the nature and degree of role consensus among occupants of certain other positions with formally prescribed ties to the focal position (e.g., teacher, principal, school-board member, finance-committee member, selectman). They collected varied data, often in metric form, on a few selected roles vis-à-vis the focal position.[5] In contrast, our own emphasis is on *closure:* we will be concerned with setting up a concept of a closed system of roles and with exploring the properties of such closed systems.[6]

Kinship suggests a second contrast. The present approach has its roots in kinship (specifically, in the Australian classificatory systems), and several of our major themes will reflect that origin (see also White 1963, chap. 1). However, those classificatory systems are less descriptions than brilliant ideologies of social structure evolved by aboriginal civilizations. In them, as in all viable kinship systems, the whole point of the system is to provide a set of rules and nomenclature for the benefit of participat-

[5] Observe how ordinary language tends to confuse differences between positions on the one hand and relations between positions on the other. Many of the operative features of social structure are lost thereby: it is possible to think of a worker as a person who works, independently of his relationship to a foreman; it is not possible to think of a vassal independently of his relationship to a lord (see Dibble [1972, p. 156], as well as Sim [1966]).

[6] In this respect, the present developments follow the pattern set by economic theory, where closed systems provide the basic model, rather than of classical sociological role theory.

ing members. Anthropologists have sensed this utilitarian point when they have talked of coding and of the information content of a kinship system.[7] By contrast, the kind of role structures which interest us here are descriptions of actual overall structure not accessible to any unaided observer, whether participant or not. Upon reflection, this should not be surprising: there is no inherent reason why the global properties of a social structure should be transparent to its members, though each may know his own position well enough.[8]

Recent developments in sociological role theory are surveyed by Mirra Komarovsky in her "Presidential Address" (1973). She differentiates sharply between "formal, Simmelian" and "substantive" aspects of roles, the latter of which enters through an accounting of prevailing "rights and obligations." Such a distinction is in fact well founded in the literature. However, it is also largely artificial and is perhaps best viewed as another manifestation of the heritage from kinship (where formal terminological studies may represent an exercise in formalism which is almost totally content free).[9] As in Part I, we try explicitly to do away with the distinction. In contrast with Komarovsky's "rights and obligations," substantive content now enters as the incidence of bonds in a blockmodel, which in turn is based on the incidence of ties in a concrete population. The present work then builds role structures in a formal way but always with this substantive foundation.

Like Gross et al., Komarovsky rightly emphasizes the importance of recognizing as a variable the degree of consensus on role expectations. However, she is handicapped by the absence of basic phenomenology. Consensus is a concept with strong connotations of equilibrium, but sociology has never developed any counterpart to the general equilibrium conditions forming the basis of economic theory (which have been operationalized through input-output tables, among other methods). Below, we treat the weakness of phenomenology as fundamental, the problem of defining and measuring consensus as secondary. Only when the concept of role structure has been operationalized can issues of expectations be approached.

[7] See Wallace (1961) for attempts to estimate upper bounds to the complexity of folk taxonomies.

[8] Once again, this observation will seem a commonplace to economists, who take for granted that the interactions of economic forces transcend the information-processing capabilities of any single agent. This is the traditional explanation of why centralized economic planning is so difficult to make efficient (Koopmans 1957; see also Kornai 1971). For a sensitive analysis of the limits of observer perception in a complex role setting, see Ekvall (1960), which is a description of the author's experiences as U.S. army interpreter for the United Nations at the Korean cease-fire talks.

[9] See, for example, Kay (1965) on Dravidian systems; Buchler (1966) on Omaha systems; and Romney (1965) for the Kalmuk case, which is a lineal type of Omaha classification.

Komarovsky also poses three main criticisms of existing role analysis. The first concerns neglect of individuality. We do not deal with this directly, but it should be pointed out that blockmodels suggest a highly natural way of measuring individual differences. Only rarely will bonds in a blockmodel image correspond to completely filled matrices in the original data; pulling back from images to data accordingly suggests that individuality may be captured by the pattern of the variability.

She directs a second criticism against the static flavor of role analyses, the awkwardness of discussing change and manipulation. Here again we treat as fundamental the descriptive problem: that of comparing blockmodels over time, and perhaps across different numbers of blocks as well. This problem is approached using the algebraic concept of homomorphism and the JNTHOM algorithm (see section on methods below).

Komarovsky's third criticism of role analysis is phrased in terms of a denial that norms determine role behavior and thus explain conformity and the social order (but see Rommetveit 1953). Our agreement with her on this point should already be clear. Returning, however, to our starting emphasis on multiple levels of structure, it should be stressed that our present approach does not exclude norms and in some instances may actually serve as a search procedure for cultural regularities. Certain aspects of role interlock (algebraic equations) will be given a significance as abstract regularities emerging from particular blockmodels. Where such equations arise and are interpretable, they may fulfill the same function as the analogous equations in classificatory kinship, of which they are reminiscent: the equations generated from a blockmodel may represent nascent cultural roles, perhaps even positive norms. More complex models will obviously be needed, but the principle is an important one: abstract cultural regularities are patterns emergent from concrete networks among particular persons.[10]

Role reciprocity.—In the sociological literature the concept of role has long been associated with a second concept, role reciprocity; one might almost say that the idea of reciprocity has been the main contribution to role theory which is distinctly sociological and not psychological in content. Included in it is the notion that alter should accept ego's expectations of him (and conversely), as well as the further notion that the converse expectations alter holds of ego should mirror in some sense (though perhaps not duplicate) ego's expectations of alter. In our terms,

[10] This agrees with the view expressed by Zetterberg (1965) that the primitive terms of sociology should arise directly from human agents and their actions. The views of Homans on this subject are well known (see Homans 1962). It is interesting that certain areas of contract law have recently begun to move in a direction suggestive of blockmodels, particularly the zeroblock concept (see the preface to Reitz 1975).

"ego" is to be thought of as a representative incumbent of one position and "alter" as that of a second (perhaps the same) position.

What has already been said suggests that reciprocity is far too restrictive and also too slippery a notion on which to build a theory of roles. Reciprocity focuses on two positions in isolation. As such, it fails to account for indirect causal effects arising from the wishes and expectations of third positions, perhaps many of them.[11] Also, the usual concept of role reciprocity remains heavily tied to subjective perceptions by the parties which are at once very difficult to establish and often not relevant to the objective structure.[12]

Suitably formulated, however, the concept of reciprocity does suggest the two main ideas of the present theory. The first is that one role generates another: A's expectations toward B generate reciprocal (though perhaps very different) expectations toward A. The second is the idea of *chaining*: taking the generation process one step further, B's expectations toward A may interact with A's original expectations toward B to produce new expectations. The difficulty with reciprocity lies in the fact that it does not push the implications of either of these ideas to their natural conclusion on a social structural level, as we now proceed to do.[13]

Compound images.—In Part I, a separate image was found for each type of tie. This image reported as a binary matrix the incidence of bonds of that type among positions. Now generate compound images: each

[11] Lévi-Strauss makes much the same point in distinguishing "restricted exchange" from "generalized exchange" (see discussion in Ekeh 1974).

[12] Is my liking for James a social fact or only a psychological quirk if any of the following hold true: (i) he does not reciprocate my liking; (ii) he does not reciprocate with any other type of tie; (iii) he is not aware of my sentiment toward him; (iv) he does not know my name; (v) he does not know whom I am tied in to; (vi) he does not recognize me; (v) no one else in the population is aware of my liking for James? Can such questions be answered by either observation or questionnaire? (see Brown 1965). For roles defined culturally, on the other hand, in abstraction from particular populations, the meaning of reciprocity may be clear. It is inconceivable that a person can be kin to me without my being kin to him. So my kinship role to him necessarily implies some reciprocal role from him to me. In addition, each of the two roles is recognized, and is known by all to be recognized, by me and by him and by all other members of our society. This is role reciprocity in the full sense. Other sorts of cultural roles imply reciprocity. It is hard to think of lawyers without clients. However, once a whole set of roles and positions is included, the issue of reciprocity becomes murkier. How many of us know the court's clerk is the bailiff's superior in the legal role frame?

[13] It is instructive to point up a contrast with the model of Friedell (1969). That model has a very similar starting point in a concern with unfolding the implications of reciprocity. However, the idea of longer chains is developed at a purely psychological level in a two-party interaction, with the chains being of the form "he thinks that I think that he thinks . . . ," etc. This kind of analysis is Schelling's forte.

such image is again a binary matrix reporting which positions are joined by a *chain* of bonds of specified types. Formally:

> *A compound image 1*2* is represented by a square binary matrix, whose (i,j)th entry is 1 if and only if there is a bond of type *1* from i to some third position k (which may coincide with i or j) and a bond of type *2* from k to j.

In an obvious way, this definition may be extended to compounds of type *1*2*3* . . . through more than one intermediate position. Much has been written about the sociological meaning of compound relations in kinship (Lévi-Strauss 1949) and in more general kinds of formal and informal social structure (White 1969; Lorrain and White 1971; Fararo 1973). The innovation is that compounds are now being formed starting with blockmodel images, not with raw sociometric or observer-reported network data (e.g., Luce and Perry 1949).

Various features of the definition deserve emphasis:

1. Any position k may be a "middleman" with respect to any other position on any type of compound image.

2. The binary coding ignores the absolute number of possible intermediaries k between a given i and a given j. This is consistent with the way in which the underlying blockmodel has already imposed a binary coding of block densities (discriminating only between bonds and zeroblocks); it is also consistent with any prior disregard of the "strength" of ties in coding the original data.

3. The constituent images of compound images need not be of different types, and any number of repetitions is allowed; for example, *3*3*1*, *2*1*2*, *1*1*1*1*, etc.

4. In particular, there will be a reflexive entry from position i to itself in the compound image of type *1*1* if i has a reciprocated bond of type *1* with some other position j. Thus reflexive entries in certain compound images may indicate a weak form of reciprocity (see also Lorrain 1975).

5. A position may appear repeatedly as an intermediary in a chain contributing to a given entry in a compound image. Indeed, a position may appear at successive steps of the chains. This is legitimate, since a bond from a position to itself in a blockmodel is comparable with a bond to a different position (contrast the sociometric level of the original data, where the diagonal of a sociomatrix is inherently meaningless).

6. The order of images in the compound is material: *1*2* need not equal *2*1* and usually will not (contrast the well-known principle of balance theory that my friend's enemies are my enemy's friends, $PN = NP$).

7. On the other hand, in computing the compound of several images, say *3*1*2*, it does not matter whether one first computes the incidence of

bonds for *3*1* and then for *(3*1)*2,* or instead first finds *1*2* and from that finds *3*(1*2)*.

In mathematical terms, the binary matrix representing a compound image of type *1*2*3 . . .* is the Boolean matrix product of the specified image matrices (see section on methods).

Just as we earlier examined different types of tie, each type of compound image is now to be examined separately in terms of the matrix which represents it. It is natural to adopt the following:

Axiom of Quality.[14]—Two types of compound image are to be identified if and only if their associated matrices coincide.

Given this axiom, there can be only a finite number of distinct types of compound image for a given blockmodel; usually, there are but a handful for the blockmodels in our case studies. Of particular importance is the special case in which one or more compounds are identical with the initial images; we now turn to such a case as our first illustration.

An example.—Go back to Sampson's (1969) study of an American monastery, and adopt Part I's three-block split of his 18-man population into Loyal Opposition, Young Turks, and Outcasts. From among Sampson's four types of sociometric choice, select the Sanction choice and impose the three-block split on two separate matrices reporting top three choices on Positive and Negative Sanction, respectively. Weighting entries by $+3$ (top choice), $+2$ (second choice), and $+1$ (third choice), obtain the following density matrices (see also table 1 below):

$$\text{(Positive Sanction)} \begin{bmatrix} .548 & .020 & 0 \\ 0 & .905 & .036 \\ .179 & .179 & .333 \end{bmatrix}$$

$$\text{(Negative Sanction)} \begin{bmatrix} 0 & .204 & .536 \\ .286 & .048 & 1 \\ .107 & .143 & .417 \end{bmatrix}$$

(Both row and column orderings follow the sequence: Opposition, Turks, Outcasts.) Observing that there are no entries in the range (.05, .1), it is natural to convert to binary form using a .1 cutoff for coding a block as a zeroblock; thus (following Part I's notation) obtain[15]

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[14] In axiomatic set theory, this same principle goes under the name of the "Axiom of Extensionality."

[15] Detailed notation is developed below in table 1. Later we show that role structure is robust with respect to refinement of blocks and to the binary coding of density matrices.

Thus each block directs a positive bond to itself, and the Outcasts also direct positive bonds toward each of the other two factions. At the same time, each pair of different blocks reciprocates a negative bond, and the Outcasts direct a negative bond to themselves.

Now form compound images: K is transitive,[16]

$$K^2 = K,$$

but this statement is comparatively vacuous since there are only three distinct blocks and K is not a linear hierarchy. Also uninteresting are the further equations

$$BK = B^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

the universal pattern in which all pairs of blocks exchange bonds.

Finally, however, one also has

$$KB = B,$$

and this last equation is neither a tautology nor trivial. It represents the intersection of two separate statements about images:

$$KB \subseteq B; \tag{1}$$

$$B \subseteq KB. \tag{2}$$

Statement (1) may be taken to imply a kind of loyalty: I will support my "friends" by taking on their "enemies." Statement (2), on the other hand, describes a pattern of social reinforcement: whoever my enemy is, I will have a friend or friends who share my enmity.

Of course, these cultural interpretations with their implicit time-ordered dynamics centered on some individual are not the only ones possible, and it is also possible to interpret $KB = B$ without cultural or psychological premises, simply as a convenient summary of certain properties of K and B at the blockmodel level. Statement 2 is then seen as reporting a pattern of social reinforcement among positions: every position which has enmities also is friendly to positions which share those enmities.

In any case, the fundamental importance of the equation lies in its *invariance* (Stevens 1975): the social structure is described in a way which is independent of the limited viewpoint of any particular ego

16 Because reflexivity is an important substantive variable (see earlier), we adopt a strict mathematical definition of transitivity for blockmodel images. An irreflexive graph K is transitive by the standard definition (Ore 1962, p. 191) if $K^2 \subseteq K$. Many earlier investigators (notably Davis 1970; Holland and Leinhardt 1976) have demonstrated that graphs representing positive affect tend to be transitive (see also the explicitly hierarchical models derived by Bernard [1974] for Coleman's high schools [1961]).

position. Such invariance means that equations will be a natural approach to comparing blockmodels with different numbers of blocks and on different populations (see below). Yet an equation need have no direct incidence upon a particular position (save possibly as a middleman): the role structure will not usually be homogeneous.

