

Lecture 2 (Chapter 2)

Linear Algebra, Course 124A, Fall, 2009

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Basics:

- ▶ **Instructor:** Prof. Peter Dodds
- ▶ **Lecture room and meeting times:**
Living and Learning, CM 216, Tuesday and Thursday, 1:00 pm to 2:15 pm
- ▶ **Office:** 203 Lord House, 16 Colchester Avenue
- ▶ **E-mail:** pdodds@uvm.edu
- ▶ **Course website:**
<http://www.uvm.edu/~pdodds/teaching/courses/2009-08UVM-124/>
- ▶ **Textbook:** “Introduction to Linear Algebra” (4th ed.)
by Gilbert Strang; Wellesley-Cambridge Press.

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Admin:

Paper products:

1. Outline

Papers to read:

1. “The Fundamental Theorem of Linear Algebra” [1]
2. “Too Much Calculus” [2]

Office hours:

- ▶ Tuesday: 2:30 pm to 4:30 pm
- Thursday: 11:30 am to 12:30 pm
- Rm 203, Math Building

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Grading breakdown:

1. Assignments (40%)

- ▶ Ten one-week assignments.
- ▶ Lowest assignment score will be dropped.
- ▶ The last assignment cannot be dropped!
- ▶ Each assignment will have a random bonus point question which has nothing to do with linear algebra.

2. Midterm exams (35%)

- ▶ Three 75 minutes tests distributed throughout the course, all of equal weighting.

3. Final exam (24%)

- ▶ \leq Three hours of joyful celebration.
- ▶ December 17, 8:00 am to 11:00 am; held in [Lafayette L108](#).

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Grading breakdown:

1. **Homework (0%)**—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.
2. **General attendance (1%)**—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline.

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How grading works:

Questions are worth 3 points according to the following scale:

- ▶ 3 = correct or very nearly so.
- ▶ 2 = acceptable but needs some revisions.
- ▶ 1 = needs major revisions.
- ▶ 0 = way off.

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Schedule:

The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

Week # (dates)	Tuesday	Thursday
1 (9/1, 9/3)	Lecture	Lecture ► A1
2 (9/8, 9/10)	Lecture	Lecture ► A2
3 (9/15, 9/17)	Lecture	Lecture ► A3
4 (9/22, 9/24)	Lecture	<i>Test 1</i>
5 (9/29, 10/1)	Lecture	Lecture ► A4
6 (10/6, 10/8)	Lecture	Lecture ► A5
7 (10/13, 10/15)	Lecture	Lecture ► A6
8 (10/20, 10/22)	Lecture	<i>Test 2</i>
9 (10/27, 10/29)	Lecture	Lecture ► A7
10 (11/3, 11/5)	Lecture	Lecture ► A8
11 (11/10, 11/12)	Lecture	Lecture ► A9
12 (11/17, 11/19)	Lecture	<i>Test 3</i>
13 (11/24, 11/26)	Thanksgiving	Thanksgiving
14 (12/1, 12/3)	Lecture	Lecture ► A10
15 (12/8, 12/10)	Lecture	Lecture

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Important dates:

1. Classes run from Monday, August 31 to Wednesday, December 9.
2. Add/Drop, Audit, Pass/No Pass deadline—Monday, September 14.
3. Last day to withdraw—Friday, November 6.
4. Reading and exam period—Thursday, December 10 to Friday, December 18.

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More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

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More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).
2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, **Academic honesty** will be expected and departures will be dealt with appropriately. See <http://www.uvm.edu/cses/> for guidelines.

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More stuff:

Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Students are encouraged to use Matlab or something similar to check their work.

Note: for assignment problems, written details of calculations will be required.

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Grading:

A+	97–100	B+	87–89	C+	77–79	D+	67–69
A	93–96	B	83–86	C	73–76	D	63–66
A-	90–92	B-	80–82	C-	70–72	D-	60–62

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Why are we doing this?

Linear Algebra is

a body of mathematics
that deals with **discrete problems**.

Many things are discrete:

- ▶ Information (0's & 1's, letters, words)
- ▶ People (sociology)
- ▶ Networks (the Web, people again, food webs, ...)
- ▶ Sounds (musical notes)

Even more:

If real data is continuous, we almost always discretize it
(0's and 1's)

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Why are we doing this?

Linear Algebra is used in many fields to solve problems:

- ▶ Engineering
- ▶ Computer Science (Google's Pagerank)
- ▶ Physics
- ▶ Economics
- ▶ Biology
- ▶ Ecology
- ▶ ...

Linear Algebra is as important as calculus.

