

Horton's Law of Stream Numbers¹

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Abstract. The possibilities for obtaining a rational explanation of Horton's law of stream numbers are reviewed. The previous explanations of the stream-number law by (1) a growth model referring to generations of rivers and by (2) statistical graph-theory are compared. The graph-theoretical explanation seems superior to the other model of Horton's law, inasmuch as it produces not only the form of the law, but even, numerically, the naturally observed bifurcation ratio. It is also independent of the structural properties of the river net. (Key words: Geomorphology; rivers)

INTRODUCTION

Some years ago, Horton [1945] stated important laws concerning the average configuration of streams in a drainage basin. All these laws are based upon the notion of 'order' of a particular stream. Horton [1945] introduced a concept of 'order,' but his laws can be stated equally well if a somewhat different concept of 'order,' due to Strahler [1957], is used. According to Strahler, first-order stream segments are those that have no tributaries on a given map, second-order stream segments are those that have as tributaries only segments of first order, etc. The term 'order' thus refers to individual river segments, not to whole streams from headwaters to mouth. Thus, the confluence of a river segment of order n with another of order n produces a channel of order $n + 1$; the confluence of a river segment of order n with one of lower order is simply ignored, i.e., it produces again a segment of order n . In any counting procedure, a river segment is counted as a single segment between such junctions that produce a change in order.

Of the three laws of Horton, the first is concerned with the topology of a river net, the others with the metric. We discuss here only his topological law. It can be stated as follows:

The numbers n_i of stream segments of order i in a given drainage basin form, on the average (ensemble average over many similar basins), a geometric sequence, so that

$$n_{i+1} = \alpha n_i$$

This law is called the law of stream numbers. The quantity $R_s = 1/\alpha = n_i/n_{i+1}$ is often called the 'bifurcation ratio.' It is interesting to note that the numerical value of R_s seems to vary from 2 to 4, but is, for mature rivers, remarkably close to 3.5. Thus, Leopold *et al.* [1964, p. 138] noted that, among many samples of basins in the United States, the above figure of 3.5 is found.

Thus, considering the law of Horton stated above, it becomes evident that a plot of stream numbers on semilogarithmic paper against order should yield, on the average, a straight line. Making such a plot for a given drainage basin is called making a 'Horton analysis.'

Horton's law, as is implicit in the statement given above, has only statistical significance. It was found by Horton [1945] from purely empirical analyses of many stream basins. It is the aim of the present paper to review the rational explanations for it.

Horton's law as stated above is expressed in terms of Horton orders. It is more convenient, for actual calculations, to use Strahler orders instead of Horton orders. In the Strahler scheme of ordering, orders are assigned to stream segments in the manner indicated above. Horton's law can also be expressed in the same fashion for Strahler stream segments as for Horton streams, without making any significant error [Scheidegger, 1967b]. Thus, in our paper, 'order'

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always means Strahler order, unless another type of order is specifically mentioned.

HORTONIAN NETWORKS

The law of stream numbers discussed in this paper is probably the most important of Horton's laws. As is evident from its formulation, it is concerned only with the topology of a river net; metric properties are completely left out of consideration.

Horton's law of stream numbers has given rise to the concept of a structurally Hortonian network. This is a network of, say, order N , in which the bifurcation ratio $R_b = n_{i-1}/n_i$ for the numbers n_i of streams of order i is constant for all $i \leq N$, and in which all complete, i.e. maximum, subbasins of order $M < N$ have this same property with the same bifurcation ratio as that valid for the main basin.

Thus, a structurally Hortonian network is made up of elements as follows: on the average, R_b first-order streams combine to form a second-order stream; on the average, R_b such second-order streams combine to form a third-order stream, etc. In such a network, one can speak of 'cycles' or 'generations' of streams. The subbasins referred to above are always taken to include all elements of a complete cycle.

It is clear that a structurally Hortonian network obeys Horton's law of stream numbers; however, such networks are not the only ones that obey this law. It is evidently possible to conceive of a basin which obeys Horton's law of stream numbers, but which is not structurally Hortonian. This was noted, for instance, by Smart [1967]. In such networks, it is no longer possible to speak of 'cycles' or 'generations' of rivers, so that the subbasins do not satisfy Horton's law of stream numbers at all, or not with the same bifurcation ratio as the main basin. Such an example is given in Figure 1.

A test of a given network for structural Hortonianity has been given by the writer [Scheidegger, 1967b].

CYCLIC MODELS

Although Horton only counted stream numbers, Horton's law has generally been taken as implying that river nets are structurally Hortonian in the sense defined above, so that one can speak of 'cycles' or 'generations' of rivers.

Horton himself [1945, p. 339] attempted a

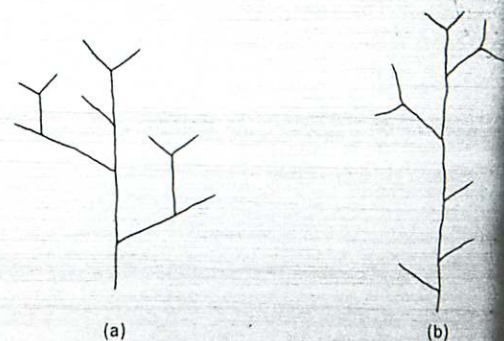


Fig. 1. Two river nets with 9 first-order streams, 3 second-order streams, and 1 third-order stream, so that the bifurcation ratio is constant and equal to 3. However, (a) is structurally Hortonian, (b) is not.

hydrophysical explanation of his law of stream numbers essentially in terms of a growth process, an idea which was formalized by Woldenberg [1966], who tried to explain this law in the expression of 'allometric growth' of a river system. Allometric growth means that the relative rate of growth of a part of the system is constant fraction of the relative rate of growth of the whole system. Woldenberg then intimates that the law of stream numbers really should use as 'order' a quantity x which is not the order N itself but is equal to the bifurcation ratio R_b to the power $N - 1$

$$x = (1/\alpha)^{N-1}$$

He referred to x as the 'absolute order' of the river segment under consideration; it would be representative of the 'size' of the river system at the point under consideration. If this 'absolute order' is used, Horton's law is expressed as a power function

$$n(x) = ax^b$$

which is characteristic of allometric growth.