Whether $KB = B$ is in some sense fundamental, as suggesting a general interlock pattern basic to the Sampson social structure, will be apparent only later, once the formalism has been fully developed. In the section on results, $KB = B$ (suitably reinterpreted employing homomorphisms) will be shown to be a robust characteristic of the Sampson structure. This characteristic will be used to distinguish the monastery from Newcomb year 2, among other data cases.

Role Structure in a Population

Multiplication tables for compounds.—We seek evidence of role structure in a population with two or more reported types of tie. Assume a particular blockmodel, which will remain fixed throughout subsequent analysis. Call the blockmodel images *generators* when they are used to form compound images; refer to both generators and compound images as *words* (drawing terminology from the theory of generative grammars). Thus, in the KB example, KB is a word; so also are K, BK, BK^2, $BKBK$, B^2. In classical sociometry, the concept of word formation is implicit in the notion of forming the transitive closure of a single relation R:

$$\mathrm{tr}(R) = R \cup R^2 \cup R^3 \cup \dots \cup R^n \cup \dots .$$

The classical approach, however, never formed compounds on more than a single generator in this way, and it never distinguished words as separate entities (but see Moon and Pullman [1967], following up the phenomenology of Landau [1951]).

A *role structure* is the set of all identifications among words obtained by applying the Axiom of Quality to the compound images formed by multiplying generators.

Mathematically, a role structure is therefore simply the Boolean matrix semigroup formed by taking Boolean matrix products of the specified generators (see section on methods).

In the KB example already introduced, the role structure has only three distinct words: K, B, and the universal element U. We have already studied role structure for this case, and our findings may be systematically summarized by means of what algebraists call the *multiplication table* of the algebra. This is the array showing all possible products of the words K, B, and U. Figure 1 displays this table in two alternative forms. The first is concrete and represents the structure as a set of matrices together

(a) Matrix products

×	$\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$
$\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$
$\begin{bmatrix}0&1&1\\1&0&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$
$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

(b) Letter products

×	(K)	(B)	(U)
(K)	(K)	(B)	(U)
(B)	(U)	(U)	(U)
(U)	(U)	(U)	(U)

FIG. 1.—*KB*3 role structure. In the notation introduced in table 1, the generators are SM(4)-3-$K(3)B(3) > .1$.

with their matrix products (by closure, these must be elements of the original set). The second is symbolic and employs letters (or later, numerals) to represent the three distinct elements of the structure. Although obtained directly from the concrete matrix table, the symbolic table retains only the abstract pattern of role interlock implied by the former one. *We employ the symbolic table as our central tool for operationalizing a new level of social structure.* This level may be inferred from blockmodels, but we will see below that recovery in the reverse direction is not necessarily or even normally possible: very different blockmodels may generate role structures giving rise to identical symbolic tables (figs. 5 and 6 below).

Before developing this point, first consider a second and much richer example. Taking all eight Sampson relations, consider our usual three-block split and code entries as bonds using only the top two choices and a strict zeroblock criterion. This leads to seven distinct generators, since Disesteem and Antagonism produce identical images and are therefore identified (fig. 2). Forming all possible compounds now produces only *four* new compound roles whose matrices are also shown (together with the informationally vacuous U pattern, which is generated as no. 10).

When there is tight-knit role structure, there should be a high incidence of compound words having multiple representations as products of generators; various alternatives are shown in figure 2. Because of this characteristic nonuniqueness, figure 3 shows the symbolic form of the multiplication table using an arbitrary integer coding for the distinct words. We have identified the generator rows in figure 3, in anticipation of the distinctive significance of generators in later comparison of tables (see section on methods).

Examine the figure 3 table more carefully. It is a complete statement of role structure in the given data; however, much of the information is redundant and only a few equations in the table convey interesting structural descriptions. We already saw this in the figure 1 table, where only one equation in nine is of interest. The following general points apply:

1. Many equations are simply definitions of a new word (although this will depend on the order in which compounds are generated). Thus, starting from words *1* and *2*, the equations $2*1 = 8$, $2*2 = 10$ simply define two new words which are not in the generator set. Once these words have been generated, however, the further equations $2*8 = 10$ and $1*10 = 2*10 = 10*1 = 10*2$ are no longer definitional (although, returning to the graph level, equations involving *10*, the filled matrix, are interpretively uninteresting).

2. Many more equations will simply be logical consequences of ones already derived, and therefore redundant. Thus if one has $L^2 = L^3$ (a weak form of transitivity found in certain of the Newcomb year 1 cases) he must also have $L^2A = L^3A$, $L^2AL = L^3AL$, etc., and these apparently new equations contain no new information. It is a mathematical fact that the complete table of any role structure with g generators and e words can be filled out from information contained in the $g \times e$ subtable corresponding to the generator rows. This is inductively clear, since any word can be built up from generators through multiplying generators by words already formed.

3. Finally, equations involving zeros assume a special significance. Formally, a (structural) "zero" in a multiplication table is an element 0 whose product by any element is itself: $0*x = x*0 = 0$. A structural zero in this sense need not correspond to a Z (empty) image pattern at the

Words	Matrix
1. $E = E^2 = EK = KE$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
2. $D = A$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
3. $I = I^2 = IK = IL = KI$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
4. N	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
5. $K = K^2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
6. $B = ED = EN = EB = KD = KB$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
7. $L = KL$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
8. $DE = DI = DK = NE = NI = NK$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
9. $IE = EI$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
10. $L\{E, D, I, N, K, B, L\} =$ $B\{E, D, I, N, K, B, L\} =$ $B^2 = D^2 = N^2 = DN = DB = ID = IB = EL$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
11. IN	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
12. KN	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

FIG. 2.—Sampson role structure on eight generators. In table 1 notation, this is SM(4)-3-$E(2)D(2)I(2)N(2)K(2)B(2)L(2)$. Each compound word is labeled by its representations as a product of two generators; there will be additional representations as products of three or more generators.

		Generators											
	×	1	2	3	4	5	6	7	8	9	10	11	12
	1	1	6	9	6	1	6	10	10	9	10	10	6
	2	8	10	8	10	8	10	10	10	8	10	10	10
	3	9	10	3	11	3	10	7	10	9	10	11	11
Generators	4	8	10	8	10	8	10	10	10	8	10	10	10
	5	1	6	3	12	5	6	7	10	9	10	11	12
	6	10	10	10	10	10	10	10	10	10	10	10	10
	7	10	10	10	10	10	10	10	10	10	10	10	10
	8	8	10	8	10	8	10	10	10	8	10	10	10
	9	9	10	9	10	9	10	10	10	9	10	10	10
	10	10	10	10	10	10	10	10	10	10	10	10	10
	11	10	10	10	10	10	10	10	10	10	10	10	10
	12	10	10	10	10	10	10	10	10	10	10	10	10

FIG. 3.—Multiplication table for Sampson generators shown in fig. 2. Numbering follows fig. 2.

blockmodel level, although any such empty pattern will act as a zero in a table. Thus *10* is a zero in figure 3 even though the graph corresponding to *10* is in fact the full graph (*U* pattern). From the standpoint of the structural information conveyed, zeros in a table are effectively "garbage" elements, since they are capable of generating no novel patterns of role interlock. If a table contains a zero, the frequency of the zero element is usually high, as is therefore the frequency of uninteresting equations. For example, in figure 1 the *U* pattern is a zero and accounts for 78% of the table; in figure 3 the frequency of the zero is about 70%.

These are three reasons to expect rather few interpretable equations in any particular table. Much of the apparent structure formally implied in a multiplication table is thus actually bogus; only a part of the table is operative in defining role interlock. At the same time, it is not always easy to exclude particular equations in preference to others, and we have seen that the outcome may be affected by purely arbitrary factors such as the order in which words are generated. These are strong grounds for rejecting a case-by-case approach to equations such as was followed above in the *KB* example and instead developing ways of treating entire tables as integrated structures. This shift in emphasis is developed below through the concept of a homomorphism.

For the moment, it should be stressed that there is no necessity that all tables be equally informative or even that a particular table reveal any significant role interlock at all. An extreme example is the degenerate table shown in figure 4*a*, where every element multiplies to give a single

(a) Week 4, Newcomb Year 2

	1	2	3
1	3	3	3
2	3	3	3
3	3	3	3

(b) Week 8, Newcomb Year 2

	1	2	3	4	5
1	3	3	3	3	3
2	4	5	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	3

FIG. 4.—Role tables with complete or partial degeneracy. In table 1 notation, *a* is NF2(4)-3-$L(3)A(3) > .1$; generators are

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Similarly, *b* is NF2(8)-3-$L(3)A(3) > .1$; generators are

$$L = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

garbage element, which is also a structural zero. In figure 4*b*, again the zero fills out most of the table. This makes excellent sense: these tables are for blockmodels obtained from weeks 4 and 8 in the Newcomb year 2 sequence, imposing a three-block split from week 15 (the last week). As was already argued in Part I, the social structure remains amorphous at the blockmodel level until about week 4 or 5. One would therefore be justified in expecting trivial role structure in figure 4*a* and perhaps rather little in figure 4*b*.

From blockmodel to role structure.—Figure 5 brings out another aspect of the kind of structural information contained in role tables. Figure 5*a*, of course, is the table familiar from classical balance theory (Abelson and Rosenberg 1958). In our scheme, every word is identified with one or the other of the generators by the Axiom of Quality. The table is unusual in

(a) Multiplication table

X	(L)	(A)
(L)	(L)	(A)
(A)	(A)	(L)

(b) Classical blockmodel

$$L = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad A = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

(c) Another blockmodel

$$L = \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \qquad A = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$

FIG. 5.—Two blockmodels, each yielding the multiplication table of classical balance theory.

that all equations are interpretable and none is excluded on any of the three grounds given above. Each individual equation has been discussed extensively in the literature (e.g., for $N^2 = P$ see Aronson and Cope 1968); collectively, the equations are always associated in the literature with a two-clique structure of the kind shown in figure 5*b* (and are interpreted at the level of individuals, not blocks).

Surely this algebra contains no surprises, although its applicability to real data may be questioned. But now look at figure 5*c*. Here are two graphs which may be interpreted as Like and Antagonism images in a six-block model. Taking these graphs as generators, proceed to fill out a compound role structure as in earlier examples. Astonishingly, for the two graphs bear little resemblance to a balanced graph, *the role structure they generate satisfies exactly the familiar equations of figure 5a!* This proposition may be made tighter: using the notation of figure 2 in Part I, and applying the BLOCKER algorithm, there is no way of collapsing the given six blocks into a (P, N) model at the two-block level (although $[V, F]$ blocking is possible). In a very strong sense, the figure 5*c* graphs are inherently different from the 5*b* ones.[17]

A parallel finding arises in many cases generated by actual data: distinct—often highly distinct—blockmodels may generate identical role tables. Figure 6 gives two examples. For instance, in figure 6*b* two different Sampson pairs of generators for three blocks give the same role table. This table is also given by a pair of generators on four blocks for Newcomb's first fraternity (shown in fig. 6*b*, *3*). *Identical equations may be satisfied by social structures that appear distinct at the blockmodel level.* Such coalescence may appear reminiscent of the way in which different networks may produce identical blockmodels. However, the latter possibility is simply an obvious by-product of aggregation. The present instance is different: a role table is a kind of "unfolding" of the generators, and it is correspondingly a strong assertion to say that two "unfoldings" coincide while the generators do not.

Passing to the role table therefore involves a fundamental loss of information, in exchange for obtaining a level of description which is invariant across positions in the sense discussed above. Assessing the net value of this transaction must await the section on results.

Similarity of Role Structures

Orientation.—Is a given role table reliable, that is, will alternate blockmodels from different codings of sociometric indicators yield essentially the same table? Is there a way to distill just the essential features of role interlock from a large table, however reliable it is? Can one determine what two role structures from different cases have in common? To develop the theory, in this section we propose answers to these three interrelated questions of robustness, aggregation, and similarity for role structures. The answers build upon one concept, homomorphism, and will lead us to

17 This example was contributed by Mr. Kazuo Seiyama (1975). As in most actual cases studied, not all bonds are reciprocated.

(a) Newcomb Year 1, Weeks 8 and 13 (NF1(i)-4-L(3)A(3) > 0.1, i=8,13)

1. Week 8

$$L = \begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \qquad A = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

2. Week 13

$$L = \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \qquad A = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

	1	2	3	4	5	6	7
1	3	5	7	6	6	6	7
2	4	6	4	6	6	6	4
3	7	6	7	6	6	6	7
4	4	6	4	6	6	6	4
5	6	6	6	6	6	6	6
6	6	6	6	6	6	6	6
7	7	6	7	6	6	6	7

(b) Monastery and fraternity models

1. SM(4)-3-I(2)N(2) (identical to SM(4)-3-I(3)N(3) > 0.1)

$$I = \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array} \qquad N = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

2. SM(4)-3-E(2)D(2)

$$E = \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \qquad D = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{array}$$

3. NF1(12)-4-L(3)A(3) > 0.2

$$L = \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \qquad A = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

	1	2	3	4	5
1	1	4	5	4	5
2	3	5	5	5	5
3	3	5	5	5	5
4	5	5	5	5	5
5	5	5	5	5	5

FIG. 6.—Identical role tables from distinct blockmodels

identify a handful of small target tables with simple substantive interpretations.

The concept of homomorphic reduction.—Start with a given role structure such as that for which the table is shown in figure *7a*. We seek a concept of aggregation which is consistent with the compounding of roles through which the table was originally generated. The following definition translates this consistency requirement into formal terms:

> A *homomorphic reduction* of a role table is a partition of its words into equivalence classes $C_1, C_2, \ldots, C_k$ in such a way that class membership

is consistent with the operation of forming the compound. That is, if $e_1, e_2 \in C_i$ and $f_1, f_2 \in C_j$, then e_1f_1 and e_2f_2 fall in the same class C_m.

Thus, a homomorphic reduction aggregates a role table by imposing new equations, those between the words in a class of the partition, in a way consistent with the equations in the full table.

An example is shown in figure 7. Here the original role structure is based on a five-block model for Sampson *I* (Influence) and *N* (Negative Influence). Following what will later be our standard convention, *1* is taken to represent the positive generator here (*I*) and *2* is taken to represent the negative generator here (*N*). There are nine further compounds, so that the entire structure contains 11 roles. There is a structural zero (element *8*) which fills out much of the table, but the upper left corner contains nondefinitional structure not involving *8* (thus $1*1 = 1*3 = 3*1 = 3*3 = 3$).

