

# Mechanisms for Generating Power-Law Size Distributions, Part 2

Last updated: 2021/10/06, 23:35:55 EDT

Principles of Complex Systems, Vols. 1 & 2  
CSYS/MATH 300 and 303, 2021–2022 | @pocsvox

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Power-Law  
Mechanisms, Pt. 2

Variable  
transformation  
Basics  
Holtmark's Distribution  
PLIPLO  
References

## General Example

- Assume relationship between  $x$  and  $y$  is 1-1.
- Power-law relationship between variables:  
 $y = cx^{-\alpha}$ ,  $\alpha > 0$
- Look at  $y$  large and  $x$  small

$$dy = d(cx^{-\alpha})$$

$$= c(-\alpha)x^{-\alpha-1}dx$$

invert:  $dx = \frac{-1}{c\alpha}x^{\alpha+1}dy$

$$dx = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}dy$$

$$dx = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy$$

Now make transformation:

$$P_y(y)dy = P_x(x)dx$$

$$P_y(y)dy = P_x\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}dy$$

- If  $P_x(x) \rightarrow$  non-zero constant as  $x \rightarrow 0$  then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \rightarrow \infty.$$

- If  $P_x(x) \rightarrow x^\beta$  as  $x \rightarrow 0$  then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \rightarrow \infty.$$



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## Example

### Exponential distribution

Given  $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$  and  $y = cx^{-\alpha}$ , then

$$P(y) \propto y^{-1-1/\alpha} + O(y^{-1-2/\alpha})$$

- Exponentials arise from randomness (easy) ...
- More later when we cover robustness.



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# Gravity

- Select a random point in the universe  $\vec{x}$ .
- Measure the force of gravity  $F(\vec{x})$ .
- Observe that  $P_F(F) \sim F^{-5/2}$ .
- Distribution named after Holtmark who was thinking about electrostatics and plasma [1].
- Again, the humans naming things after humans, poorly.<sup>1</sup>



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## Transformation:

Using  $r \propto F^{-1/2}$ ,  $dr \propto F^{-3/2}dF$ , and  $P_r(r) \propto r^2$

$$\begin{aligned} P_F(F)dF &= P_r(r)dr \\ &\propto P_r(\text{const} \times F^{-1/2})F^{-3/2}dF \\ &\propto (F^{-1/2})^2 F^{-3/2}dF \\ &= F^{-1-3/2}dF \\ &= F^{-5/2}dF. \end{aligned}$$



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## Gravity:

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$$P_F(F) = F^{-5/2} dF$$

$$\gamma = 5/2$$

- Mean is finite.
- Variance =  $\infty$ .
- A **wild** distribution.
- **Upshot:** Random sampling of space usually safe but can end badly...

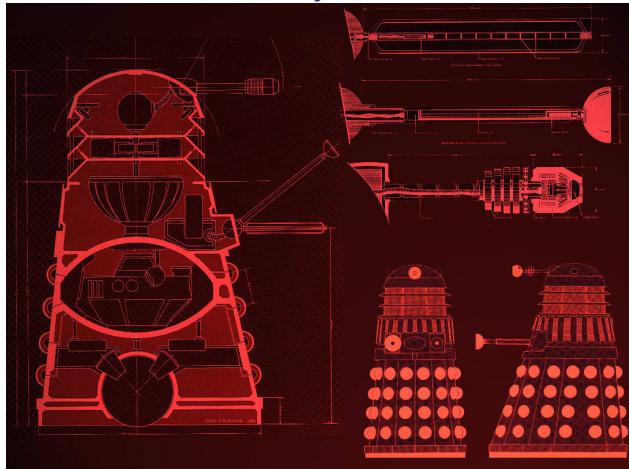
## Extreme Caution!

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- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument...
- Don't do this!!! (slap, slap)
- MIWO = Mild in, Wild out is the stuff.
- In general: We need mechanisms!

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Todo: Build Dalek army.



## References I

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