o of gradients. A similar relationship
rder streams have one more order of
ns, it follows that on the whole $\bar{Z}_{1}$ ...

the overall frequency of streams, the
streams of different orders.

am Lengths represent the geomorphic
llow automatically from the definition
system. The value of the coefficients
he influence of some of these factors

vater Storage and Base Flow, *Journal of*

d Their Drainage Basins; Hydrophysical
ull., **56**, pp. 275-370.
ntropy in Landscape Evolution, *U.S. Geol.*

Patterns, *Am. Jour. Sci.*, **262**, pp. 325-329.
s Law of Stream Numbers, *Trans. Am.*

niversity of Melbourne.
ne Watersheds in the Appalachian Plateau.

Factors in Erosional Terrain, *Trans. Am.*

A COMPLETE THERMODYNAMIC ANALOGY FOR LANDSCAPE EVOLUTION [1]

A.E. SCHEIDEGGER [2]

ABSTRACT

The analogy embodying an entropy concept for landscape evolution can be extended to all other thermodynamic functions.

INTRODUCTION

Leopold and Langbein (1962) recently postulated an analogy between landscape evolution and the nonsteady-state temperature distribution in a planar medium. Their argument is based on the concept of entropy and thus on a formal analogy with regard to the second principle of thermodynamics, between landscape, and temperature fields. Scheidegger (1964) has shown that there is a statistical justification for this, originally purely formal, analogy.

Since the success in explaining the evolution of landforms by means of the entropy-analogy is profound, the question arose as to whether the analogy with thermodynamics could not be extended further than originally envisaged. As has been noted above, the analogy, up to this point, pertains only to the entropy concept, i.e. it involves only the second principle of thermodynamics. One might expect that there also ought to be phenomena in landscape evolution that would be governed by a corollary of the first principle of thermodynamics. In other words, it might be expected that there is a complete analogy between landscape evolution and the (two-dimensional) nonsteady state temperature distribution in an ideal gas.

It is the aim of this paper to investigate the possibility of such a complete temperature analogy, and to show that the latter, indeed, exists.

THE COMPLETE CORRESPONDENCES

In ordre to fix the background of our investigation, we recall the analogy relations of Leopold and Langbein (1962) between a temperature field and a landscape.

The temperature field is described by the temperature T; the quantity of heat Q is associated with a temperature. The planar Cartesian coordinates are x and y.

The landscape is described by the elevation h of a point above sea level; the mass M is associated with an elevation. The planar Cartesian coordinates are again x and y.

The analogy between a thermal field and a landscape then maintains the following correspondences:

$$T \leftrightarrow h$$

$$\mathrm{d}Q \leftrightarrow \mathrm{d}M$$

Based on the above, it is possible to define corresponding entropies ($\mathrm{d}S = \mathrm{d}Q/T \leftrightarrow \mathrm{d}M/h$) and other thermodynamic properties. Furthermore, the quantity of heat introduced in a given substance is given by

$$\mathrm{d}Q = \gamma\, \mathrm{d}T$$

(1) Publication authorized by the Director, U.S. Geological Survey.
(2) U.S. Geological Survey, Address of author: c/o University of Illinois, Urbana, Illinois, U.S.A.

with γ being a heat capacity coefficient. The analog of this in a landscape is

$$\mathrm{d}M = \gamma \mathrm{d}h$$

where γ is now an analog of the heat-capacity coefficient.

Our task is now to extend the above correspondences to energy terms. For a regular thermodynamic system, the first principle of thermodynamics states (see e.g. Planck, 1945)

$$U_2 - U_1 = Q + W$$

or, in differentials

$$\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W$$

where U is the internal energy, Q is the quantity of heat introduced from outside and W the work performed externally on the system. In landscape evolution, one would like to have, therefore, a similar relation, viz.

$$U_2 - U_1 = M + W$$

or, in differentials

$$\mathrm{d}U = \mathrm{d}M + \mathrm{d}W$$

where U now signifies some potential, M the mass that was introduced and W some "fictitious work" whose physical meaning has yet to be defined.