Figure 7*b* now shows a homomorphic reduction of the structure. There are three classes C_i,

C_1: (1, 3)
C_2: (2, 5, 9)
C_3: (4, 6, 7, 8, 10, 11),

and it is directly apparent that the aggregation preserves consistency of compounds. Relying on this consistency, we may in turn replace the 7*b* table with the three-element table in figure 7*c*. Note that this last table is very simple in structure and that we have already seen it in a different context, through the *KB* example introduced in figure 1.

We will refer to the 7*c* table as a *homomorphic reduction* of the original table in 7*a*.

Homomorphic reduction will serve as our operational concept of role structure aggregation. Two comments should be made:

1. Aggregation is not unique: role structures will always possess numerous alternative reductions (compare the general observations on aggregation in Leijonhufvud [1968]). Each reduction may be viewed as one possible simplified description of the original structure, and not all such simplifications need be equally useful. For example, it is obviously possible to reduce the 7*a* table down onto the degenerate table

	1	*2*
1	*1*	*2*
2	*2*	*2*,

but this further reduction is clearly uninformative.

Later, in the section on results, we will compare Sampson and Newcomb role tables by examining their alternative aggregations into nondegenerate three-word tables.

(a) Role table -- original form

	1	2	3	4	5	6	7	8	9	10	11
1	3	5	3	8	9	11	8	8	9	8	11
2	4	6	7	8	10	8	8	8	8	8	8
3	3	9	3	8	9	11	8	8	9	8	11
4	7	10	7	8	8	8	8	8	8	8	8
5	8	11	8	8	8	8	8	8	8	8	8
6	8	8	8	8	8	8	8	8	8	8	8
7	7	8	7	8	8	8	8	8	8	8	8
8	8	8	8	8	8	8	8	8	8	8	8
9	8	11	8	8	8	8	8	8	8	8	8
10	8	8	8	8	8	8	8	8	8	8	8
11	8	8	8	8	8	8	8	8	8	8	8

(b) Role table -- permuted and partitioned

	1	3	2	5	9	4	6	7	8	10	11
1	3	3	5	9	9	8	11	8	8	8	11
3	3	3	9	9	9	8	11	8	8	8	11
2	4	7	6	10	8	8	8	8	8	8	8
5	8	8	11	8	8	8	8	8	8	8	8
9	8	8	11	8	8	8	8	8	8	8	8
4	7	7	10	8	8	8	8	8	8	8	8
6	8	8	8	8	8	8	8	8	8	8	8
7	7	7	8	8	8	8	8	8	8	8	8
8	8	8	8	8	8	8	8	8	8	8	8
10	8	8	8	8	8	8	8	8	8	8	8
11	8	8	8	8	8	8	8	8	8	8	8

(c) Image table

	(1)	(2)	(3)
(1)	(1)	(2)	(3)
(2)	(3)	(3)	(3)
(3)	(3)	(3)	(3)

FIG. 7.—Homomorphic reduction: mapping a Sampson *IN*5 table down onto a three-word table. In *a*, the role table is shown in unpermuted form as originally generated. In later notation, *a* is for SM(4)-5-$I(3)N(3) > .1$. In *b*, the same table is shown in permuted and partitioned form. In *c*, the multiplication table of the reduction is shown: this is the same table directly obtained in the *KB*3 example of fig. 1.

2. Aggregation by homomorphic reduction is a purely algebraic concept: the reduction possibilities make use of structural information in the role table only, and there need not be any consistent parallel reduction at the level of blockmodel graphs.[18] In particular, it is not usually true that a homomorphic reduction may be generated as a Boolean matrix semigroup from matrices which are unions of words mapped into generators under the homomorphism. A more abstract way of making the same observation is to lay emphasis on the decoupling of structural levels. There is a first level of graphs, starting with blockmodel images and later forming compounds in Boolean arithmetic. There is a second level of equations, summarized in the role table. In all cases we have met thus far, the abstract form of a role table may be introduced merely as a shorthand for the matrix form of the table (see again fig. 1). In defining the concept of homomorphic reduction, we are making the promised transition to a genuinely separable level of social structure—one where certain operations may be possible that need have no counterpart at the level of graphs.

The joint table of two role structures.—Continuing to develop this point of view, we turn to the problem of comparing two role tables. The two tables may come from blockmodels for the same case derived by different codings of the data, or they may be from entirely distinct cases. It must be possible to identify generator symbols across cases to be compared. Thus, in figure 8, cases *II* and *III* are each generated by elements *1* and *2* (which, by reference to the data, correspond, respectively, to positive and negative sentiment). Now define:

> The *joint reduction* of two role structures on the same set of generators is the largest table consistent with both original role tables, that is, the most refined role table which is a homomorphic reduction of both given tables.

In substantive terms, the joint reduction is the outcome of abstracting structure common to both original role systems; it is the common denominator of the two role structures being compared. In mathematical terms, the joint reduction may be characterized as the intersection (great-

[18] There is such a parallel reduction in the example of fig. 7. Consider the Boolean union of the words in each of the three classes C_i. This gives:

$$C_1 = \begin{matrix} \bar{1} & 1 & 1 & \bar{1} & 0 \\ 1 & 1 & 1 & \bar{1} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & \bar{1} & 1 & 1 & 1 \end{matrix} \qquad C_2 = \begin{matrix} \bar{1} & \bar{1} & 1 & \bar{1} & 1 \\ \bar{1} & \bar{1} & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ \bar{1} & 1 & 0 & 0 & 1 \\ \bar{1} & 1 & 1 & \bar{1} & \bar{1} \end{matrix} \qquad C_3 = \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix};$$

a bar has been added to the four 1's in C_1 which are not already present in the first generator, and similarly for C_2, to show the impact of unioning. It is easy to verify that C_1 and C_2 as generators yield exactly the three-word table of fig. 7*c*.

est lower bound) of two given semigroups in the reduction lattice (see section on methods). In figure 8, table *IV* is in fact the joint reduction of tables *II* and *III* (and of *I* and *III* as well). Thus the simple three-word table of figure 1 is just what the monastery and fraternity cases of figure 8 have in common.

I: 11 words -- table shown in Fig. 7.

II:	1	2	3	4	5
1	1	4	5	4	5
2	3	5	5	5	5
3	3	5	5	5	5
4	5	5	5	5	5
5	5	5	5	5	5

III:	1	2	3	4
1	1	2	3	4
2	3	4	3	4
3	3	4	3	4
4	3	4	3	4

ϕ ψ

IV:	1	2	3
1	1	2	3
2	3	3	3
3	3	3	3

(IV is the joint reduction of II and III)

FIG. 8.—Homomorphisms and the joint table. In each of *I–III, 1* is the positive sentiment generator and *2* is the negative sentiment generator. Table *II* is SM(4)-3-*I*(2)*N*(2) and also SM(4)-3-*I*(3)*N*(3) $>$ .1. Table *III* is NF2(10)-3-*L*(3)*A*(3) $>$ .1. Partitions:

$$II \rightarrow IV\colon\ (1)(2,4)(3,5) \leftrightarrow (P_\phi)$$

$$III \rightarrow IV\colon\ (1)(2)(3,4) \leftrightarrow (P_\psi)$$

In the section on methods, we develop a numerical measure (δ) of how far apart two role tables are through their joint reduction (see Boorman and Olivier [1973] for the mathematical basis of this strategy). The distance is used to assess reliability across different codings of one case, as well as to measure the similarity of different cases. When reliability is high, the joint reduction table is the robust statement of role structure.

Inclusion orders and generators.—Before the joint reduction of role tables from two cases can be obtained, the respective generators must be identified with one another. The face definitions of sociometric indicators can be misleading, and their various labels are often incomparable. A generalization of the Axiom of Quality suggests one objective criterion: if one image wholly contains another, the former type of tie is a weakened form of the latter and a candidate for homomorphic identification with it. This criterion cannot be directly used to compare different cases, the blockmodels for which may be on different numbers of blocks. Instead, the overall structure of inclusions among distinct words in one blockmodel can be compared with that for another. The rationale will be further developed in the next subsection.

Figure 9 reports inclusion orderings typical for two generators representing positive and negative affect. They are for the fraternity data Newcomb designed to pick up precisely this affect dimension and come from the blockmodels of the last five successive weeks of the first experiment (year 1), when clear social structure had emerged among the four blocks. The similarities among these five orderings are striking, and there seems to be a progression over time.

Now consider figure 10, the inclusion order for the monastery blockmodel in figure 2. There are *eight* generators, as defined by Sampson (or seven, since D and A coincide). Yet in its gross features figure 10 resembles the typical inclusion ordering for a *pair* of generators: D, A, and B are weakened forms of N; I, E, and L are weakened forms of K. The positions of the compounds fit with the former four representing negative affect and the latter four representing positive affect: cross-products appear as additional weakened forms of N whereas powers and products of the positive four are weakened forms of K.

In the section on results we exploit this simplification of the Sampson case. A large role table by itself can be opaque: it may not only be hard to identify the nature of the generators relative to other cases but may also be difficult to see the main features of role interlock within the single table itself.

Inclusions, equations, and social economy.—There may be no comparable cases, and no alternate blockmodels for a given case, from which to derive a joint homomorphism. Imposing a single additional equation upon a role table defines a homomorphism, sometimes a sweeping one with just a few equivalence classes because of the large number of other equations implied by the given one. In such an attempt to identify main features of role interlock, one wishes to equate two words which are similar. A word W just above another word X in the inclusion order is a weakened form of it, as suggested earlier, and more similar to X than words elsewhere in the inclusion order.

Such a procedure can also help to identify which of the equations imposed on a role table by a joint reduction are the strategic ones sufficient to imply the rest. The likely candidates are equations of compound words

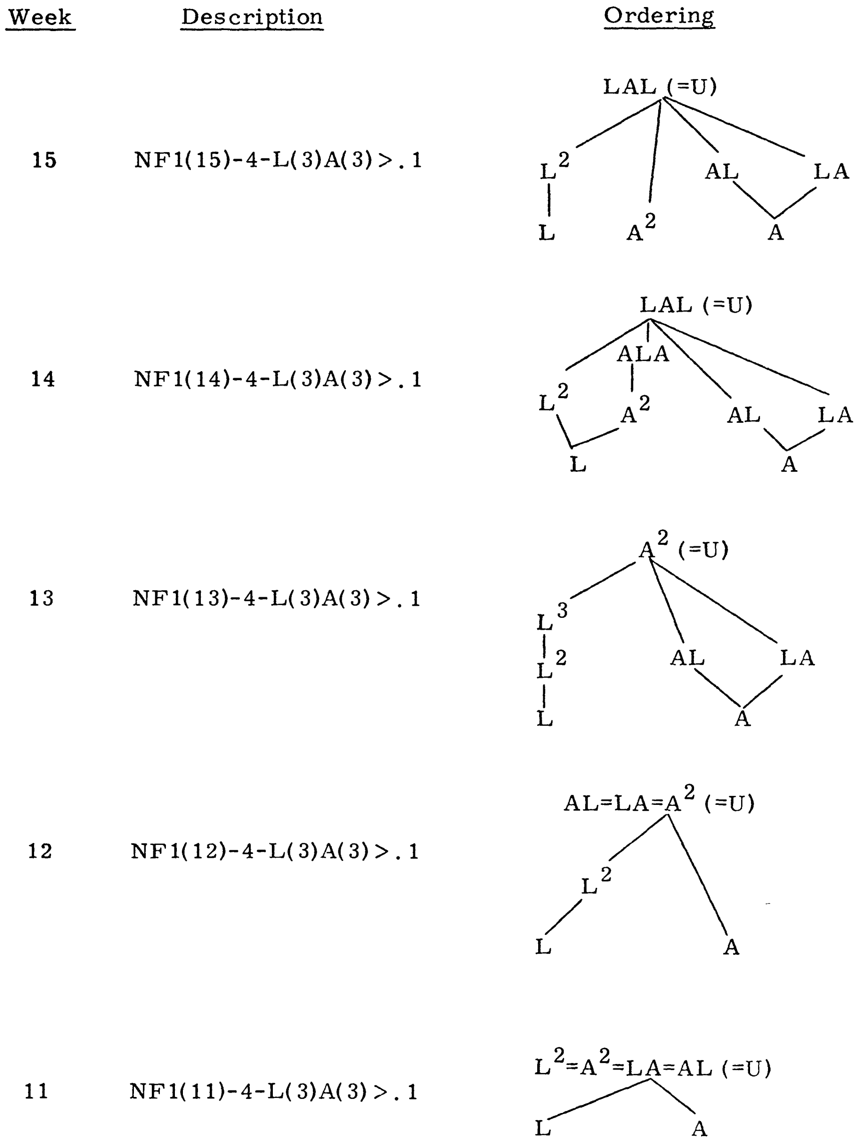

FIG. 9.—Inclusion orderings for Like and Antagonism in final weeks of Newcomb year 1. Notation follows table 1.

to generators. (Note that the words high in the inclusion ordering are "garbage" words in the sense discussed earlier for U.) If an imposed equation is inconsistent with the detailed interlock in the full table, the resulting reduced table will be degenerate.

Take the inclusion orderings of figure 9 as an example. One obvious

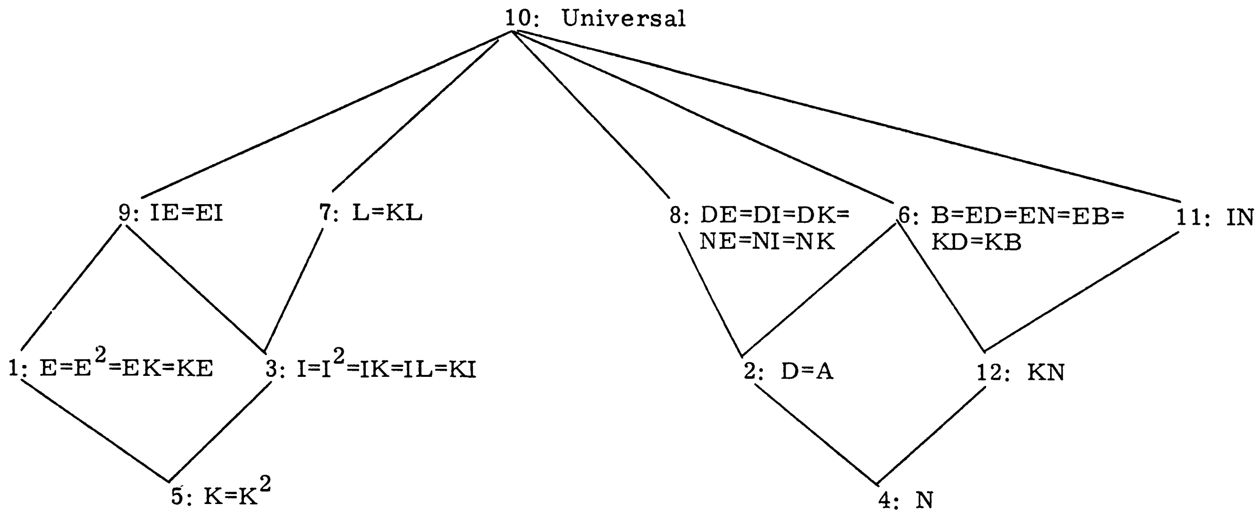

FIG. 10.—Inclusion ordering for words in Sampson eight-generator case (fig. 2). Note the "butterfly" pattern. The ordering forms an upper semilattice.

candidate for a simplifying equation is transitivity of the positive generator: $L^2 = L$. There are two competing candidates for the other strategic equation:

$$AL = A, \quad \text{and} \quad LA = A.$$

The latter is the one observed in the full role table of figure 1, the one imposed successfully in figure 7, and the one resulting from the joint reduction in figure 8. In the discussion of results we shall always see degenerate results from imposing both at once, for all Sampson and Newcomb cases.

