

The Scaling of the Numbers of Types and the Numbers of Things

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Principles of Complex Systems,
Vols. 1, 2, 3D, 4 Fourever, V for Vendetta

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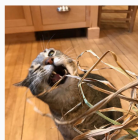
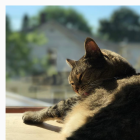
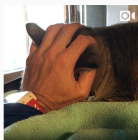
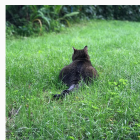
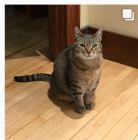
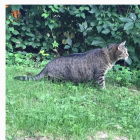
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Types and Tokens:



In linguistics, words are described on the two levels of types and tokens  ^[4].



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 In “Pride and Prejudice”, for example, there are 498 ‘!’s, 4,058 ‘the’s, 90 ‘love’s, and 0 ‘spork’s.



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Beyond language:

Lift out and expand the type-token framing to complex systems in general.

Three Four possible parts:



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²Fame.

Language:

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Atoms:

1. Type: Atom
2. Thing: Element (stuff made of a given atom; e.g., gold)
3. Measure: Mass; could be Number.
4. Experience: Atomic bonds.



Water:

1. Type: Water molecule, H^2O .
2. Thing: Water.
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Moneyspace:



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-  Apple and Microsoft may be viewed as components of the publicly-owned corporate world.
 -  The sizes of corporations may be broken down into many rankable dimensions such as annual revenue or number of employees worldwide.
 -  In principle, market capitalization represents a kind of current collective belief in terms of money.



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-  Perceived size is a combination of Measure (what exists) and Experience (what is measured).
-  Important: We may also have rankings where we do not know the underlying ‘size’ (e.g., book/thing sales on Amazon).



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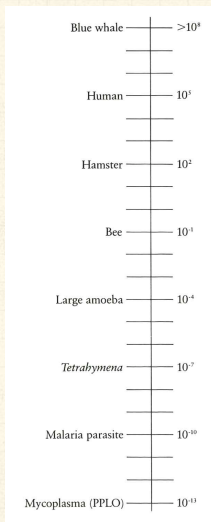
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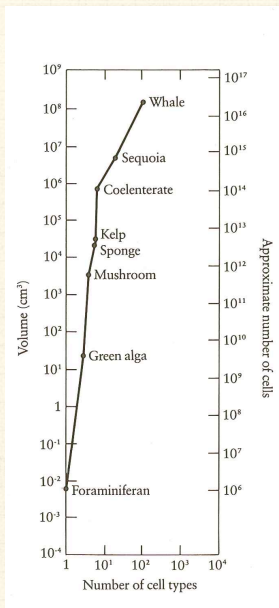
⁴Universal: Identical twins look the same until they don't.

From scaling: Size range (in grams) and cell differentiation:



10^{-13} to 10^8 g, p. 3,

McMahon and Bonner [3]



Scaling of Specialization:



“Scaling of Differentiation in Networks: Nervous Systems, Organisms, Ant Colonies, Ecosystems, Businesses, Universities, Cities, Electronic Circuits, and Legos”

Changizi, McDannald, and Widders,
J. Theor. Biol, **218**, 215–237, 2002. ^[1]

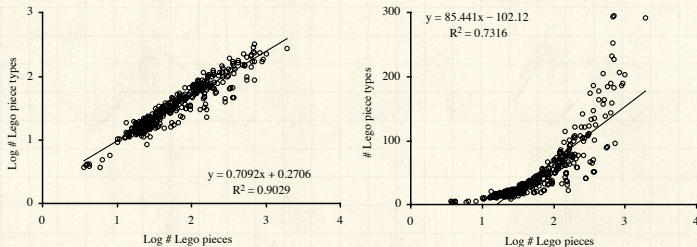


FIG. 3. Log-log (base 10) (left) and semi-log (right) plots of the number of Lego piece types vs. the total number of parts in Lego structures ($n = 391$). To help to distinguish the data points, logarithmic values were perturbed by adding a random number in the interval $[-0.05, 0.05]$, and non-logarithmic values were perturbed by adding a random number in the interval $[-1, 1]$.



Independent observation:

$$C \sim N^{1/d}, d \geq 1:$$

 C = network differentiation = # node types.

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-  Claim: Natural selection produces high d systems.
-  Claim: Engineering/brains produces low d systems.