The allometric growth process of Woldenberg can be further specified in various ways. Thus, for instance, one can envisage a birth and-death process that leads from generation to generation of rivers [Scheidegger, 1966]. It is clear that any model in which the process leading from one generation or cycle of rivers to the next is the same for every order will produce a structurally Hortonian network and therewith Horton's law of stream numbers.

However, no such growth processes can account for the possible existence in nature of non-Hortonian networks. It is not clear how an allometric growth process can be defined when a given river net is not structurally Hortonian, so that one cannot speak of generations or cycles of rivers. The whole allometric growth-model is dependent upon the idea that only structurally Hortonian networks should exist in nature.

GRAPH ENSEMBLE STATISTICS

Horton's law of stream numbers can also be obtained by a statistical model that is not based on the concept of cycles or generations of rivers. Thus, the writer has proposed [Scheidegger, 1967a] that a particular river network is the realization of a particular graph among all possible graphs (arborescences) with the same numbers of pendant vertices (first-order streams). The only reasonable assumption regarding the probability distribution for graphs (arborescences) with a given number of pendant vertices is that every possible graph is equally likely. This corresponds statistically to a 'microcanonical' distribution: Every state of the system that has a given value for the constant of the motion involved (the number of pendant vertices) is equally probable in the ensemble of all possible states. In the paper cited, the possible graphs (arborescences) up to an order of 10 were enumerated; for graphs with a higher number of pendant vertices, a Monte Carlo method was proposed [Scheidegger, 1967a; Liao and Scheidegger, 1968] to sample the conditions. The ensemble approach does not make the assumption that river nets are structurally Hortonian; in fact, among the possible graphs there are certainly graphs that are structurally non-Hortonian. These are counted in and averaged.

The result of using this procedure is startling. Not only does one obtain Horton's law of stream-order numbers, implying a roughly constant expectation value for the bifurcation ratio (there is a slight decrease with order), but the expectation value for the bifurcation ratio is very close to that observed in the mature rivers of the United States, viz., it is very close to 3.5. This is a great achievement of the ensemble-statistical theory, inasmuch as the theories based on generation of rivers cannot

produce a numerical value for the bifurcation ratio.

In context with the above, it may be noted that Shreve [1966], in a discussion of Horton's law of stream numbers, also introduced the hypothesis of 'randomness' of the topological configuration of drainage basins. However, although Shreve calculated probabilities of occurrences of certain channel configurations, he did not take the next logical step, involving the introduction of a microcanonical 'ensemble' over which observable quantities have to be averaged, so that expectation values can be calculated in entirely general terms.

DISCUSSION

We have seen above that essentially two different types of models have been advanced to explain the existence of Horton's law of stream numbers.

In the first type, one assumes that all river nets are structurally Hortonian. Horton's law of stream numbers is then a direct consequence of this assumption. In structurally Hortonian nets, one can speak of 'generations' of rivers, and an explanation is sought for the existence of such 'generations.' Such an explanation is obtained by assuming that the conditions leading from one 'generation' to the next are statistically invariable.

In the second type of model, no hypotheses about the intrinsic structure of river nets are formed, other than that such nets are simply topologically bifurcating arborescences. If, then, nature operates completely at random, all possible arborescences must be equally probable. These arborescences may or may not be structurally Hortonian. This approach produces not only Horton's law of stream numbers, but even the correct value for the actually observed bifurcation ratio.

In both models discussed above, one is dealing with statistical expectation values. However, it should be noted that the two possible types of models do not produce the same statistical populations: The cyclic type models produce only a part of the theoretically possible bifurcating arborescences, namely only those which are structurally Hortonian, whereas in the ensemble-statistical approach the possible non-Hortonian networks are counted in. It is evident, thus, that there is a fundamental difference

between the two types of models, inasmuch as the statistical populations are quite different in them.

At this juncture, it is a matter of speculation which of the two possibilities is realized in nature. Horton's law of stream numbers has generally been assumed as equivalent to the statement that river nets are structurally Horton nets, and Horton [1945, p. 339] has already attempted to give a rational explanation of his law upon this basis. However, he already noted the existence of 'adventitious streams' [Horton, 1945, p. 341], which do not fit into this pattern. Thus, even early observations seem to favor the graph theoretical approach. Theoretically, there does not seem to be any justification for the assumption that only those graph patterns that are structurally Hortonian should be naturally possible. Surely, nature must be held to be impartial and to produce all possible configurations equally often. The fact that the graph-ensemble theory produces the correct numerical value for the bifurcation ratio constitutes a further indication that this theory is probably correct.

Therefore, there is certainly a conceptual, if not a factual (i.e. observational) reason to prefer the graph-theoretical explanation of Horton's law of stream numbers over the others.

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