As a second example, recall the Bank Wiring Room blockmodel which splits the population into three hierarchical strata. The role structure generated by Like and Antagonism has 11 words, and figure 11 shows both the multiplication table and the inclusion ordering. The latter is strikingly different from figures 9 and 10. There is no overall division into words of positive and of negative quality; there is a word, LAL, with empty image (Z); and AL, as well as LA, is contained in A instead of the reverse. The only equation clearly suggested is transitivity of L; the main features of interlock are otherwise not obvious.

An equation is the intersection of two inclusions. The equation in the first example (fig. 1) was discussed in those terms, and the two inclusions were named. In default of the equation, does an inclusion hold, and if so is it always the same one? The typical inclusion orderings for friendship and enmity (figs. 9 and 10) show that the inclusion called social reinforcement holds, but not the one called loyalty. That is, typically any position antagonistic to some given position can count on having friends also antagonistic to the latter position ($A \subseteq LA$, but not the reverse). Figures 9 and 10 also show another, less auspicious, face of this social reinforcement: a position can also count on having other enemies who are friends with a given enemy ($A \subset AL$ but not the reverse).

Against this background the equation $LA = A$ can be interpreted as a development of social economy in affective structure, a tightening of the pattern of social reinforcement into the discipline of a role structure. In the example of figure 1, not only do friends share one's own enmities but also each position shares all the enmities of its friends; the latter tightening could be sought by a particular ego either through adopting as his own enemies all enemies of friends, and/or by dropping friends who have enemies not already his own.

It does not follow that the cognate equation, $AL = A$, need develop from *its* pattern of social reinforcement. Not only can one equation hold without the other, as illustrated by figures 1, 7*c*, and 8*IV*, but also their developments would be quite different. An ego does not control the bonds sent by intermediary positions; so in achieving the economy of $AL = A$

(a) Generators (BW-3-LA)

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Role table

	1	2	3	4	5	6	7	8	9	10	11	12
1	3	5	3	7	5	10	7	3	7	10	5	7
2	4	6	4	8	9	6	7	8	11	12	11	6
3	3	5	3	7	5	10	7	3	7	10	5	7
4	4	9	4	7	9	12	7	4	7	12	9	7
5	7	10	7	3	7	10	7	3	5	7	5	10
6	8	6	8	8	11	6	7	8	11	6	11	6
7	7	7	7	7	7	7	7	7	7	7	7	7
8	8	11	8	7	11	6	7	8	7	6	11	7
9	7	12	7	4	7	12	7	4	9	7	9	12
10	3	10	3	3	5	10	7	3	5	10	5	10
11	7	6	7	8	7	6	7	8	11	7	11	6
12	4	12	4	4	9	12	7	4	9	12	9	12

(c) Inclusion ordering

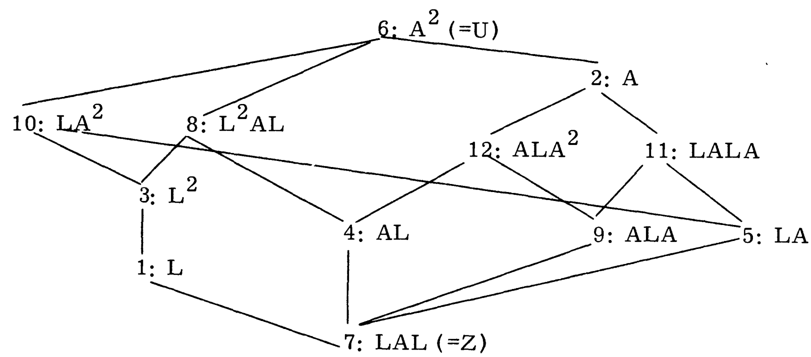

FIG. 11.—Bank Wiring Room: Like and Antagonism on three blocks (stratified model) with zeroblock cutoff.

from the base of $A \subset AL$, each ego would have a most delicate synchronization task whether he added or dropped enemies. Nor is the motivation apparent for making enemies of all friends of one's enemies.

The previous paragraphs illustrate a general point: the interpretation of any one equation in a role table depends on what other equations also hold—and thus on what equations do not hold. The phrase "role interlock" refers to just this point. The main kinds of role interlock are brought out next through systematic examination of possible small role tables.

Target tables.—We construct a guide to the outcomes of joint reductions and other homomorphisms for role tables with two generators. It takes the form of an annotated set of eight tables, each with just three words. Two of the tables are typical in cases with positive and negative affect, three occur where neither generator is of negative quality, and three may result from data sets of either sort.

Joint reductions to tables with *no* distinct compound words mean there is nothing, or very little, in common between the original role tables. This is so if the tables are

$$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}, \qquad \begin{matrix} 1 & 2 \\ 2 & 2 \end{matrix}, \quad \text{or the equivalent.}$$

(Hereafter we suppress the row and column headings for these multiplication tables in the symbolic form introduced by figure 3, since these will always be the integers in ascending order.) The first table above, for example, means that all compounds had to be equated to a generator in order to find a common, and nil, role table.

There is a very strong structure in common, however, if the joint reduction is to the table shown in figure 5*a:* viz.,

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}.$$

This is the table for classical balance theory, with *1* the positive generator, which is transitive, and *2* the generator for negative affect. It has not emerged from any of the joint or other reductions applied to our collection of empirical cases; in particular, there is rarely any basis for equating *2*2* with *1* (enemies' enemies with friends).

There are two other significant possibilities for tables of two words alone:

$$\text{First Letter} = \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix}, \quad \text{and} \quad \text{Last Letter} = \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}.$$

By the First Letter table, any compound word of whatever length is equated to the generator which occurs first in the word (that is, leftmost). A substantive interpretation is that any type of indirect bond takes on the quality of the direct tie to the first intermediary. The Last Letter table is, formally, simply the dual in which the last generator determines the quality of each type of compound. The First Letter table, or a refinement of it, is often found when one generator is for an objective, though positive, sort of tie (e.g., similar policy), whereas the other denotes positive affect: thus one's tie to any kind of contact of one's business associate takes on the color of a business association, whereas one views in affective terms any kind of contact through a friend. This kind of ego-oriented interpretation does not extend to the Last Letter table, where the compound tie is colored by the kind of tie from the last middleman in the compound.

When not all compounds are equated to one or another of the generators, a broader spectrum of role interlock can be specified through tables with just one distinct compound. In particular, significant refinements of the two useful smaller tables (First Letter and Last Letter) may be distinguished. The resulting tables will be assigned labels T_1–T_8 for reference in the section on results. At least one generator, for convenience the first, will always be assumed transitive: $1*1 = 1$.

Two of these tables, found for positive and negative affect cases, have already been discussed and contrasted at some length:

$$T_1 = \begin{matrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{matrix}\,, \quad \text{and} \quad T_2 = \begin{matrix} 1 & 3 & 3 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{matrix}\,.$$

In all eight tables, the distinct compound, labeled *3*, acts as the garbage word discussed earlier: entries of *3* in a table do not convey positive information but permit and contrast with equations to the generators. The natural contrast to both T_1 and T_2 is

$$\mathrm{T}_3 = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{matrix}\,,$$

in which the two generators commute. In classical balance the generators also commute, but T_3 does not have the classical balance table above as a homomorphic reduction, because *2*2* in this table cannot be mapped into *1* without inducing total degeneracy.

James A. Davis (1968) has developed an alternative to classical balance in which, in our terms, he rejects the equation of *2*2* with *1* but does not specify a unique alternative. The population must be split into at least three blocks, and with three blocks the extreme form of the Davis theory, expressed as a blockmodel, is

$$L = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\,, \quad \text{and} \quad A = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\,.$$

The role table of this extreme Davis blockmodel is precisely T_3. We shall see that T_3 also does not emerge from our empirical reductions, so that commutativity of the generators is suspect, even without the mapping of $2*2$ into 1.[19]

When neither generator represents negative affect, a role table showing no interlock between the generators, but only transitivity for each, is T_8, shown below with four other possibilities:

$$T_4 = \begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 3 & 2 & 3 \end{matrix}\,, \quad T_5 = \begin{matrix} 1 & 3 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{matrix}\,, \quad T_6 = \begin{matrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{matrix}\,, \quad T_7 = \begin{matrix} 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{matrix}\,, \quad T_8 = \begin{matrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 3 \end{matrix}\,.$$

The other four are variants of the First Letter table (T_5 and T_6) or the Last Letter table (T_4 and T_7): T_4 is like T_1 but with an additional important equation, namely transitivity of 2, which changes the significance of the existing equation; T_4 will be found to result also from cases with negative generators, where the negative bonds are so directed toward one block as to make the negative image transitive (compare fig. 16 below for Newcomb year 2).[20]

It is not necessary to turn to tables with two or more distinct compound words in order to distinguish the main kinds of role interlock found in our cases. There are other tables on three words than the eight above, but none have been found useful. Associativity, a requirement imposed

[19] Like classical balance, Davis balance assumes, in our terms, that positive and negative images have no overlap; there is no ambiguity. Other blockmodels consistent with Davis's theory yield the multiplication table

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 3 \end{matrix};$$

but so does the blockmodel

$$L = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

and

$$A = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{matrix},$$

the generators of which do overlap. This table, in contrast to T_3, has the classical balance table as a reduction, because the table equates $2*2*2$ with 2 (semitransitivity) although not $2*2$ with 1. Thus some cases of Davis's theory are consistent with the multiplication table of classical balance, but so are other blockmodels exhibiting ambiguity.

[20] As emphasized earlier, many blockmodels can yield any given role table. Each of these eight tables of three words can be produced by some blockmodel on just two blocks: in the notation of fig. 2 in Part I, such pairs of generators are $T_1(V, F)$; $T_2(V, E)$; $T_3(P, F)$; $T_4(V, T)$; $T_5(V, G)$; $T_6(G, E)$; $T_7(H, E)$; and $T_8(V, W)$.

by the nature of matrix multiplication, precludes many possibilities. For example, T_3 with the transitivity entry replaced by $1*1 = 3$ is impossible: commutativity of generators is possible only if in addition at least one is transitive (or if one is semitransitive; see n. 19).

METHODS

Our models for role structure are *semigroups* (Clifford and Preston 1961; Kurosh 1965).

Definition 1. An *abstract semigroup* is an ordered pair $(S,*)$ where S is a set and $*$ is a binary operation $*: S \times S \rightarrow S$ which satisfies associativity, that is, for which

$$a*(b*c) = (a*b)*c$$

for all $a,b,c \in S$.

As long as S is finite, the entire structure of the semigroup can be summarized through the *multiplication table* of the operation $*$. We have already seen examples (e.g., fig. 1).

We make use only of concretely presented semigroups in which *generators* have been identified.

Definition 2. A *semigroup with generators* is a triple $(S, G, *)$ where $(S, *)$ is a semigroup, $G \subseteq S$, and any element a of S may be expressed as a product of elements in G:

$$a = g_1 * g_2 * \ldots * g_k, \quad g_i \in G.$$

Thus in the examples of figure 1, $G = \{K, B\}$, while in figure 2, $G = \{E, D, L, A, I, N, K, B\}$, and in figure 6*a*, $G = \{L, A\}$, etc. As those examples made clear, the selection of generators is a substantive matter.

A useful alternative way to characterize the structure of a semigroup with generators proceeds through the intermediate concept of a *free semigroup*:

Definition 3. The *free semigroup* generated by a set G consists of all possible finite strings of elements $g_i \in G$ with multiplication $*$ given by the operation of concatenating strings.

Thus if $W_1 = ALA$, and $W_2 = AL^2$ in the free semigroup on $G = \{L, A\}$, $W_1 * W_2$ is ALA^2L^2, the result of concatenating the two sequences.

Designate the free semigroup on G by FS(G). Note that FS(G) will always be infinite even though G contains only one letter. In category theory (Mitchell 1965), FS(G) may also be characterized as a universal object in the relevant semigroup category.

Then if $(S, G, *)$ is any semigroup with generators, the structure of $(S, G, *)$ is uniquely characterized by the partition $\boldsymbol{P}$ of FS(G) obtained by putting two words in FS(G) in the same equivalence class of the partition if and only if they evaluate to the same element of S in the multiplication table of $(S, G, *)$.

Note that the partition P will have a finite number of cells as long as S is finite, but certain cells must then have an infinite number of members (pigeonhole principle). This means that although some words may have unique expressions as a product of generators, at least one must have an infinite number of alternative expressions. Accordingly, the letter form of the multiplication table is in part arbitrary. This is a basis for preferring the numerical form of tables (e.g., fig. 3), even though it is more abstract and less immediately interpretable.

Our semigroups result from computations with matrices.

Definition 4. A *Boolean matrix semigroup* is a semigroup generated by one or more specified binary matrices under Boolean matrix multiplication.

In all our examples, the generator matrices are blockmodel images rather than matrices representing raw network data; this is a basic difference between the point of departure of the present paper and the earlier approaches of White (1969) and Lorrain and White (1971).

It is important to distinguish between the *matrix representation* of a particular role structure obtained from Definition 4 and the purely algebraic interpretation of the same structure under Definition 2. The matrix representation obviously subsumes all information contained in the abstract multiplication table. It also contains additional information associated with the matrices representing particular compound words. Both levels should be clearly kept in view at all times, since certain interpretations consistent with the algebraic level will be inconsistent with the matrix level, as was discussed in the section on theory.

