Complex Networks, CSYS/MATH 303—Assignment 6
University of Vermont, Spring 2011

Dispersed: Thursday, March 10, 2011.
Due: By start of lecture, 2:30 pm, Thursday, March 31, 2011.
Some useful reminders:
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Office hours: 3:45 pm to 4:15 pm Tuesday post class; 12:00 pm to 2:00 pm, Wednesday
Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2011-01UVM-303`

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related variant).

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1. Generating functions and giant components: In this question, you will use generating functions to obtain a number of results we found in class for standard random networks.

   (a) For an infinite standard random network (Erdös-Rényi/ER network) with average degree $\langle k \rangle$, compute the generating function F_P for the degree distribution P_k.
   (Recall the degree distribution is Poisson: $P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$, $k \geq 0$.)

   (b) Show that $F'_P(1) = \langle k \rangle$ (as it should).

   (c) Using the joyous properties of generating functions, show that the second moment of the degree distribution is $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.

2. (a) Continuing on from Q1 for infinite standard random networks, find the generating function $F_R(x)$ for the $\{R_k\}$, where R_k is the probability that a node arrived at by following a random direction on a randomly chosen edge has k outgoing edges.

   (b) Now, using $F_R(x)$ determine the average number of outgoing edges from a randomly-arrived-at-along-a-random-edge node.

   (c) Given your findings above, what is the condition on $\langle k \rangle$ for a standard random network to have a giant component?

3. (a) Find the generating function for the degree distribution P_k of a finite random network with N nodes and an edge probability of p.
   (b) Show that the generating function for the finite ER network tends to the generating function for the infinite one. Do this by taking the limit $N \to \infty$ and $p \to 0$ such that $p(N-1) = \langle k \rangle$ remains constant.

4. (a) Prove that if random variables U and V are distributed over the non-negative integers then the generating function for the random variable $W = U + V$ is

$$F_W(x) = F_U(x) F_V(x).$$

   Denote the specific distributions by $\mathbf{Pr}(U = i) = U_i$, $\mathbf{Pr}(V = i) = V_i$, and $\mathbf{Pr}(W = i) = W_i$.

   (b) Using the your result in part (a), argue that if

$$W = \sum_{j=1}^{U} V^{(j)}$$

   where $V^{(j)} \stackrel{d}{=} V$ then

$$F_W(x) = F_U(F_V(x)).$$

   Hint: write down the generating function of probability distribution of $\sum_{j=1}^{k} V^{(j)}$ in terms of $F_V(x)$.

5. Again, given

$$W = \sum_{i=1}^{U} V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

   where we know that

$$F_W(x) = F_U\left(F_V(x)\right),$$

   determine the mean of W in terms of the means of U and V.