For an ideal gas, W is

$$W = -\int_v p \mathrm{d}V$$

Here, V is the geometric domain in which the variables vary, and p the pressure. Because of the ideal gas law, the latter can be expressed as follows

$$p = \frac{RT}{V} = \text{const}\,\frac{T}{V}$$

The last relation yields a means of setting up an analogy to "pressure" in landscapes. In the latter, V corresponds to the area A under consideration, T is the height h (see above) so that one has

$$p_{\text{landscape}} = \text{const}\,\frac{h}{A},$$

at least in the equilibrium case.

If we are essentially interested in an "average" geographic cross section across a landscape, we have only one space-coordinate (x); denoting the total length of the section by L, we have (denoting the constant by α)

$$p_{\text{landscape}} = \text{const}\,\frac{h}{L} = \frac{h}{L}\alpha$$

The analog of work is then

$$W = -\int p \mathrm{d}V = -\int \alpha(h/L)\mathrm{d}L$$

is in a landscape is

nces to energy terms. For a regular
namics states (see e.g. Planck, 1945

roduced from outside and W the work
on, one would like to have, therefore,

as introduced and W some "fictitious

ry, and p the pressure. Because of the

gy to "pressure" in landscapes. In the
is the height h (see above) so that one

phic cross section across a landscape.
l length of the section by L, we have

$\frac{h}{L}\alpha$

$L)\mathrm{d}L$

The equilibrium case, which is here under discussion, would therefore be illustrated, so to ay, by a "box of sand", or by an alluvial fan with a straight surface-section (fig. 1) and h denoting average height.

The relationships for the "pressure" in a landscape and the potential already establish the mplete analogy between thermodynamics and certain variables in landscape theory.

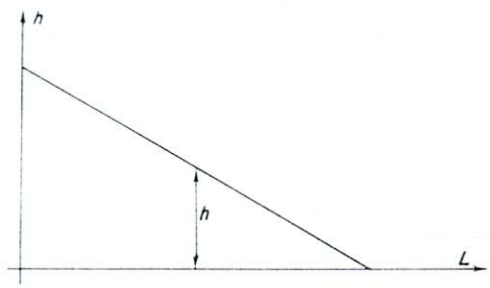

Fig. 1 — Alluvial fan (in section) representing a landscape of average height h.

ARNOT CYCLE

We illustrate the analogy established above on the example of a Carnot cycle. In ordinary ermodynamics involving an ideal gas, the Carnot cycle demonstrates the fundamental principles volved in a classical way.

The Carnot cycle for an ideal gas consists of the following processes (see e.g. Planck, 1945, 65):

1. An isothermal expansion of a certain quantity of gas at temperature T_1. During this phase, the quantity of heat Q_1 must be taken from a heat reservoir, as the gas performs the work W_1;
2. An adiabatic compression. The gas temperature goes from T_1 to T_2; at the same time the work W_2 must be done on the gas;
3. An isothermal compression. An amount of heat Q_2 enters a reservoir at T_2, and the work W_3 is done on the gas;
4. An adiabatic expansion until the temperature drops in the gas from T_2 to T_1; the gas performs the work W_4.

For the above Carnot cycle, the first principle of thermodynamics states (note that all the uantities Q, W are defined as positive)

$$Q_2 - Q_1 = W_2 - W_1 + W_3 - W_4,$$

e second principle implies

$$\frac{Q_2}{T_2} - \frac{Q_1}{T_1} = 0$$

To set up the analogy of the above process in a landscape, we assume that a landscape section composed of a certain mass of rock of length L with the average height h_1 at surface above ome given base level. The steps of the "Carnot" cycle (see fig. 2) are then:

1. The landscape section is extended from L_1 to L_2, with h being held at h_1. In order to do this, the mass M_1 must be added to the landscape, and the value of W is

$$W_1 = \int_{L_1}^{L_2} \alpha(h_1/L)\mathrm{d}L = \alpha h_1 \log\frac{L_2}{L_1}$$