After some brief comments on the mathematical and statistical properties of Boolean matrix semigroups, we will move to the concept of a homomorphism and its application to measuring distance between semigroups.

Boolean Matrix Semigroups

Computation of all distinct product words generated from a particular blockmodel, and thereby the multiplication table of the associated role structure, was speeded by use of a computer program named TABWDL.[21] Normally, all distinct compounds were found using only words of three or fewer letters, and never of more than five letters. The semigroups obtained are small.[22] In cases involving three to five blocks, most sizes are in the

[21] TABWDL, in the APL language, is available upon request. It was adapted by the second author from an annotated package of APL programs created by Mr. G. H. Heil (see Heil and White 1976) to analyze homomorphisms of blockmodel semigroups.

[22] Working in the context of the old approach using sociometric matrices as generators (Lorrain and White 1971), we performed a Monte Carlo investigation of semigroup sizes. Taking populations of sizes about 10, two random square matrices of fixed dimension were formed on each trial, using a fixed probability p for a 1 in

range 3–15, with a mild tendency for more blocks to generate larger semigroups (see the several series of semigroup sizes in the second columns of figs. 15–18). As is also reasonable, semigroups on three generators are somewhat larger than those on two, but sizes are still in the indicated range. We have seen one example (fig. 3) in which the semigroup of an eight-generator model gives only 12 words; various alternative cutoff densities for the same generators give semigroups of sizes up to 25.

Inclusion orderings.—Given any Boolean matrix semigroup, the associated inclusion ordering is the partial ordering obtained from the natural ordering of matrices:

$$M_1 \leqslant M_2 \Longleftrightarrow M_1(i,j) \leqslant M_2(i,j) \text{ for all } (i,j).$$

This suggests the following class of structures:

Definition 5. A *partially ordered (p.o.) semigroup with generators* is a quadruple $(S, G, *, \leqslant)$ where $(S, G, *)$ is a semigroup with generators and $\leqslant$ is a partial ordering of S.

This is a somewhat more general characterization than usually adopted by algebraists, where monotonicity of $*$ with respect to $\leqslant$ is assumed (thus $a \leqslant b \Rightarrow c^*a \leqslant c^*b$ and $a^*c \leqslant b^*c$ for all $a,b,c \in S$ [see Fuchs 1963; Vinogradov 1969]).

It is obvious that $(S, G, *, \leqslant)$ contains new information that cannot be obtained from $(S, G, *)$. In the cases we consider, it is not usually possible to place additional a priori restrictions on the ordering, such as requiring it to be a lattice or an upper semilattice (Friedell 1967). Notice, however, that a U pattern will always be a maximum element and a Z pattern a minimum element (cf. fig. 11*c*, where both elements appear). Because they tend to be more densely filled as matrices, longer words usually occupy higher positions in a semigroup p.o.; generators are often (though not necessarily) minimal elements.

Homomorphisms

Starting from Definition 2, one may define a homomorphism as follows:

Definition 6. If $(S_1, G_1, *)$, $(S_2, G_2, \circ)$ are semigroups with specified generator sets G_1 and G_2, a homomorphism $\phi: S_1 \rightarrow S_2$ is a mapping which: (i) preserves semigroup operations: $\phi(a^*b) = \phi(a) \circ \phi(b)$; (ii) maps generators to generators: $\phi(g_1) = g_2 \in G_2$ for all $g_1 \in G_1$.

each entry. The results indicated that semigroups would be quite small for p under about .08 and above about .25. In the .08–.25 range, sizes were larger and it was difficult to compute the mean size since limited storage capacity compelled truncation of semigroup sizes at $N = 65$ elements. (Interestingly, many sociometric relations have densities in exactly the difficult .08–.25 range.) It may be possible to obtain at least asymptotic estimates and bounds on semigroup sizes by adapting the probabilistic methods of the Hungarian school of combinatorics. We are indebted to Paul Erdös for discussion of technical aspects of this problem.

Our attention is confined to homomorphisms having two additional properties as follows:

Definition 7. A homomorphism ϕ is a ***homomorphic reduction*** of $(S_1, G_1, *)$ if ϕ maps S_1 onto S_2, that is, for any $c \in S_2$ one has $\phi(a) = c$ for some $a \in S_1$.

As already explained, a homomorphic reduction is an algebraic concept of aggregation, and $(S_2, G_2, \circ)$ is a coarser version of the structure $(S_1, G_1, *)$.

Definition 8. A homomorphism ϕ *preserves generators* if ϕ maps G_1 onto G_2 in a 1–1 way, that is, ϕ sets up a 1–1 correspondence between generators.

Definition 8 clearly implies that $(S_2, G_2, \circ)$ is also a homomorphic reduction of $(S_1, G_1, *)$, since every word in S_2 can be expressed as a product of generators in G_2. In earlier discussion of homomorphisms, we assumed Definition 8 implicitly, because the identification of a generator from one empirical case with one of the generators from another case is an important substantive issue.

The Joint Reduction of Two Semigroups

Given any semigroup with generators, we have already noted that it is possible to interpret it as a partition of $\mathrm{FS}(G)$. If two semigroups $(S_1, G, *)$ and $(S_2, G, \circ)$ have the same generator set (or identified generator sets), each may therefore be interpreted as a partition of the same free semigroup. Since partitions form a natural lattice (Birkhoff 1967), this suggests that the set of all semigroups generated by G may also be endowed with a lattice ordering. This is in fact correct and is the formal basis of our approach to comparing role structures.

Without loss of generality, we will now assume that all semigroups to be compared are formed from a common generator set G. In this case, by a *generator-preserving* (GP) homomorphism, we will mean a homomorphism which is the identity mapping on generators, that is, $\phi(g) = g$ for all $g \in G$.

In order to define the semigroup which we call the *joint reduction* of $(S_1, G, *)$ and $(S_2, G, \circ)$, it is necessary to characterize the natural semigroup lattice in more exact terms.

Definition 9. Consider the family of all finite semigroups $(S, G, *)$ on the fixed generator set G, and define the relation $\circledless$ where

$$(S_1, G, *) \circledless (S_2, G, \circ) \Longleftrightarrow (S_1, G, *) \text{ is a GP homomorphic reduction of } (S_2, G, \circ).$$

It is immediately obvious that $\circledless$ is a partial ordering. The statement that $\circledless$ also forms a lattice is embodied in the following proposition:

Proposition: $\circledless$ is a lattice ordering, that is, for any two semigroups

$(S_1, G, *)$ and $(S_2, G, \circ)$, (i) there is a semigroup (S_{LUB}, G, x) which is a least upper bound for $(S_1, G, *)$ and $(S_2, G, \circ)$ in the ordering, that is, for which (1) there are GP homomorphic reductions ϕ and ψ

$$\begin{tikzcd}
 & (S_{\mathrm{LUB}}, G, x) \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
(S_1, G, *) & & (S_2, G, \circ)
\end{tikzcd}$$

and (2) if one is given GP reductions f and g from some third semigroup down onto $(S_1, G, *)$ and $(S_2, G, \circ)$,

$$\begin{tikzcd}
 & (S_3, G, \cdot) \arrow[d, dashed, "h"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
 & (S_{\mathrm{LUB}}, G, x) \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
(S_1, G, *) & & (S_2, G, \circ)
\end{tikzcd}$$

it is possible to construct the indicated GP reduction h. (ii) There is a semigroup (S_{GLB}, G, x) which is a greatest lower bound for $(S_1, G, *)$ and $(S_2, G, \circ)$: that is, for which (1) there are GP reductions θ and λ

$$\begin{tikzcd}
(S_1, G, *) \arrow[dr, "\theta"'] & & (S_2, G, \circ) \arrow[dl, "\lambda"] \\
 & (S_{\mathrm{GLB}}, G, x) &
\end{tikzcd}$$

and (2) if $\hat{f}$, $\hat{g}$ are also GP reductions

$$\begin{tikzcd}
(S_1, G, *) \arrow[dr, "\theta"] \arrow[ddr, "\hat{f}"'] & & (S_2, G, \circ) \arrow[dl, "\lambda"'] \arrow[ddl, "\hat{g}"] \\
 & (S_{\mathrm{GLB}}, G, x) \arrow[d, dashed, "j"] & \\
 & (S_3, G, \cdot) &
\end{tikzcd} ,$$

the indicated GP reduction j exists.

Proof: See Birkhoff (1967), where the same statement is established for an arbitrary abstract algebra, of which semigroups are one special case.

Given this proposition, the definition of the joint reduction immediately follows:

Definition 10. The *joint reduction* of two semigroups $(S_1, G, *)$ and $(S_2, G, \circ)$ is the semigroup (S_{GLB}, G, x) whose existence and uniqueness follows immediately from the above proposition.

In looser but more intuitive terms, it is possible to characterize the joint reduction as the result of imposing the union of all equations implied by each of the multiplication tables of $(S_1, G, *)$ and $(S_2, G, \circ)$. Viewed as a partition of FS(G), the GLB semigroup is hence the *union* of the partitions corresponding to $(S_1, G, *)$ and $(S_2, G, \circ)$ (Birkhoff 1967). This makes clear that the joint reduction is generally a coarser structure than either of the original semigroups, since in it more words are forced to be equal. Obviously, the size of the joint reduction is bounded above by the sizes of S_1 and S_2. Note that a fixed 1–1 mapping of generators in S_1 to those in S_2 is implied throughout the construction: *if a different matching is used, an entirely different joint reduction will probably result.*

A special case exists when one of $(S_1, G, *)$ and $(S_2, G, \circ)$ is already a GP reduction of the other. Without loss of generality, assume that $(S_1, G, *)$ is a GP reduction of $(S_2, G, \circ)$. Then the joint reduction is itself $(S_1, G, *)$, as may be verified formally from the above proposition. This makes sense: if two semigroups are already homomorphically comparable, the result of imposing the equations of both should lead identically to the coarser structure.

Observe that if $(S_1, G, *)$ and $(S_2, G, \circ)$ are each Boolean matrix semigroups, there is no implication that the joint reduction will also possess a naturally induced matrix representation. The joint reduction is therefore a purely algebraic measure of the consistency of two role structures and has no necessary correlates at the matrix level.

Actual computation of a joint reduction may be complicated. It is effected by a computer program, JNTHOM, which contains a separate subprogram for effecting arbitrary homomorphisms.[23] The resulting intersection table may itself be large and complex, and it may appear much closer to one than to the other of the original tables. For both reasons an objective measure of distance between the original tables is desirable.

Distance measure.—Using the joint reduction, we seek a numerical measure of distance between two role structures. We require the respective generator sets to be in one-to-one correspondence. From data already given, it is apparent that the raw size of a role structure is not a stable measure across closely related blockmodels. Accordingly, we are led to

[23] The JNTHOM program is available upon request; it is written by the second author in the APL language. We are indebted to Dr. François Lorrain for supplying an alternative program, MASTERGLB, based on an independent approach, together with an elegant proof of its validity.

reject the naive alternative of defining similarity (and thus distance) through the size of the joint reduction.

Instead, return to figure 8 and consider the *distributions* of words in $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ which are implied by the joint reduction. Specifically, consider the partitions of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively, that are obtained from the *inverses* of the reductions ϕ and ψ:

$$a \cong b \ \text{(in } \boldsymbol{S}_1) \Longleftrightarrow \phi(a) = \phi(b)$$

$$c \cong d \ \text{(in } \boldsymbol{S}_2) \Longleftrightarrow \psi(c) = \psi(d).$$

We are accordingly led to equate words in $\boldsymbol{S}_1$ (respectively, in $\boldsymbol{S}_2$) which map similarly under ϕ (respectively, ψ). This gives rise to a partition of $\boldsymbol{S}_1$ and a partition of $\boldsymbol{S}_2$; it makes sense to treat the coarseness of these partitions as a measure of the extent of aggregation in passing from $\boldsymbol{S}_1$ or $\boldsymbol{S}_2$ to the joint reduction, in other words, as a measure of distance to the joint reduction (Boorman 1970; Boorman and Arabie 1972).

The coarseness of a partition is a standard concept familiar from the entropy measure of information theory. For present purposes, we adopt a somewhat different definition, which has been shown to have more desirable numerical properties than an entropy measure when used in conjunction with multidimensional scaling (Arabie and Boorman 1973). This measure is:

$$h(P) = \frac{\sum_{c \epsilon P} \binom{|c|}{2}}{\binom{N}{2}}$$

where

$P = (c_1, c_2, \ldots, c_m)$ is a partition of a finite set $\boldsymbol{S}$ into nonempty and disjoint subsets c_i;
$|c_i| =$ size of c_i;
$N =$ size of $\boldsymbol{S}$.

The measure $h(P)$ assumes a maximum value of 1 if P is the "lumper" partition in which $\boldsymbol{S}$ is not divided at all; $h(P)$ assumes a minimum value of 0 if P is the "splitter" partition which places each element of $\boldsymbol{S}$ in a separate cell. This last possibility will occur when one of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ is itself the joint reduction, so that one of the mappings ϕ, ψ is the identity mapping.

Applying this measure to the present problem, define P_ϕ and P_ψ to be the partitions of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively, which are induced by the joint reduction (see fig. 8 for an example). Define $h(P_\phi)$ (respectively, $h[P_\psi]$) to be the distance between $(\boldsymbol{S}_1, G, *)$ (respectively, $[\boldsymbol{S}_2, G, \circ]$) and the common joint reduction. Finally, therefore, one has:

Definition 11. The distance between two semigroups $(S_1, G, *)$ and $(S_2, G, \circ)$ is $\delta([S_1, G, *], [S_2, G, \circ]) = h(P_\phi) + h(P_\psi)$, where P_ϕ and P_ψ are determined by the joint reduction.

The measure δ will be our basic tool of numerical comparison. Its range is from 0 to 2. The maximum value is attained when all entries in the joint reduction table are the same; that is, maximum degeneracy. For the case shown in figure 8, $h(P_\phi) = 0.200$ and $h(P_\psi) = 0.167$, whence $\delta = 0.367$. This indicates that the semigroups are quite close, even though the multiplication tables are superficially quite different.

