

# Optimal Supply Networks I: Branching

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CSYS/MATH 6701, 6713, & a pretend number, 2024–2025

The PoCverse  
Optimal Supply  
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Optimal  
transportation

Optimal branching

Murray's law


Murray meets Tokunaga

References

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The PoCverse  
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Murray's law

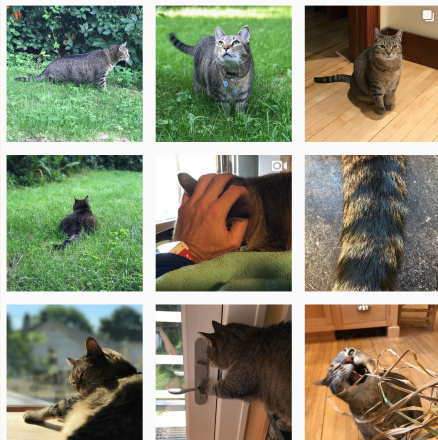
Murray meets Tokunaga



References



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# Outline

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# Optimal supply networks

What's the best way to distribute stuff?

 Stuff = medical services, energy, people, ...

 **Some** fundamental network problems:

1. Distribute stuff from a **single source** to **many sinks**
2. Distribute stuff from **many sources** to many sinks
3. **Redistribute** stuff between nodes that are both sources and sinks


 Supply and Collection are equivalent problems





# Single source optimal supply

Basic question for distribution/supply networks:

 How does flow behave given cost:


$$C = \sum_j I_j^\gamma Z_j$$

where

$I_j$  = current on link  $j$

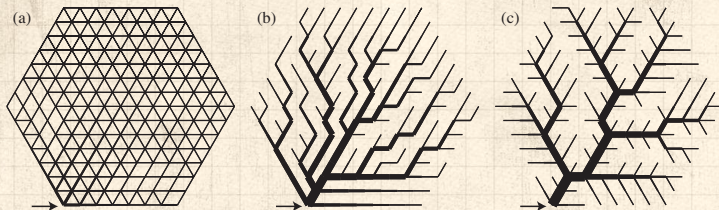
and

$Z_j$  = link  $j$ 's impedance.

 Example:  $\gamma = 2$  for electrical networks.




# Single source optimal supply



(a)  $\gamma > 1$ : Braided (bulk) flow

(b)  $\gamma < 1$ : Local minimum: Branching flow

(c)  $\gamma < 1$ : Global minimum: Branching flow

 Note: This is a single source supplying a region.

From Bohn and Magnasco <sup>[3]</sup>

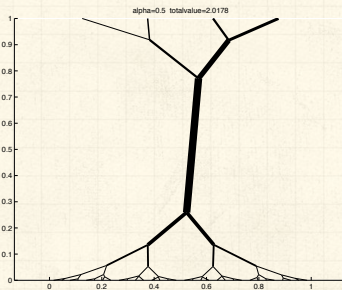
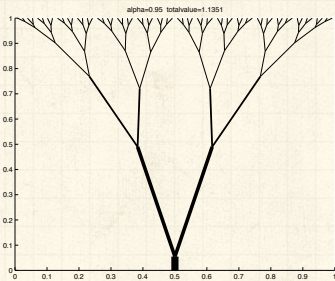
See also Banavar *et al.* <sup>[1]</sup>: “Topology of the Fittest Transportation Network”; focus is on presence or absence of loops—same story





# Single source optimal supply

## Optimal paths related to transport (Monge) problems



Optimal  
transportation


Optimal branching

Murray's law

Murray meets Tokunaga

References



“Optimal paths related to transport problems” 

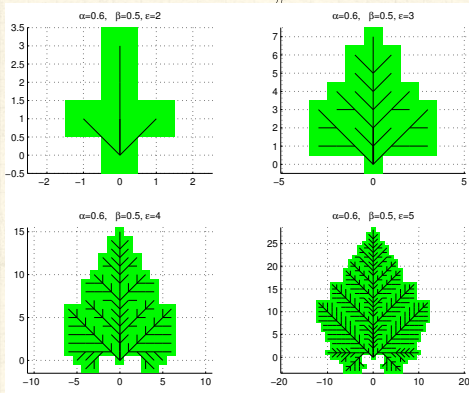
Qinglan Xia,


Communications in Contemporary Mathematics, **5**,  
251–279, 2003. <sup>[20]</sup>




# Growing networks—two parameter model: [21]

FIGURE 1.  $\alpha = 0.6, \beta = 0.5$



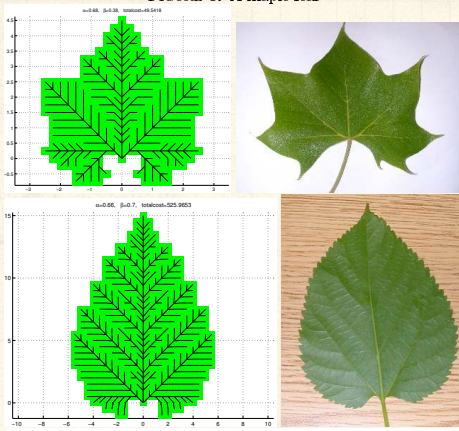
 Parameters control impedance ( $0 \leq \alpha < 1$ ) and angles of junctions ( $0 < \beta$ )


 For this example:  $\alpha = 0.6$  and  $\beta = 0.5$



# Growing networks: [21]

FIGURE 3. A maple leaf



 Top:  $\alpha = 0.66, \beta = 0.38$ ; Bottom:  $\alpha = 0.66, \beta = 0.70$



# Single source optimal supply

An immensely controversial issue ...

