Lognormals and friends

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Outline


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THE MANIFESTO
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THE FAMILY OF SPHERICAL COWS
THE IMPROBABLE PACHYDERM
THE ILL-SCALING OF ALLOMETRIC BODIES
THE SUPPLY MOST FAIR
THE SUPPLY TO THE MOST
THE DARK ART OF GENERATING FUNCTIONS
THE RANDOM SUM OF RANDOMNESS
THE TWO-SIDED WORLDS
THE STORIES AND THE TROPES
THE TRUTH OF DIFFUSION
THE UNVEILING OF HIDDEN POWERS
THE SCRYING FOR THE SHAPES OF THINGS
THE LIMITS OF ORGANIZATIONS
THE OVERCOMING OF CATASTROPHE
THE TROUBLES ILL-FORMED
THE NETWORKS OF MANY ILKS
THE CHAOS OF FETTERED CONTAGION
THE END


THE MANY TALES OF MANY TAILS

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right)$$

2. Weibull distributions

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

   CCDF = stretched exponential.

3. Also: Gamma distribution, Erlang distribution, and more.

lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma}\mathrm{exp}\left(-\frac{(\mathrm{ln}x-\mu)^2}{2\sigma^2}\right)$$

- $\mathrm{ln}x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma}\text{exp}\left(-\frac{(\text{ln}x-\mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu+\frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.

Derivation from a normal distribution

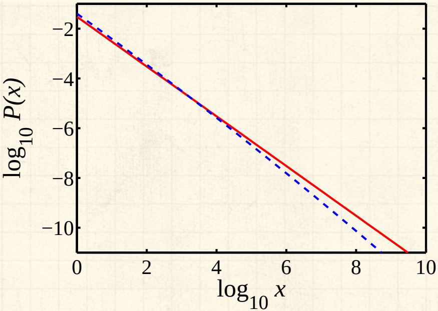
Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\mathrm{d}y$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

$$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$

$$= -\ln x - \ln\sqrt{2\pi}\sigma - \frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}.$$

If the first term is relatively small,

$$\boxed{\ln P(x) \sim -\left(1-\frac{\mu}{\sigma^2}\right)\ln x + \text{const.}} \Rightarrow \boxed{\gamma = 1-\frac{\mu}{\sigma^2}}$$

Confusion

- If $\mu < 0$, $\gamma > 1$ which is totally cool.
- If $\mu > 0$, $\gamma < 1$, not so much.
- If $\sigma^2 \gg 1$ and μ,

$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$:

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu)\log_{10} e \simeq 0.05(\sigma^2 - \mu)$$

- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

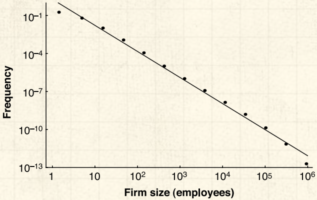
  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?

- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



Freq $\propto$ (size)$^{-\gamma}$
$\gamma \simeq 2$

- One piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1].

An explanation

- Axtel cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

Some math later...

Insert question from assignment 7

- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$

- where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$

N = total number of firms.

- Now, if $c/N \ll 1$ and $\gamma > 2$ $\quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$

- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$

- Groovy... c small $\Rightarrow \gamma \simeq 2$

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

Averaging lognormals

$$P(x) = \int_{t=0}^{\infty} a e^{-a t} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln\frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

- Insert fabulous calculation (team is spared).
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln\frac{x}{m})^2}}$$

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}}$$
- Depends on sign of $\ln \frac{x}{m}$, i.e., whether $\frac{x}{m} > 1$ or $\frac{x}{m} < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } \frac{x}{m} < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } \frac{x}{m} > 1 \end{cases}$$
- ‘Break’ in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website

Summary of these exciting developments:

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

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