From a formal standpoint, the measure δ has two relevant formal properties:

1. It makes full use of the structure of both $(S_1, G, *)$ and $(S_2, G, \circ)$, since it depends on P_ϕ and P_ψ and these partitions depend on the reductions ϕ and ψ, respectively.
2. It is a semimetric, that is, it has the following two properties:

$$\text{i)}\quad \delta([S_1, G, *], [S_2, G, \circ]) \geqslant 0, \text{ and } = 0 \text{ if and only if}$$
$$[S_1, G, *], [S_2, G, \circ] \text{ coincide (i.e., are isomorphic).}$$
$$\text{ii)}\quad \delta([S_1, G, *], [S_2, G, \circ]) = \delta([S_2, G, \circ], [S_1, G, *]).$$

The measure δ is *not* a metric, that is, it is possible to construct examples violating the triangle inequality so that

$$\delta([S_1, G, *], [S_2, G, \circ]) + \delta([S_2, G, \circ], [S_3, G, x]) > \delta([S_1, G, *], [S_3, G, x]).$$

It would be possible to eliminate this apparent difficulty directly by an appropriate transformation of δ. Since, however, we are going to use δ exclusively in conjunction with scaling, we will use the uncorrected measure as input and allow the scaling itself to determine an appropriate transformation through the Shepard diagram.

Multidimensional scaling.—When a large number of blockmodel role structures are to be compared with one another, it would be hopeless to try to discuss explicitly all the possible joint reductions. Even the numerical distances among all possible pairs may be quite difficult to assess as a set (see fig. 12 below for an example). In such situations, a natural strategy is to scale the half-matrix of distances using some form of multidimensional scaling (see Shepard 1962*a*, 1962*b*; Kruskal 1964; Shepard 1974).

In the present paper, all scaling applications use the MDSCAL-5 algorithm with the Euclidian metric ($r = 2$ in the scaling solution), stress formula 1 (Kruskal and Carroll 1969) and the primary approach to ties (Kruskal 1964). Scalings were obtained from a substantial number of alternative random initial configurations, and the solution giving the best (i.e., lowest) stress was chosen for presentation. It was found that many of the starting configurations tried gave rise to clearly nonoptimal local minima, thus supporting the earlier findings of Arabie (1973) on other data sets.

RESULTS

The results to be considered fall naturally into two parts. First, surveys of distances within large sets of role tables, including variants of given cases, are used to assess the robustness of our approach and to locate different case studies in relation to one another. Second, role structures for particular cases are examined in detail and compared via their joint reductions. As a preliminary, we lay out a consistent notation for identifying role table generators for the numerous cases, over time and with variant codings.

Nomenclature

All of the case studies in Part I involve multiple networks; some (Sampson, Newcomb) involve data over time; each can be described by a number of blockmodels, some involving different numbers of blocks. In addition, while Part I does not formalize it, there is also a concept of *cutoff density* for coding zeroblocks: in many cases (as exemplified by the Similar Policy relation in the Firth-Sterling data) a given block may contain only one or two ties, so that measurement error in the data suggests coding as a zeroblock rather than as a bond. All of these factors contribute to a proliferation of blockmodels, each possessing an associated semigroup. A system of notation is developed in table 1.

We draw a strict distinction between the determination of a blockmodel partition on the one hand and the coding of bonds on the other. *All blockmodel partitions used in this paper are shown in panel IV of table 1; how they are determined is considered to be a task for the methods of Part I and will not be further discussed here* (see also Breiger, Boorman, and Arabie 1975; Schwartz 1977). Accordingly, the notation merely reminds the reader of the number of blocks and is directed toward the coding of bonds (see panel II of table 1 for an example).

For given density matrices one may define a one-parameter (piecewise constant) family of semigroups $S(\alpha)$, $\alpha > 0$, where α is the density cutoff. If $\alpha = 0$ a strict zeroblock criterion is being employed. As indicated by the basic blockmodel phenomenology of Part I, we are interested only in α values that are small compared to the average density. (As α increases further, more and more blocks are coded as zeroblocks until

TABLE 1

NOTATION FOR GENERATORS

I. The general format is:

(case study initials)(time period)-(number of blocks)-$X(i)Y(j) > \alpha$ where cutoff density α is being imposed on both generators X and Y, taking top i and top j choices, respectively, *weighted* linearly (see example). The first letter in the name of that type of tie is X or Y. The specific partition of persons into blocks for each case study is reported in IV below.

Conventions:

1. If a strict zeroblock criterion is being employed, $\alpha = 0$ and the final inequality is suppressed. Thus SM(4)-5-$I(2)N(2) > 0$ becomes shortened to SM(4)-5-$I(2)N(2)$.
2. When the data are observer reported (e.g., in the Bank Wiring group), there is no ordering of choices and ties are not weighted in computing the density. Then i and j are not required: thus BW-3-*LA* identifies Like and Antagonism in the Bank Wiring group, using a three-block model.
3. If the data are not longitudinal, time period is suppressed.
4. Each density is computed relative to the number of possible entries in a block, excluding the diagonal if the block is a diagonal block. Thus the density in an $r \times r$ diagonal block is

$$\Sigma\ w(i,j)/r(r-1)$$

entries where $w(i,j)$ is the weight of the choice in the (i,j)th entry within the block. The density of an $r \times c$ off-diagonal block is

$$\Sigma\ w(i,j)/r \cdot c.$$

II. Example: SM(4)-5-$E(3)D(3) > 1$

This identifies the matrix semigroup formed from the Sampson data, time 4, five-block model (see IV below), using cutoff density $\alpha = .1$ on the weighted top three choices for the Esteem and Disesteem relations (weight 3 for top choice, etc.). The weighted densities are (to three figures):

$$\begin{bmatrix} 0.167 & 0.833 & 0.111 & 0 & 0 \\ 0.667 & 1.17 & 0 & 0 & 0 \\ 0.222 & 0.417 & 1.67 & 0.083 & 0 \\ 0 & 0 & 1.5 & 0.917 & 0 \\ 0.25 & 0.063 & 0.5 & 0.125 & 1.17 \end{bmatrix} \text{(Esteem)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0.083 & 0.75 & 0.375 & 0.688 \\ 0 & 0.25 & 0.333 & 0.25 & 0.333 \\ 0 & 1.12 & 0 & 0 & 1.06 \\ 0.083 & 1.31 & 0.25 & 0 & 0.083 \end{bmatrix} \text{(Disesteem).}$$

Cutoff $\alpha = .1$ leads to:

$$E = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The semigroup is of size 10.

III. Case study names

- BW: Bank Wiring Room
- SM: Sampson monastery ("Praise" in fig. 4 of Part I has been changed to "Kudos" to avoid "*P*" as the initial letter).
- NF1: Newcomb fraternity, year 1
- NF2: Newcomb fraternity, year 2
- FS: Firth-Sterling Corporation, management
- GS: Griffith scientists (here C stands for mutual contact ties, and A for asymmetric unawareness. See Part I).

TABLE 1 (*Continued*)

IV. Standard blockmodel partitions	
BW-2:	(W1 W3 W4 S1 I1 W2 W5 I3)(W7 W8 W9 S4 W6 S2) *Numbering:* Follows Homans (1950); see also Part I.
BW-3:	(W3 W4 S1 W8 W9)(W1 I1 W7 S4)(W2 W5 S2 W6 I3)
BW-6:	(W4 S1 W3)(W1 I1)(W2 W5 I3)(W8 W9)(W7 S4)(W6 S2)
SM-3:	(10 5 9 6 4 11 8)(12 1 2 14 15 7 16)(13 3 17 18) *Numbering:* Follows Sampson (1969); see also Part I.
SM-5:	(10 5 9)(6 4 11 8)(12 1 2)(14 15 7 16)(13 3 17 18)
NF1-4:	(1 3 4 6 12)(2 11 13)(5 9 10 14 15)(7 8 16 17) *Numbering:* Follows Nordlie (1958) and Newcomb (1961).
NF2-3:	(13 9 17 1 8 6 4)(7 11 12 2)(14 3 10 16 5 15) *Numbering:* Follows Nordlie (1958) and Newcomb (1961); see also Part I.
FS:	(11 9 13)(2 4 6 7 8 1 15)(5 10 12 14 3 16) *Numbering:* Follows White (1961); see also Part I.
GS:	(9 26 23 4 1)(12 7 6 2 24 19)(14 28 11 10 18 22 15)(16 20 17 5 8 13 21 27 25 3) *Numbering:* Follows Part I; see also Breiger 1976.

eventually, when α exceeds the maximum density, all generators become trivial Z patterns.) For example, the average density is .375 in the three-block model for the final week (week 15) of the Newcomb year 2 data, with top three choices weighted. The density matrices for the generators are[24]

$$L:\ \begin{matrix} 0.950 & 0.071 & 0.048 \\ 0.357 & 1.08 & 0.042 \\ 0.571 & 0.417 & 0.067 \end{matrix} \qquad A:\ \begin{matrix} 0 & 0 & 1 \\ 0.071 & 0 & 0.917 \\ 0.095 & 0.083 & 1 \end{matrix}$$

Here there is a clear dichotomy between low density (<0.1) and high density (>0.35) blocks, indicating a preferred cutoff which separates these two ranges. More typical, however, is the Sampson *ED* case illustrated in table 1. Here the density sequence begins (0, .063, .083, .111, .125, .167, .222, .250, . . .) and there is no clearly preferred cutoff. In such cases, we have accordingly explored a variety of alternative cutoffs, with standard α values being .1, .15, and .2.

A Global Geometry for Types of Role Structures

Results for two sets of role structures lead to an interpretation of the relations among the main case studies.

Table 2 identifies the first set of role structures, using the nomenclature of table 1. This first sample is intended as a potpourri from all our main

24 Compare with the bottom panel of fig. 7 in Part I, where unweighted densities of choices are used taking only the top two and bottom three choices for each man. Generally, a cutoff of $\alpha = 0.1$ with weighted top three choices yields the same or nearly the same generators as for the strict zeroblock cutoff on the top two choices which was emphasized in Part I.

TABLE 2

IDENTIFICATIONS OF ROLE TABLES IN FIGURES 12 AND 13

1. BW-2-*LA*	10. SM(4)-5-*L*(2)*A*(2)
2. BW-3-*LA*	11. SM(4)-5-*E*(2)*D*(2)
3. BW-6-*LA*	12. SM(4)-5-*I*(2)*N*(2)
4. NF1(13)-4-*L*(2)*A*(2)	13. SM(4)-5-*K*(2)*B*(2)
5. NF1(13)-4-*L*(3)*A*(3) > .1	14. SM(4)-5-*E*(2)*I*(2)
6. NF1(9)-4-*L*(3)*A*(3)	15. SM(4)-5-*E*(3)*D*(3) > .1
7. NF2(15)-3-*L*(2)*A*(3)	16. SM(3)-5-*E*(3)*D*(3) > .1
8. NF2(15)-3-*L*(3)*A*(3) > .1	17. SM(4)-3-*E*(2)*D*(2)
9. NF2(10)-3-*L*(3)*A*(3) > .1	18. FS-3-*FU*
	19. GS-4-*CA*

cases. All these role structures are based on two generators, which with two exceptions have the quality of a positive versus negative affect pair, evidenced both by face definitions and by their inclusion orders. The test-case exceptions are #14, SM(4)-5-*E*(2)*I*(2), with Esteem and Influence as the types of ties, and #19, GS-4-*CA*, based on Griffith's scientists (see table 1, III).

Figure 12 now shows the pairwise distances. Even without scaling, it is already apparent that Sampson and the two Newcomb years are quite close to one another, while the Bank Wiring Room cases and the Firth-Sterling case are outliers. Note also that the values of δ are quite unevenly spread over the [0, 2] range: there are 35 entries where $\delta = 2$, and very few with $\delta < 0.1$.

Using the figure 12 matrix as input, one obtains the two-dimensional nonmetric scaling shown in figure 13. The stress value $S = 12.8\%$ is well within the acceptable range for two dimensions and 19 points (Arabie and Boorman 1973). More important, it is clear that the scaling is coherently discriminating the main data cases, which are shown delineated by clusters superimposed on the scaling solution. The clusters shown may be further supported by an independent hierarchical clustering of the figure 12 lower halfmatrix. Using the HICLUS algorithms (diameter methods and connectedness method) described by Johnson (1967), one obtains clusters which discriminate Newcomb year 2, Firth-Sterling, the Bank Wiring Room, and (#4, #6) of Newcomb year 1. This is in perfect agreement with the clustering by data cases, with the main exception that #5 (a Newcomb year 2 case) is located with the Sampson cluster.

Note that the Sampson *LA*5 case (#10 in fig. 13) is an outlier to the main Sampson cluster, as is to be expected when one compares the *LA* blockmodel images with other Sampson images. It is possible that this outlying position may be a reflection of the unwillingness of Sampson's monks to reveal their preferences on social relationships as blatant as Liking Most and Liking Least. Note that the *EI*5 case (#14 in fig. 13)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	×																		
2	.02	×																	
3	.05	.03	×																
4	1.4	.98	.75	×															
5	1.5	1.2	.69	.39	×														
6	1.2	.68	.76	.30	.48	×													
7	2.0	2.0	2.0	2.0	.76	2.0	×												
8	2.0	2.0	2.0	2.0	.76	2.0	.67	×											
9	2.0	2.0	2.0	2.0	.93	2.0	.17	.17	×										
10	1.6	1.3	.68	.33	.48	.62	2.0	2.0	2.0	×									
11	1.6	1.3	.93	.61	.25	.65	.35	.27	.20	.49	×								
12	1.7	1.4	.78	.42	.29	.68	.38	1.0	.55	.54	.10	×							
13	1.5	1.2	.94	.78	.48	.48	.24	.24	.14	.75	.09	.29	×						
14	1.6	1.3	1.1	.88	.42	.58	.83	.83	1.0	.88	.66	.85	.59	×					
15	1.6	1.3	.88	.33	.20	.62	.98	.29	.46	.44	.06	.11	.18	.77	×				
16	1.6	1.3	.83	.28	.21	.58	.28	.83	.44	.39	.27	.28	.28	.56	.33	×			
17	1.4	1.4	1.5	1.2	.29	.93	.20	.93	.37	1.4	.18	.20	.29	.66	.37	.29	×		
18	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	1.3	2.0	2.0	2.0	×	
19	1.5	1.2	.94	.78	.95	.48	2.0	.76	.76	.75	.33	.83	.19	1.1	.30	1.1	1.3	.96	×