2. The landscape material is compressed until the average height reaches h_2. No mass is added or subtracted. The quantity W_2 is

$$W_2 = \int_{h=h_1}^{h=h_2} \alpha \frac{h}{L(h)} \mathrm{d}L(h)$$

and since mass is constant, $hL = \text{const} = \beta$. The quantity W_2 follows:

$$W_2 = \int \alpha \frac{h^2}{\beta} \frac{\beta}{h^2} = \mathrm{d}h = \int_{h_1}^{h_2} \alpha \mathrm{d}h = \alpha(h_2 - h_1)$$

or

$$W_2 = \alpha(h_2 - h_1)$$

3. The landscape material is "compressed" some more at constant average height h_2. An amount M_2 of mass is taken from the system while W_3 is

$$W_3 = -\int_{L_3}^{L_4} \alpha \frac{h_2}{L} \mathrm{d}L = \alpha h_2 \log \frac{L_3}{L_4}$$

4. An expansion occurs with no mass added or subtracted until h drops from h_2 to h_1. We have

$$W_4 = \alpha(h_2 - h_1)$$

Note that all quantities W are defined so as to be positive. The cycle is now closed and the model is in its original state. The first principle of thermodynamics (i.e. its analog in the present case) states

$$\begin{aligned} M_2 - M_1 &= W_2 - W_1 + W_3 - W_4 \\ &= \alpha(h_2 - h_1) - \alpha h_1 \log \frac{L_2}{L_1} + \alpha h_2 \log \frac{L_3}{L_4} - \alpha(h_2 - h_1) \\ &= \alpha\left[-h_1 \log \frac{L_2}{L_1} + h_2 \log \frac{L_3}{L_4}\right]; \end{aligned}$$

note that $L_3 > L_4$, $L_2 > L_1$, $h_2 > h_1$. Thus

$$M_2 - M_1 = \alpha\left[h_2 \log \frac{L_3}{L_4} - h_1 \log \frac{L_2}{L_1}\right]$$

The last is a relation valid for a Carnot process of the type envisaged.

Next, the second fundamental principle of thermodynamics (i.e. its analogy in landscape evolution) states

$$\frac{M_2}{h_2} - \frac{M_1}{h_1} = 0$$

The latter is nothing but the expression of the fact that an addition of material to a landscape will make its elevation proportionately higher, so that the last equation can be taken as an expression of the law of conservation of mass. It is clear that this must be so, since the analogy of entropy in landscape evolution is justifiable (as shown by Scheidegger, 1964) by assuming that the process is a statistical one with mass being conserved.

Thus, the second principle leads to a further confirmation of the analogy postulated by Leopold and Langbein (1962) and verifies the contention of Scheidegger (1964) that this analogy is possible because mass must be conserved.

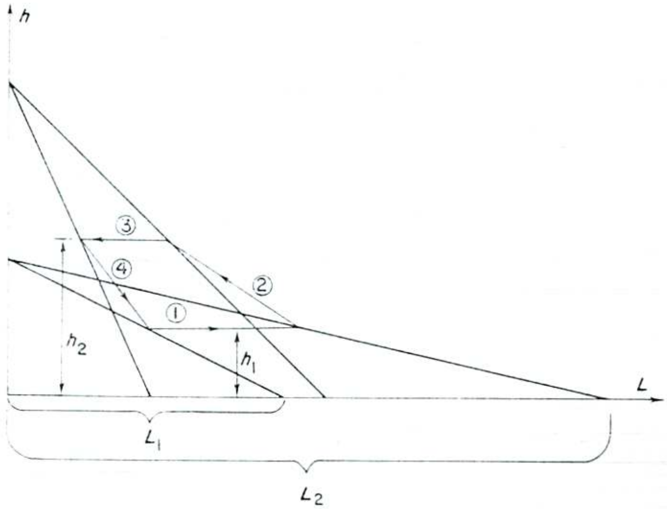

Fig. 2 — Carnot cycle in a landscape section.