- 🧱 The form of natural branching networks:  
Random, optimal, or some combination? [6, 19, 2, 5, 4]
- 🧱 River networks, blood networks, trees, ...


Two observations:


- 🧱 Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- 🧱 Real networks **differ** in **details of scaling** but reasonably agree in **scaling relations**.



# River network models


## Optimality:


 Optimal channel networks <sup>[13]</sup>

 Thermodynamic analogy <sup>[14]</sup>

versus ...

## Randomness:

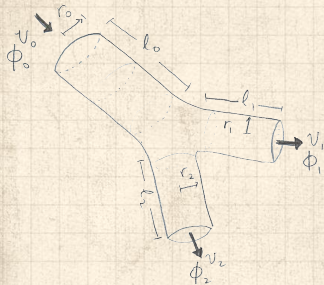
 Scheidegger's directed random networks


 Undirected random networks






# Optimization—Murray's law





 Murray's law (1926) connects  
branch radii at  
forks: [11, 10, 12, 7, 17]

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$


where  $r_{\text{parent}}$  = radius of  
'parent' branch, and  $r_{\text{offspring1}}$   
and  $r_{\text{offspring2}}$  are radii of the  
two 'offspring' sub-branches.

 Holds up well for outer branchings of blood networks [15].

 Also found to hold for trees [12, 8] when xylem is not a  
supporting structure [9].

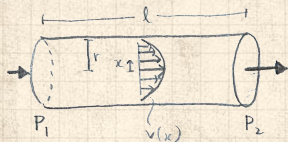
 See D'Arcy Thompson's "On Growth and Form" for  
background and general inspiration [16, 17].






 Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$


where  $\Delta p$  = pressure difference,  $\Phi$  = flux.




 Fluid mechanics: Poiseuille impedance  for smooth Poiseuille flow  in a tube of radius  $r$  and length  $\ell$ :

$$Z = \frac{8\eta\ell}{\pi r^4}$$

  $\eta$  = dynamic viscosity  (units:  $ML^{-1}T^{-1}$ ).

 Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z.$$

 Also have rate of energy expenditure in maintaining blood given metabolic constant  $c$ :


$$P_{\text{metabolic}} = cr^2\ell$$








# Optimization—Murray's law


## Aside on $P_{\text{drag}}$

 Work done =  $F \cdot d$  = energy transferred by force  $F$

 Power =  $P$  = rate work is done =  $F \cdot v$

  $\Delta p$  = Pressure differential = Force per unit area


  $\Phi$  = Volume flow per unit time (current)  
= cross-sectional area  $\cdot$  velocity

 So  $\Phi \Delta p$  = Force  $\cdot$  velocity





# Optimization—Murray's law



## Murray's law:

 Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell$$

 Observe power increases linearly with  $\ell$


 But  $r$ 's effect is nonlinear:

-  increasing  $r$  makes flow easier **but increases metabolic cost** (as  $r^2$ )
-  decreasing  $r$  decrease metabolic cost **but impedance goes up** (as  $r^{-4}$ )




# Optimization—Murray's law

## Murray's law:

 Minimize  $P$  with respect to  $r$ :

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left( \Phi^2 \frac{8\eta\ell}{\pi r^4} + cr^2\ell \right)$$

 Flow rates at each branching have to add up (else our organism is in serious trouble ...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches




# Optimization—Murray's law

## Murray's law:

 Find:

$$\Phi = kr^3$$


 Insert assignment question 


 All of this means we have a groovy cube-law:

$$r_{\text{parent}}^3 = r_{\text{offspring1}}^3 + r_{\text{offspring2}}^3$$




## Murray meets Tokunaga:


  $\Phi_\omega$  = volume rate of flow into an order  $\omega$  vessel segment

 Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

 Using  $\phi_\omega = kr_\omega^3$

$$(r_\omega)^3 = 2(r_{\omega-1})^3 + \sum_{k=1}^{\omega-1} T_k (r_{\omega-k})^3$$

 Same form as:

$$n_\omega = \underbrace{2n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}n_{\omega'}}_{\text{absorption}}$$



## Murray meets Tokunaga:


- Find Horton ratio for vessel radius  $R_r = r_\omega / r_{\omega-1}$ .
- Find  $R_r^3$  satisfies same equation as  $R_n$  and  $R_v$  ( $v$  is for volume):

$$R_r^3 = R_n = R_v$$

- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?





## Murray meets Tokunaga:


 Isometry:  $V_\omega \propto \ell_\omega^3$

 Gives

$$R_\ell^3 = R_r^3 = R_n = R_v$$





 We need one more constraint ...

 West *et al.* (1997) <sup>[19]</sup> achieve similar results following Horton's laws (but this work is a disaster).

 So does Turcotte *et al.* (1998) <sup>[18]</sup> using Tokunaga (sort of).



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