TABLE 1
Summary of results*

Network	Node	No. data points	Range of log N	Log-log R^2	Semi-log R^2	$p_{\text{power}}/p_{\text{log}}$	Relationship between C and N	Comb. degree	Exponent ν for type-net scaling	Figure in text
<i>Selected networks</i>										
Electronic circuits	Component	373	2.12	0.747	0.602	0.05/4e-5	Power law	2.29	0.92	2
Legos tm	Piece	391	2.65	0.903	0.732	0.09/1e-7	Power law	1.41	—	3
Businesses										
military vessels	Employee	13	1.88	0.971	0.832	0.05/3e-3	Power law	1.60	—	4
military offices	Employee	8	1.59	0.964	0.789	0.16/0.16	Increasing	1.13	—	4
universities	Employee	9	1.55	0.786	0.749	0.27/0.27	Increasing	1.37	—	4
insurance co.	Employee	52	2.30	0.748	0.685	0.11/0.10	Increasing	3.04	—	4
Universities										
across schools	Faculty	112	2.72	0.695	0.549	0.09/0.01	Power law	1.81	—	5
history of Duke	Faculty	46	0.94	0.921	0.892	0.09/0.05	Increasing	2.07	—	5
Ant colonies										
caste = type	Ant	46	6.00	0.481	0.454	0.11/0.04	Power law	8.16	—	6
size range = type	Ant	22	5.24	0.658	0.548	0.17/0.04	Power law	8.00	—	6
Organisms	Cell	134	12.40	0.249	0.165	0.08/0.02	Power law	17.73	—	7
Neocortex	Neuron	10	0.85	0.520	0.584	0.16/0.16	Increasing	4.56	—	9
<i>Competitive networks</i>										
Biotas	Organism	—	—	—	—	—	Power law	≈ 3	0.3 to 1.0	—
Cities	Business	82	2.44	0.985	0.832	0.08/8e-8	Power law	1.56	—	10

* (1) The kind of network, (2) what the nodes are within that kind of network, (3) the number of data points, (4) the logarithmic range of network sizes N (i.e. $\log(N_{\text{max}}/N_{\text{min}})$), (5) the log-log correlation, (6) the semi-log correlation, (7) the serial-dependence probabilities under, respectively, power-law and logarithmic models, (8) the empirically determined best-fit relationship between differentiation C and organization size N (if one of the two models can be refuted with $p < 0.05$; otherwise we just write “increasing” to denote that neither model can be rejected), (9) the combinatorial degree (i.e. the inverse of the best-fit slope of a log-log plot of C versus N), (10) the scaling exponent for how quickly the edge-degree δ scales with type-network size C (in those places for which data exist), (11) figure in this text where the plots are presented. Values for biotas represent the broad trend from the literature.



Language:



The naturally-incorrectly-attributed⁵ Heaps' Law 

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⁵Plus one for Stigler's Law of Eponymy. 

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For words: $N_{\text{types}} \sim N_{\text{tokens}}^{\beta}$ where $0 \leq \beta \leq 1$.



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-  Applies when reading through a sufficiently large coherent text (e.g., a book)
-  Applies when sampling from a fixed power-law size distribution
-  Misapplies when reading through sequential texts.
-  Implied by this PoCS generated paper:



“Text mixing shapes the anatomy of rank-frequency distributions” 

Williams et al.,

Physical Review E, **91**, 052811, 2015. ^[5]



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The type-thing scaling law:

 Most generally: $N_{\text{types}} \sim N_{\text{things}}^{\beta}$ where $0 < \beta \leq 1$.

$$t = \sum_{r=1}^{N_{t,\alpha}} S_{r,t,\alpha} \simeq N_{t,\alpha}^{\alpha} \int_{z=1}^{N_{t,\alpha}} z^{-\alpha} dz \sim \frac{N_{t,\alpha}^{\alpha}}{1-\alpha} [N_{t,\alpha}^{1-\alpha} - 1] , \quad (1)$$



Type-token scaling for finite systems:



“Zipf’s Law leads to Heaps’ Law: Analyzing their relation in finite-size systems” 

Lü, Zhang, and Zhou,
PLOS ONE, 5, 1–11, 2010. ^[2]

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Types and Things

Type-thing scaling law

References



In a somewhat complicated way which we will fix up, Lü et al. determine that given a size-rank distribution:

$$S_r \sim r^{-\alpha} \quad (2)$$

for $r = 1, 2, \dots, N_{\text{types}}$ then

$$N_{\text{things}} \simeq \frac{N_{t,\alpha}^{\alpha}}{1-\alpha} [N_{t,\alpha}^{1-\alpha} - 1] . \quad (3)$$

Insert assignment question 



Given:

$$N_{\text{things}} \simeq \frac{N_{t,\alpha}^\alpha}{1-\alpha} [N_{t,\alpha}^{1-\alpha} - 1], \quad (4)$$

then

$$N_{\text{types}} \rightarrow \begin{cases} N_{\text{types}} & \text{for } \alpha = 0, \\ (1-\alpha)N_{\text{types}} & \text{for } 0 < \alpha \ll 1, \\ e^{W(N_{\text{types}})} \sim \frac{N_{\text{types}}}{\ln N_{\text{types}}} & \text{for } \alpha = 1, \\ (\alpha N_{\text{types}})^{1/\alpha} & \text{for } \alpha \gg 1, \\ 1 & \text{for } \alpha \rightarrow \infty, \end{cases} \quad (5)$$

where W is the Lambert⁶ W function.



⁶There can be only one  multivalued Lambert function .

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