FIG. 12.—Lower halfmatrix of distances between role tables identified in table 2

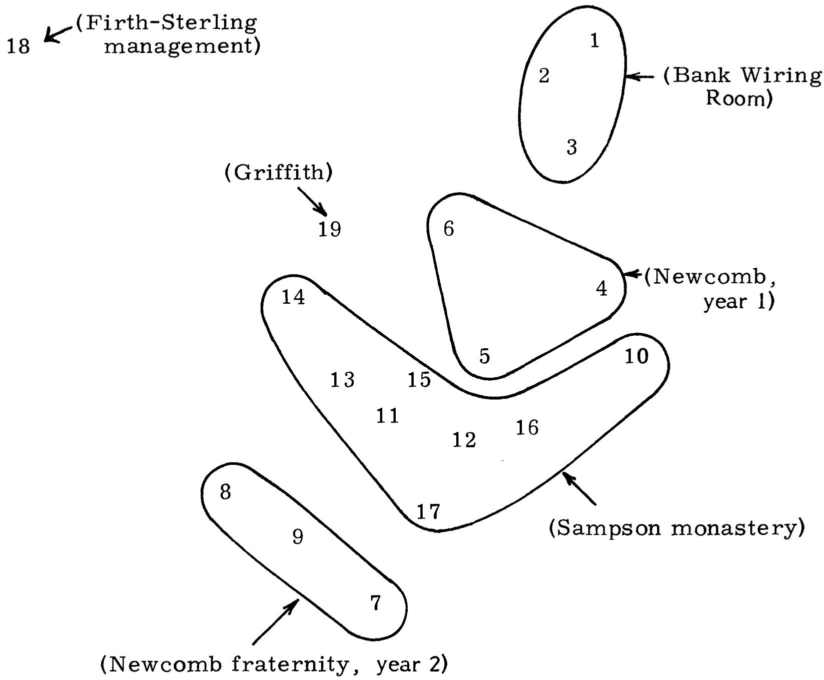

FIG. 13.—Multidimensional scaling of 19 role tables from different data sets, identified in table 2. MDSCAL-5 algorithm applied to lower halfmatrix of dissimilarities (fig. 12) based on JNTHOM algorithm. Two dimensions, Euclidean metric, stress (formula 1) = 12.8% (best of 20 random initial configurations; for theoretical background, see Arabie [1973]).

is the other Sampson outlier, as is consistent with the fact that E and I are both naturally interpreted as coding positive sentiment. Note that EI is situated on a ray leading from the center of the plot and directed toward the Firth-Sterling case. The three-dimensional solution (not shown) places EI somewhat farther from the main cluster, and correspondingly closer to Firth-Sterling.

Finally, note that the Griffith Most Contact and Asymmetric Unawareness semigroup is also placed in the same quadrant as $EI5$ and Firth-Sterling. This is internal grounds for arguing that this Griffith case is not picking up any true negative sentiment relation.

Figure 14 shows a second population of 20 role structures identified in table 3, now consisting entirely of positive versus negative generator pairs. The scaling shown in figure 14 is obtained by applying the MDSCAL-5 algorithm in the same manner as figure 13. This second population is constituted to focus more closely on alternate codings of the Sampson monastery and the two Newcomb years. In particular, the $ED5$ images

TABLE 3

IDENTIFICATIONS OF ROLE TABLES IN FIGURE 14

1. SM(4)-5-$E(2)D(2)$	10. SM(4)-3-$E(3)D(3) > .1$
2. SM(4)-5-$E(3)D(3)$	11. SM(4)-3-$K(2)B(2)$
3. SM(4)-5-$E(3)D(3) > .1$	12. NF1(13)-4-$L(2)A(2)$
4. SM(4)-5-$E(3)D(3) > .2$	13. NF1(13)-4-$L(2)A(2) > .1$
5. SM(4)-5-$E(3)D(3) > .5$	14. NF1(13)-4-$L(3)A(3) > .1$
6. SM(4)-5-$E(3)D(3) > 1.0$	15. NF1(13)-4-$L(3)A(3) > .2$
7. SM(3)-5-$E(3)D(3) > .1$	16. NF1(9)-4-$L(3)A(3) > .1$
8. SM(4)-5-$I(3)N(3) > .1$	17. NF2(15)-3-$L(2)A(2)$
9. *SM(4)-5-SUMPOS(3)SUMNEG(3) $> .4$	18. NF2(15)-3-$L(3)A(3) > .1$
	19. NF2(10)-3-$L(3)A(3) > .1$
	20. BW-2-LA

* Computed by adding the weighted block densities of the four positive and four negative images, and using a cutoff density corresponding to .1 on individual generator pairs.

are now introduced with *five* distinct cutoffs ($\alpha = 0$, .1, .2, .5, 1); see the example in table 1 for the density matrices, from which the generator images for these cutoffs may be coded. Remarkably, the *ED*5 semigroups remain tightly clustered in the center of figure 14 until $\alpha = 1$, when the corresponding point (#6) moves out to the northeast. By the present measure of distance, it therefore appears that the structure of the semigroups is quite insensitive to varying cutoffs; the insensitivity is certainly not apparent from simply considering the images.

Because output of the MDSCAL algorithm has no preferred orientation, the scaling solution in figure 14 has been rotated to approximately the same orientation as figure 13 (using the CONGRU algorithm of D. C. Olivier). The agreement between the two figures is close, with each of the four principal case studies occupying similar relative positions in each solution. The clusters shown in figure 14 also can be largely replicated by a HICLUS analysis applied to the lower halfmatrix input (not shown). Once again, the chief difficulty consists in discriminating some of the Newcomb year 1 cases from the Sampson cases, as is also apparent from the scaling. Note also that cases #11 and #17 give identical blockmodels, though based on different case studies. This is of course reflected in the scaling, where #11 and #17 correspond to the same point in the solution.

Outcasts and leading crowds.—Return to the figure 13 scaling. There is a clear linear arrangement of the four main case studies: Bank Wiring Room :: Newcomb year 1 :: Sampson :: Newcomb year 2, and this ordering is replicated consistently in figure 14 as well. It is in fact almost possible to separate each of these four principal cases in the linear Minkowski sense, that is, by means of a separating hyperplane (a line in two dimensions); this is not possible only because of certain extreme cases, such as #6 in figure 14 (*ED*5 with the aberrant cutoff value $\alpha = 1$).

It is particularly interesting that the Sampson cluster clearly separates

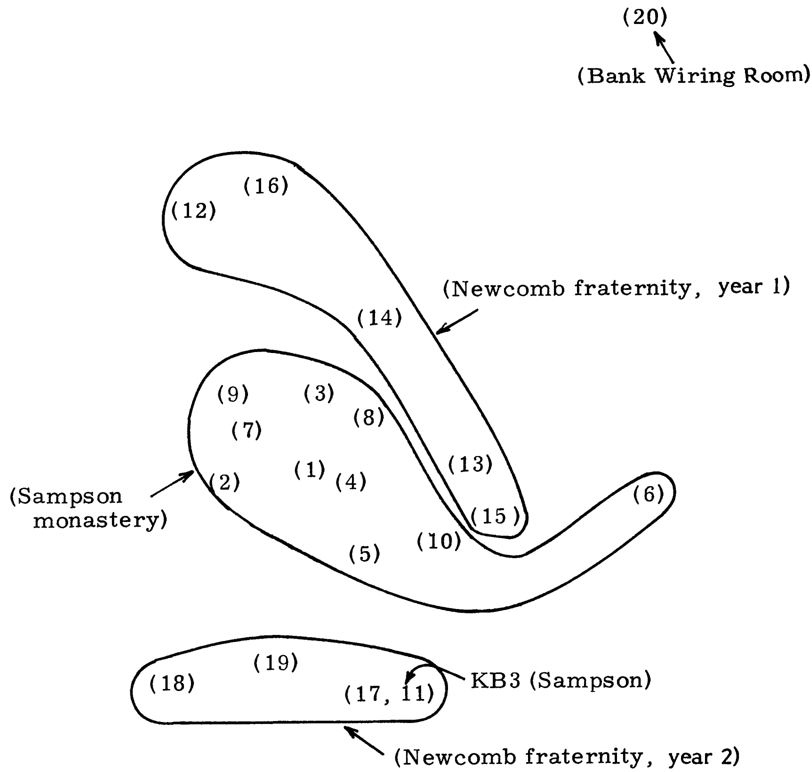

FIG. 14.—Closer view of selected Sampson and Newcomb cases, identified in table 3. Cases 11 and 17 generate the same role table. Case 6 was generated using cutoff = 1, far higher than the values used elsewhere, and one would accordingly expect aberrant results. Multidimensional scaling in two dimensions, Euclidean metric, stress (formula 1) = 13.7%; rotated to approximately the same orientation as fig. 13.

the two Newcomb years. Since Part I does not describe the Newcomb data for his first experimental fraternity (year 1), it is worth briefly introducing a blockmodel. As in year 2, the system reaches approximate equilibrium after the first few weeks (see also figs. 16 and 17 below). A basic blockmodel description can therefore be based on any of the later weeks with little change in result; we chose week 13. With the four blocks produced by CONCOR, the densities are (weighted top three choices):

$$L: \begin{matrix} 0.70 & 0.87 & 0.08 & 0.05 \\ 0.33 & 1.3 & 0.07 & 0 \\ 0.04 & 0.20 & 0.96 & 0.10 \\ 0.10 & 0.08 & 0.50 & 0.69 \end{matrix} \qquad A: \begin{matrix} 0.30 & 0.13 & 0.48 & 0.50 \\ 0.53 & 0 & 0.27 & 0.50 \\ 0.84 & 0 & 0.08 & 0.35 \\ 1.0 & 0.25 & 0 & 0.06 \end{matrix}$$

This structure can be interpreted as two cliques ($I + II$ versus $III + IV$), each divided into a core group of leaders (II or III) and a hangers-on group (*I* or *IV*). Moreover, as in the Bank Wiring case, it is possible to present internal evidence for concluding that one of the cliques ($I + II$) is in fact the dominant one. Observe, for instance, that the hangers-on of the subordinate clique (*IV*) direct sharp antagonism toward the hangers-on of the top dogs of the dominant clique (*I*); whereas the hangers-on of the dominant clique (*I*) direct relatively far less attention to the members of the subordinate clique (*III, IV*) (cf. Breiger [1976] discussing "visibility"; Stinchcombe [1968] discussing the structure of "attention").

This overall blockmodel structure is generally quite similar to the Bank Wiring blockmodel on the same number of blocks, although possibly because the data are sociometric and not observer reported the pattern is not quite so clear cut (e.g., in the Bank Wiring case there is no Antagonism either within or between core cliques; in the present Newcomb example, there is some antagonism directed from the core we have called dominant to the subordinate core, although the density .27 remains fairly low). On the other hand, the Newcomb year 1 pattern differs in various noticeable respects from the basic three-block model for the late weeks of year 2. Thus the year 1 hangers-on scapegoat themselves far less than does the bottom block in the year 2 model, as evidenced separately on both Like and Antagonism images.

Looking directly at the blockmodels, it makes sense for the Sampson role structures to be intermediate between the Newcomb cases: the basic Sampson pattern, identifiable in all generator pairs, involves two separate main cliques (the Young Turks and the Loyal Opposition), each possessing the familiar internal organization of leaders and hangers-on; but there is also a peripheral group, the Outcasts, whose lack of received positive sentiment from the top blocks is analogous to that of the scapegoats in Newcomb year 2, but whose position in other respects is closer to that of the hangers-on in Newcomb year 1.

At the same time, certain regularities in the figure 13 and 14 scalings remain anomalous when evaluated solely at the blockmodel level. Why, for example, does the scaling place Sampson so clearly between the Newcomb cases when a strong argument can also be made that the Sampson outcasts are not analogous to any Newcomb position? (Thus with respect to positive sentiment directed outside the block, the average density for the Sampson outcasts is far lower than for the Newcomb year 2 scapegoats, and also for the Newcomb year 1 hangers-on.) Again, why does the scaling locate all three Bank Wiring cases in figure 13 in essentially the same relation to Newcomb year 1, even though BA-3-*LA* conveys a quite different aspect of the group's structure than that described by the cliques (namely, a three-tiered stratification system obtained by aggregating across cliques)?

These anomalies bear on one main point: there is coherence in the purely algebraic structure on which figures 13 and 14 are based, and this coherence is reflected only to a limited extent in similarities at the blockmodel level. Even the most elementary comparisons at the blockmodel level run into difficulties when alternative cases are most naturally described by different numbers of blocks (e.g., four for Newcomb year 1 but three for Newcomb year 2). One may hope that it may become possible to speak with confidence of "the" role structure found in a particular case study, even though thus far we have established no formal criteria for obtaining such a unique description. To carry the analysis one step further, it is time to look inside role tables on a purely algebraic level and to seek a nucleus of common role structure through selected homomorphic images.

Role Interlock

Our results will be based on role tables constructed from pairs of generators. For pairs interpretable as describing positive-negative affect, figures 13 and 14 have suggested that three of the case studies—the Firth-Sterling management, the Bank Wiring Room, and Griffith's scientists—are quite unlike the others; this divergence is reinforced by the inclusion orders (thus see fig. 11). These three populations are much more constrained than the others by formal organization and outside pressures (see Part I; and see Breiger [1976] for a further report on the Griffith scientists case). In such a population, we suspect that any role interlock which includes negative-affect generators will be idiosyncratic, shaped by particular constraints not accounted for in the formalism. Among the other three populations, however, it is possible to identify role interlocks which have common features.

Where neither generator is negative, we will identify role interlock features common to all the populations with such data (only Newcomb's fraternities are excluded, since here there is only a single generator connoting positive affect). Apparently, it is chiefly role interlock involving *negative* affect which is sensitive to bureaucratic and other outside pressures.

Positive-negative pairs of generators.—Sampson's case is central in figures 13 and 14, and we build from it. We begin with the strict zeroblock cutoff version on three blocks which was emphasized in Part I. The earlier analysis of figure 10, as well as the clustering in figures 13 and 14, suggests seeing whether his four different pairs of generators exhibit similar role interblock.

The Influence pair and the Esteem pair, *IN*3 and *ED*3, have the same role table, which is shown in figure 6*b*. Note again that these two pairs of generators (also shown in fig. 6*b*) are distinct as blockmodels (depending on whether it is the Opposition or the Turks who are placed on top of the hierarchy); this difference was important in our concrete account of the monastery in Part I, yet we now find it compatible with identical role interlock. For the Kudos pair $KB3$ (#5 and #6 in fig. 2) the role table proves to be just the target table T_1, discussed in detail at the end of the section on theory.

The joint reduction of $IN3$-$ED3$-$KB3$, the coarser form of interlock common to them, is itself precisely T_1; *thus* T_1 *is a homomorphism of the IN3-ED3 table, as well as being the KB3 table.* This finding of underlying T_1 structure is now confirmed and extended by three series of tests. First, the joint reduction of the role table for each pair of generators is computed for each of the seven other target tables. Figure 15 reports whether or not these others are also homomorphisms, and in addition gives the distances through the joint reduction.[25] Only the LA case gives problems, and this must happen since the original role structure on L and A is itself degenerate.