It is possible to illustrate the Carnot cycle, for instance, with a hypothetical "alluvial fan" (see fig. 2). The height h refers to the mid-point of the fan ("average height"). Of course, step 2 is not one that is likely to occur in nature without external action, but neither is the corresponding step in gas-thermodynamics!

Since, in thermodynamics the Carnot process is one transforming work into heat (or icev ersa), in our analogy it connects the variables W and M.

Thermodynamic potentials

With the definition of W, it is now possible to set up a complete analog to thermodynamic functions in landscape theory.

The "potential" U is defined by

$$dU = dM + dW$$

Since we have been able to give a meaning to the quantity W in landscapes, it is clear that one now is able to assign a meaning to the potential U. The analog of entropy, S, was already defined earlier by Leopold and Langbein; it is clear that all ordinary thermodynamic functions have now found an equivalent in landscape theory. To recapitulate, we have (with α and γ being lag-one and lag-two auto-correlations respectively,

$$a = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}, \qquad b = \frac{\rho_2-\rho_1^2}{1-\rho_1^2} \tag{2}$$

If $\varrho_1^2 = \varrho_2$, then $a = \varrho_1$, $b = 0$, and equation (1) reduces to the first order Markov process

$$x_{t+2} = \rho_1 x_{t+1} + \varepsilon_{t+2} \tag{3}$$

If $\varrho_1 = 0$, then equation (3) reduces to

$$x_{t+2} = \varepsilon_{t+2} \tag{4}$$

a purely random process.

For a given record suppose that there are 50 observations which make up the record. If a first order process can be fitted to the record, then ϱ_1 indicates the amount of dependence among the 50 observations. Another way of saying this is that the 50 observations are time dependent, and because of this dependence each observation does not contribute as much information as if it were a random event. Hence, 50 random events would contribute more information than 50 dependent events. And in general, for $\varrho_1 > 0$, N random events contribute more information than N dependent events.

For the second order Markov process this is not always true—under certain conditions N dependent events will contribute more information than N random events, and in some cases the magnitude of the contribution is much larger than the sample size N would indicate. The following paragraphs will illustrate this point.

Let $x_1, x_2, \ldots, x_N$ be a series of size N which follows a second order Markov process as given in equation (1). For different values of N, ϱ_1, and ϱ_2 it is possible to determine the effective sample size N' of random events which yield the same amount of information as the N dependent events. This can be done by constructing the variance of the mean of the N dependent events, setting it equal to the variance of the mean of the N' random events and solving for N'.

Let $\bar{x}$ be the mean of the dependent series. Then

$$\text{Var}(\bar{x}) = E[\bar{x} - E(\bar{x})]^2 \tag{5}$$

Without loss in generality assume $E(\bar{x}) = 0$, then

$$\begin{aligned}\text{Var}(\bar{x}) &= E(\bar{x})^2 \\ &= E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right]^2 \\ &= \frac{1}{N^2} E\left[\sum_{i=1}^{N} x_i^2 + \sum_{\substack{i=1 \\ i<j}}^{N}\sum_{j=2}^{N} x_i x_j\right] \\ &= \frac{1}{N^2}\left[N\sigma^2 + 2E\left(\sum_{\substack{i=1 \\ i<j}}^{N}\sum_{j=2}^{N} x_i x_j\right)\right]\end{aligned} \tag{6}$$

where σ^2 is the variance of both the de
cross products can be written

$$\sum_{\substack{i=1 \\ i<j}}^{N}\sum_{j=2}^{N} x_i x_j = \; + \; \ldots \; +$$

Hence

$$E\left(\sum_{\substack{i=1 \\ i<j}}^{N}\sum_{j=2}^{N} x_i x_j\right) =$$

where ϱ_k is the kth autocorrelation. For
(Kendall: 1951)

$$\rho_k$$

where

co

$$\tan\psi$$

where a and b are the coefficients given in
Now ϱ_k can be rewritten by expanding sin

$$\rho_k = c^k \cot$$

Sum up ϱ_k over $\sin k\theta$ and $\cos k\theta$ (Jolle
summation in equation (8),