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Matrices as gadgets:

A transforms \vec{x} into \vec{x}' through multiplication

$$\vec{x}' = A\vec{x}$$

Can use matrices to:

- ▶ Grow vectors
- ▶ Shrink vectors
- ▶ Rotate vectors
- ▶ Flip vectors
- ▶ Do all these things to different directions

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Three key problems of Linear Algebra

1. Given a matrix A and a vector \vec{b} , find \vec{x} such that

$$A\vec{x} = \vec{b}.$$

2. Eigenvalue problem: Given A , find λ and \vec{v} such that

$$A\vec{v} = \lambda\vec{v}.$$

3. Coupled linear differential equations:

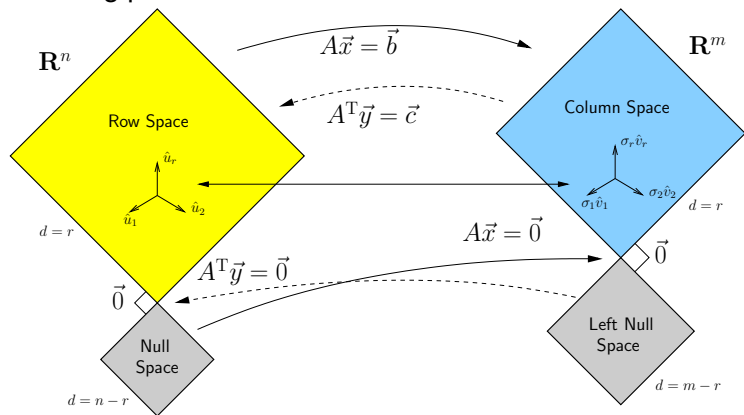
$$\frac{d}{dt}y(t) = Ay(t)$$

- ▶ Our focus will be largely on #1, partly on #2.

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Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:



What is going on here? We have 26 lectures to find out...

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Our friend $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:

- ▶ \vec{b} represents reality (e.g., music, structure)
- ▶ A contains building blocks (e.g., notes, shapes)
- ▶ \vec{x} specifies how we combine our building blocks to represent \vec{b} .

How can we disentangle an orchestra's sound?

What about pictures, waves, signals, ...?

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Our friend $A\vec{x} = \vec{b}$

What does knowing \vec{x} give us?

If we can represent reality as a superposition (or combination) of simple elements, we can do many things:

- ▶ Compress information
- ▶ See how we can alter information
- ▶ Find a system's simplest representation
- ▶ Find a system's most important elements
- ▶ See how to adjust a system in a principled way

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Three ways to understand $A\vec{x} = \vec{b}$:

- ▶ Way 1: The **Row** Picture
- ▶ Way 2: The **Column** Picture
- ▶ Way 3: The **Matrix** Picture

Example:



$$\begin{array}{rclcl} -x_1 & + & x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 4 \end{array}$$

- ▶ Call this a 2 by 2 system of equations.
- ▶ 2 equations with 2 unknowns.
- ▶ Standard method of solving by adding and subtracting multiples of equations from each other
= **Row Picture**

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Three ways to understand $A\vec{x} = \vec{b}$:

Row Picture—what we are doing:

- ▶ (a) Finding intersection of two lines
- ▶ (b) Finding the values of x_1 and x_2 for which both equations are satisfied (true/happy)
- ▶ A splendid and deep connection:
(a) Geometry \Leftrightarrow (b) Algebra

Three possible kinds of solution:

1. Lines intersect at one point — **One, unique solution**
2. Lines are parallel and disjoint — **No solutions**
3. Lines are the same — **Infinitely many solutions**

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Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

See

$$\begin{array}{rcl} -x_1 & + & x_2 = 1 \\ 2x_1 & + & x_2 = 4 \end{array}$$

as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

General problem

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

- ▶ Column vectors are 'building blocks'
- ▶ **Key idea:** try to 'reach' \vec{b} by combining multiples of column vectors \vec{a}_1 and \vec{a}_2 .

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Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- ▶ Intuitive.
- ▶ Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \not\parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \not\parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions

Assuming neither \vec{a}_1 or \vec{a}_2 are $\vec{0}$.

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Three ways to understand $A\vec{x} = \vec{b}$:

Difficulties:

- ▶ Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- ▶ **No!** We can still find the \vec{x} that gets us as close to \vec{b} as possible.
- ▶ Method of approximation—very important!
- ▶ We may not have the right building blocks but we can do our best.

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Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:

Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

as

$$A\vec{x} = \vec{b} : \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

A is now an operator:

- ▶ A transforms \vec{x} into \vec{b} .
- ▶ In general, A does two things to \vec{x} :
 1. Rotation
 2. Dilation (stretching/contraction)

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The Matrix Picture

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Key idea in linear algebra:

- ▶ Decomposition (or factorization) of matrices.
- ▶ Matrices can often be written as products or sums of simpler matrices
- ▶ $A = LU$, $A = QR$, $A = U\Sigma V^T$, $A = \sum_i \lambda_i \vec{v} \vec{v}^T$, ...

The truth about mathematics

Ch. 2: Lec. 2

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References

The Colbert Report on Math (February 7, 2006)

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References I

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