Second, the same procedure is carried out for the refined models with the three blocks split into five. The results, also given in figure 15, again show that T_1 is the form of role interlock common to these generator pairs. Again, only LA is the outlier. Running reductions of the five-block models in this way is not redundant with the three-block reductions just reported, since a refinement of a blockmodel need not yield a semigroup which maps homomorphically onto the three-block blockmodel's role structure.

Third, we test reliability by seeing whether T_1 remains the common form of role interlock for an alternate coding of the generators. Apply a zeroblock cutoff of 0.1 to the weighted top three choices. The resulting generators for $KB3$, shown in figure 1, again identically yield T_1 as their role table. For $IN3$ and $ED3$ the generators, and thus the role table, are the same as with the strict zeroblock cutoff, hence must also reduce to T_1 *a fortiori.* (As for the refined models on five blocks, figure 7 has already shown that the role table for the Influence pair reduces to T_1; the same is true for $KB5$ though not for $ED5$.) Reliability was tested further, using still higher values of 0.15 and 0.2 for the cutoff density, with much the same results for these three generator pairs; and, for the first time, the Like-Antagonism pair also has a role table reducing to T_1.

Because of relative position in the scalings of figures 13 and 14, Newcomb's two fraternities should contrast in opposite ways with the monastery. The T_1 role interlock common to all of Sampson's pairs of generators can be compared with that for the one generator pair we have

[25] Where T_i is not a homomorphic reduction of the original semigroup, the joint reduction in each case is one of the trivial 2×2 tables shown in the discussion of target tables at the end of the section on theory.

Role Table	Size	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
		1 2 3 3 3 3 3 3 3	1 3 3 2 3 3 3 3 3	1 2 3 2 3 3 3 3 3	1 2 3 3 2 3 3 2 3	1 3 3 2 2 2 3 3 3	1 1 1 3 3 3 3 3 3	1 3 3 1 3 3 1 3 3	1 3 3 3 2 3 3 3 3
SM(4)-3-E(2)D(2)	5	Y(.35)	Y(.20)	N(.93)	N(.93)	N(.93)	N(.93)	N(.93)	N(.96)
SM(4)-5-E(2)D(2)	11	Y(.20)	Y(.29)	N(.90)	Y(.27)	N(.90)	N(1.2)	Y(.44)	N(.90)
SM(4)-3-I(2)N(2)	5	Y(.38)	Y(.20)	N(.93)	N(.93)	N(.93)	N(.93)	N(.93)	N(.93)
SM(4)-5-I(2)N(2)	12	Y(.20)	Y(.38)	N(1.0)	N(1.0)	N(1.0)	N(1.2)	N(1.2)	N(1.0)
SM(4)-3-K(2)B(2)	3	ISO(0)	N(.67)	N(.67)	N(.67)	N(.66)	N(2.0)	N(2.0)	N(.67)
SM(4)-5-K(2)B(2)	7	Y(.24)	N(.86)	N(.86)	Y(.24)	N(.86)	N(1.1)	Y(.33)	N(.86)
SM(4)-3-L(2)A(2)	3	---------- (degenerate) ----------							
SM(4)-5-L(2)A(2)	10	N(2.0)	N(2.0)	N(2.0)	N(2.0)	N(2.0)	N(1.1)	N(1.1)	N(2.0)

FIG. 15.—Alternative homomorphisms of role tables from the Sampson data. Distance δ to the target tables are shown parenthetically. Y = yes (is a homomorphic reduction), N = no (is not), ISO = isomorphic.

identified in Newcomb's data, Like and Antagonism, using a cutoff density of 0.2. In addition, longitudinal stability of the results for each Newcomb fraternity can be shown. (This is not really possible for the monastery: Newcomb collected data week by week, whereas Sampson has only recall data for earlier periods; see Part I.) The results are shown in figures 16 (year 2) and 17 (year 1) in a format parallel to that of figure 15.

From an early week, the role structure for the second fraternity either is or reduces to T_4, but not to T_1 or the other six. The distinctive equation in T_1, $LA = A$, is also present in T_4, but the role interlock is different because in addition the second generator is transitive, $A^2 = A$. This latter equation reflects the scapegoating of a bottom block found here but not in the monastery (e.g., contrast the Antagonism matrix in the text of the Nomenclature section with the Disesteem matrix in table 1).

From an early week, the role structure for the *first* fraternity reduces to T_1 and to T_2, but never to T_3. A T_4 reduction is not obtained until very late in the sequence, and it is not possible to know whether its presence is robust. Recall that T_3 is the table for Davis's weaker form of balance theory, in which the commutativity of A with L is retained. Thus, the two key equations of T_3 cannot be obtained together, but can be obtained separately in reductions of the role table to T_1 and T_2. Unlike the second fraternity, the first has role interlock akin to balance theory, but a form of balance even weaker than Davis's. This conclusion is consistent with the earlier interpretation in the section on outcasts and leading crowds. The above conclusions for each fraternity are essentially unchanged if a zeroblock cutoff density of 0.1 is used instead of 0.2. The main effect of thus lowering α is that *none* of T_1–T_8 becomes a homomorphic image until the late weeks—role interlock is not characterized as rapidly.

Other pairs of generators.—The first lines in figure 18 already suggest the main conclusion to be drawn: for a range of populations, including those with formal organizations, role interlock between a generator for Similar Policy (or some similar type of tie) and a generator for Liking reduces to the predominance of one's direct ties, whose quality subsequently carries over to indirect ties as well. This is the First Letter interlock discussed earlier under theory, and there exemplified by target tables T_5 and T_6 (see figure 18).

In addition to Friendship and Similar Policy for the Firth-Sterling management, we also report results for the LH pair of generators in the Bank Wiring Room and for the LK pair of generators in the monastery (H signifies "Helping" in the Bank Wiring data and is the only major tie in these data which is not symmetric; see Part I). Figure 19 shows the joint reductions of pairs of these role tables. To show reliability, three

Week	Size	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
		1 2 3 3 3 3 3 3 3	1 3 3 2 3 3 3 3 3	1 2 3 2 3 3 3 3 3	1 2 3 3 2 3 3 2 3	1 3 3 2 2 2 3 3 3	1 1 1 3 3 3 3 3 3	1 3 3 1 3 3 1 3 3	1 3 3 3 2 3 3 3 3
0	3	N	N	N	N	N	N	N	N
1	3	N	N	N	N	N	N	N	N
2	4	N	N	N	N	N	N	N	N
3	5	Y	N	N	Y	N	N	Y	N
4	5	Y	N	N	Y	N	N	Y	N
5	6	Y	N	N	Y	N	N	Y	N
6	3	N	N	N	Y	N	N	N	N
7	6	Y	N	N	Y	N	N	Y	N
8	3	N	N	N	Y	N	N	N	N
10	3	N	N	N	Y	N	N	N	N
11	3	N	N	N	Y	N	N	N	N
12	5	N	N	N	Y	N	N	Y	N
13	5	N	N	N	Y	N	N	Y	N
14	3	N	N	N	Y	N	N	N	N
15	3	N	N	N	Y	N	N	N	N

FIG. 16.—Alternative homomorphisms across Newcomb weeks, year 2. Data are NF2(i)-3-$L(3)A(3) > .2$ for $i = 0, 1, \ldots, 15$ (week 9 missing); $Y =$ yes, $N =$ no.

Week	Size	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
		1 2 3 3 3 3 3 3 3	1 3 3 2 3 3 3 3 3	1 2 3 2 3 3 3 3 3	1 2 3 3 2 3 3 2 3	1 3 3 2 2 2 3 3 3	1 1 1 3 3 3 3 3 3	1 3 3 1 3 3 1 3 3	1 3 3 3 2 3 3 3 3
1	5	N	N	N	N	N	N	N	N
2	5	N	N	N	N	N	N	Y	N
5	7	N	N	N	N	N	N	Y	N
6	7	N	N	N	N	N	N	Y	N
7	7	Y	Y	N	N	N	N	N	N
8	9	Y	Y	N	N	N	N	N	N
9	6	Y	Y	N	N	N	N	N	N
11	9	Y	Y	N	N	N	N	N	N
12	5	Y	Y	N	N	N	N	N	N
13	6	Y	Y	N	N	N	N	N	N
14	10	Y	Y	N	Y	N	N	Y	N
15	6	Y	Y	N	Y	N	N	Y	N

FIG. 17.—Alternative homomorphisms across Newcomb weeks, year 1. Data are NF1(i)-4-$L(3)A(3) > .2$ for indicated weeks

Role Table	Size	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
		1 2 3 3 3 3 3 3 3	1 3 3 2 3 3 3 3 3	1 2 3 2 3 3 3 3 3	1 2 3 3 2 3 3 2 3	1 3 3 2 2 2 3 3 3	1 1 1 3 3 3 3 3 3	1 3 3 1 3 3 1 3 3	1 3 3 3 2 3 3 3 3
FS-3-FS	6	N	N	N	N	N	Y	N	N
FS-3-SF	6	N	N	N	N	Y	Y	N	N
SM(4)-5-E(3)I(3) > .1	9	N	Y	N	N	N	N	N	Y
SM(4)-5-I(3)E(3) > .1	9	N	Y	N	N	N	N	N	Y
SM(4)-5-E(3)I(3) > .2	9	N	Y	Y	N	N	N	N	Y

FIG. 18.—Alternative homomorphisms for role tables with neither generator negative; $Y =$ yes, $N =$ no

$$
\begin{tikzcd}
 & \text{(6) SM(4)-5-L(3)K(3)} > .25 \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{matrix} 3&3&3&3\\4&4&4&4\\3&3&3&3\\4&4&4&4 \end{matrix} & \begin{matrix} 3&3&3&3\\4&4&4&4\\3&3&3&3\\4&4&4&4 \end{matrix} & \begin{matrix} 1&4&1&4\\3&2&3&2\\1&4&1&4\\3&2&3&2 \end{matrix} \\
\text{(6) FS-3-FS} \arrow[u] \arrow[d] & \text{(11) SM(4)-5-L(3)K(3)} > .1 \arrow[u] \arrow[d] & \text{(10) BW-6-LH} \arrow[u] \arrow[d] \\
\begin{matrix} 1&1\\2&2 \end{matrix} & \begin{matrix} 1&1\\2&2 \end{matrix} & \begin{matrix} 1&4&1&4\\3&2&3&2\\1&4&1&4\\3&2&3&2 \end{matrix} \\
 & \text{(5) SM(4)-5-L(3)K(3)} > .5 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

FIG. 19.—Joint reductions of role tables from pairs of positive generators. The size of each role table is shown in parentheses before its identification. Lines lead from each member of a pair to their joint reduction table.

different codings for the *LK* pair are used. It is clear that the First Letter table is the predominant feature of role interlock in the joint reductions.

Return to figure 18. The first line shows that the FS pair for Firth-Sterling reduces *only* to a First Letter interlock, to T_6. In the second line, order of generators is reversed: the FS pair is reversed to become the SF pair, which has a reduction to T_5 as well as to T_6, both First Letter interlocks. The role tables for SF and for FS are therefore *not* the same (see the discussion of Definition 10 in the methods section) and need not have the same homomorphisms. (With positive-negative pairs of generators, generator reversal customarily leads to entirely different homomorphisms: indeed, the distance δ between such a pair and its reverse through their joint reduction often assumes the maximum value of 2.0.)

First Letter interlock need not occur for a pair of generators even though neither represents negative affect. The remaining lines of figure 18 analyze an example. The Esteem versus Influence pair is interesting in its own right: recall the earlier discussion of *ED* versus *IN* role tables above. We see that regardless of whether $\alpha = .1$ or $.2$ the *EI* role table reduces to the T_8 form of transitive generators with no interlock between them.[26] And the reversed pair *IE* yields a similar pattern of reductions. But all these cases reduce also to T_2, which is the natural dual to T_1.

Further work.—As more case studies become available, it should be possible to correlate the main features of role interlock with further properties of concrete populations. We have made initial analyses of two further sets of data of high quality.[27] Given such correlations, it will be worthwhile to try to verify more detailed descriptions of role interlock for specific cases. And with more cases it should be possible to find comparable sets of three and more generators: the existing analytic techniques and computer programs already described are directly applicable, but further development of target tables will be necessary.

CONCLUSION

What makes a society human? Speaking as a sociologist, one is tempted to seek the answer in the existence of roles. The problem is not so simple: whatever the distinctive features of the invertebrate societies (Haskins 1939; Grassé 1959; Wilson 1971), it is clear that at least the higher primates have well-developed complexes of stable social relationships which seem to behave much like human roles, at least to primatologists (Kummer 1967; Blaffer Hrdy 1976). A somewhat more sophisticated hypothesis is that the characteristic features of human society lie in the

[26] The full role table for the coarser three-block version is T_8.

[27] Kindly supplied by A. P. M. Coxon and by F. Lorrain.

peculiarly intricate complexes of interlocking roles which only men can sustain. The present work tries to take seriously what Durkheim saw but most of his followers did not: that the organic solidarity of a social system rests not on the cognition of men but rather on the interlock and interaction of objectively definable social relationships.

There is a moral cast to the study of roles (Emmet 1966). We see support even in the present limited study for a stance we think important: Humans can and do build complex and subtle social structure without the need for a directing hand or an acknowledged plan (Geertz 1965). Official structures can impede solutions to structural problems (Burns and Stalker 1955), not least by the drain of energy involved in reconciling reality with facade. One thinks of law, and also of formal organizations (Boorman 1975*b*).

We see at present no intelligent way to develop role interlock for open networks extending through large populations, even though this topic is much closer to the heart of sociology than is small-group structure (Milgram 1967; White 1970; Granovetter 1976). From an analytic standpoint, the present machinery is suggestive of social castes and classes and their interrelations on a macroscopic level of large-scale social structure (Mayer 1960; Boyd 1969*a*). Various classical hypotheses lend themselves to possible blockmodel and algebraic reformulations: can the classical Aristotelian theory of revolution (as the result of separation between political and economic elites) be made operational through tracing the emergence of certain kinds of zeroblocks? The next analytic task is to provide ways to probe how role structures of the kind we have identified actually come into being, through the continuing accommodations and manipulations of all individuals acting simultaneously (Leach 1954). Some first steps have recently been taken in this direction (Lorrain 1971; White 1973; Spence 1974; Boorman 1975*a*).

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