$$E(\Sigma\Sigma x_i$$

where

$$A = \frac{\cot\psi}{1 - 2c\cos\theta + c^2}\begin{bmatrix} Nc\sin\theta - \\ [2c^{N+2}\sin N \end{bmatrix}$$

$$B = \frac{1}{(1 - 2c\cos\theta + c^2)^2}\begin{bmatrix} c^{N+3}\cos(N - \\ - c^3\cos\theta + \\ + 3Nc^3\cos\theta \end{bmatrix}$$

where σ^2 is the variance of both the dependent and independent series. The summation of the cross products can be written

$$\sum_{\substack{i=1\\i<j}}^{N}\sum_{j=2}^{N} x_i x_j = \begin{aligned}&x_1x_2+x_1x_3+\ldots+x_1x_{N-1}+x_1x_N\\&+x_2x_3+x_3x_4+\ldots+x_2x_N\\&\ldots\\&+x_{N-1}x_N\end{aligned} \tag{7}$$

Hence

$$E(\sum_{\substack{i=1\\i<j}}^{N}\sum_{j=2}^{N} x_i x_j) = \sigma^2[\sum_{k=1}^{N-1}\rho_k + \sum_{k=1}^{N-2}\rho_k + \ldots + \sum_{k=1}^{1}\rho_k] \tag{8}$$

where ρ_k is the kth autocorrelation. For the second order Markov process ρ_k can be written as (Kendall: 1951)

$$\rho_k = \frac{c^k \sin(k\theta+\psi)}{\sin\psi} \tag{9}$$

where

$$c = \sqrt{-b}$$

$$\cos\theta = \frac{a}{2\sqrt{-b}} \tag{10}$$

$$\tan\psi = \frac{1+c^2}{1-c^2}\tan\theta$$

where a and b are the coefficients given in equation (1).
Now ρ_k can be rewritten by expanding $\sin(k\theta+\psi)$,

$$\rho_k = c^k \cot\psi \sin k\theta + c^k \cos k\theta \tag{11}$$

Sum up ρ_k over $\sin k\theta$ and $\cos k\theta$ (Jolley: 1961), thus giving compact expressions for each summation in equation (8),

$$E(\Sigma\Sigma x_i x_j) = \sigma^2(A+B) \tag{12}$$

where

$$\left.\begin{aligned}A &= \frac{\cot\psi}{1-2c\cos\theta+c^2}\left[\begin{aligned}&Nc\sin\theta-\\&[2c^{N+2}\sin N\theta - c^{N+1}\sin(N+1)\theta-\\&\qquad - c^{N+3}\sin(N-1)\theta + c(1-c^2)\sin\theta]\\&\qquad[1-2c\cos\theta+c^2]^{-1}\end{aligned}\right]\\ B &= \frac{1}{(1-2c\cos\theta+c^2)^2}\left[\begin{aligned}&c^{N+3}\cos(N-1)\theta - 2c^{N+2}\cos N\theta + c^{N+1}\cos(N+1)\theta\\&-c^3\cos\theta + Nc\cos\theta - Nc^2 - 2Nc^2\cos^2\theta - Nc^4+\\&+3Nc^3\cos\theta - c\cos\theta + 2c^2\end{aligned}\right]\end{aligned}\right\} \tag{13}$$

Hence

$$\mathrm{Var}(\bar{x}) = \frac{1}{N^2}[N\sigma^2 + 2\sigma^2(A+B)] \tag{14}$$

The effective number of observations N' is the number of random events whose variance of the mean equals the variance of the mean for a sequence of autocorrelated events.

The variance of the mean $\bar{x}'$ for the N' random events is

$$\mathrm{Var}(\bar{x}') = \frac{\sigma^2}{N'} \tag{15}$$

Equating (14) to (15) and solving for N',

$$N' = N\left[1 + \frac{2}{N}(A+B)\right]^{-1} \tag{16}$$

Tables 1-7 give values of N' for $N=5$, 10, 20, 30, 40, 50, 100 and for $0 \leq \rho_1 \leq 0.9$ and $-0.9 \leq \rho_2 \leq 0.9$. No negative values of ρ_1 are considered because hydrologic phenomena yield positive first order autocorrelations. The asterisks in the table represent certain values of ρ_1 and ρ_2 which are inadmissable due to mathematical constraints on the second order Markov process. These constraints are (Kendall: 1951),

$$\left.\begin{aligned}
&-1 < \frac{\rho_2-\rho_1^2}{1-\rho_1^2} < 0 \\
&\left[\frac{\rho_1-\rho_1\rho_2}{1-\rho_1^2}\right]^2 < 4\left|\frac{\rho_2-\rho_1^2}{1-\rho_1^2}\right| \\
&\left|\frac{\rho_1-\rho_1\rho_2}{1-\rho_1^2}\right| \bigg/ 2 < 1 \\
&\left|\sqrt{-\left(\frac{\rho_2-\rho_1^2}{1-\rho_1^2}\right)-\frac{(\rho_1-\rho_1\rho_2)^2}{4(1-\rho_1^2)^2}}\right| < 1
\end{aligned}\right\} \tag{17}$$

It is evident from tables 1-7 that negative values of ρ_2 yield high values of N' such that $N' > N$. Only for $\rho_2 = -0.1$ are there some values of N' such that $N' \leq N$. In many cases N' is much greater than N, and as ρ_2 increases, so does N'. For example, when $N=30$, $\rho_1=0.3$, $\rho_2=-0.8$, then $N'=850$. That is, 30 dependent events are contributing as much information as 850 random events—a startling result indeed! This is an extreme situation, however. For a more realistic example let $N=30$, $\rho_1=0.3$, $\rho_2=-0.2$. Then $N'=31$. One can speculate as to what is happening to produce such high values of N', and it would appear that the negative serial correlations are responsible.

As an example of the effect negative serial correlation has on a system, consider the case where $\rho_2=\rho_1^2$, so that the Markov process is first order. The effective sample size N' for this process (Dawdy and Matalas: 1964) is

$$N' = N\left[1 + \frac{2}{N}\left(\frac{N\rho_1(1-\rho_1)-\rho_1(1-\rho_1^N)}{(1-\rho_1)^2}\right)\right]^{-1} \tag{18}$$

en $N=30$, $\varrho_1=0.3$, then $N'=24$. But when $N=30$ and $\varrho_1=-0.3$, then $N'=54$. The negative al correlation is adding more information than the positive serial correlation is taking away.

MMARY AND CONCLUSIONS

From the values of N' in tables 1-7, it would seem that for the second order Markov process negative values of ϱ_2 are working more for the investigator than the positive values of ϱ_1 and re working against him. That is, sequences generated by the second order Markov process d more information than sequences generated by the first order Markov process. Whether the nd order process could accomodate certain hydrologic time series, as well as meteorologic geochronologic sequences, requires further research.

REFERENCES

DY, D.R. and N.C. MATALAS, 1964, Analysis of variance, covariance, and time series: Chap. 8-III, *Handbook of applied hydrology*, Edited by Ven Te Chow, Mc Graw-Hill, New York, N.Y.
AN, P.R., 1961, A study of the statistical predictability in stream runoff in the upper Colorado River Basin: High Altitude Observatory, U. of Colo., Boulder, Colo.
DALL, M.G., 1951, *The advanced theory of statistics*, Vol. II; Hafner Press, New York, N.Y.
ALAS, N.C. 1963, Autocorrelation of rainfall and streamflow minimums: U.S. Geological Survey, Prof. Paper 434-B.
PO, R.G., 1967, Stochastic model of daily river flow sequences: Hydrology Paper No. 18, Colorado tate University, Ft. Collins, Colorado.
JEVICH, V.M., 1964, Fluctuations of wet and dry years, Part II: Hydrology Paper No. 4, Colorado tate University, Ft. Collins, Colarado.